Quasiconformal maps in the plane, Problem set II

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1 Problems

1. Prove Weyl's lemma: If U is an open set and $f : U \to \mathbb{C}$ is a distribution, then f is analytic in U if and only if $\partial_{\bar{z}} f = 0$ in the distributional sense.

2. In this problem you will construct a homeomorphism f such that $\partial_{\bar{z}} f = 0$ a.e., yet f is not (quasi-)conformal.

   (a) Let $g(x) : \mathbb{R} \to \mathbb{R}$ be the "devil's staircase" aka Cantor function on $[0, 1]$ and set $g = 0$ for $x < 0$ and $g = 1$ for $x > 1$. (That is, if $C \subset [0, 1]$ is the standard Cantor 1/3-set, then for $x \in C$ define $g(x) = \sum_{n=1}^{\infty} a_n/2^n$ if $x = \sum_{n=1}^{\infty} 2a_n/3^n$ for $a_n \in \{0, 1\}$ and if and $x \in [0, 1] \setminus C$ we define $g(x) = \sup_{y \le x : y \in C} g(y)$. There are other constructions that you may look up. This function is continuous, but not absolutely continuous.)

   (b) Show that $g'(x) = 0$ on $\mathbb{R} \setminus C$. Conclude that $g'(x) = 0$ a.e.

   (c) Define a function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z + ig(x)$, $(z = x + iy)$.

   (d) Show that f is a homeomorphism which is differentiable except on the set $C \times i\mathbb{R}$, that is area-a.e. in $\mathbb{C}$. Conclude that $\partial_{\bar{z}} f = 0$ a.e. in $\mathbb{C}$.

   (e) Show that f is not quasiconformal. What conclusion can you draw about the distributional derivatives of f?

3. Let $(\mu_n)_{n=1}^{\infty}, \mu$ be Beltrami coefficients such that $\mu_n \to \mu$ a.e. Suppose further that for all n, $\|\mu\|_\infty, \|\mu_n\|_\infty \le \kappa < 1$ and that there is some $R < \infty$ so that the supports of $\mu, \mu_n, n \ge 1$, are contained in $B(0, R)$. Let f_n, f be the corresponding normal (in the sense of Ahlfors p55) solutions to the Beltrami equation. Prove that (a) there exists $p > 2$ such that $\|\partial_z f_n - \partial_z f\|_p \to 0$ as $n \to \infty$ and (b) $f_n \to f$ as $n \to \infty$ uniformly on compact sets.

4. Optional, but a good exercise: derive the versions of Green's formula in complex coordinates that was stated in Lecture 4:

$$\int_{\partial D} f dz = 2i \int_D \partial_{\bar{z}} f d^2 z$$

and

$$\int_{\partial D} f d\bar{z} = -2i \int_D \partial_z f d^2 z$$

(You may assume D is a smooth Jordan domain, f is smooth in a neighborhood of D.) Deduce the generalized Cauchy formula for $f \in C_0^\infty(\mathbb{C})$.