Quasiconformal mappings and Teichmüller Theory

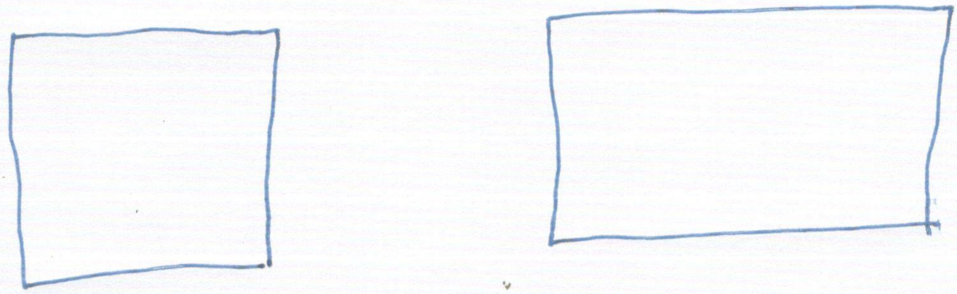
Dec 2020

- Grötzsch problem
- Quasiconformal mappings
- Ahlfors - Bers theorem
- Teichmüller theory

Grötsch

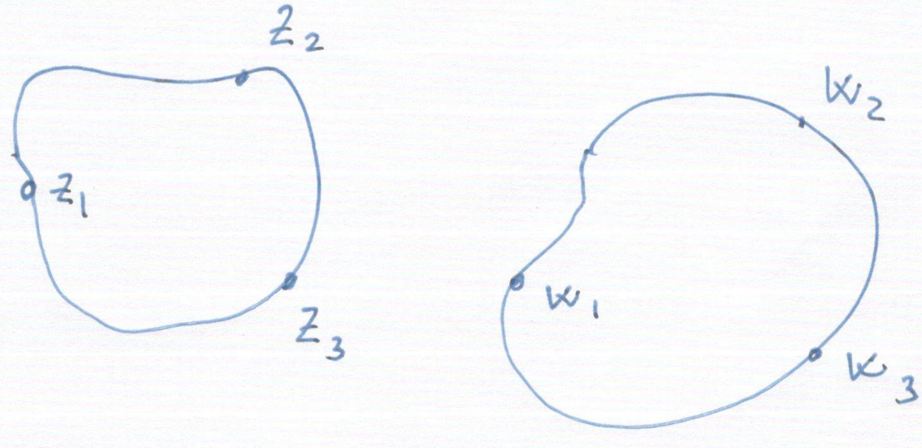
Q is a square

R is a rectangle $\neq$ square



What is the most near conformal map from Q to R?

Well known: For a conformal map one can prescribe 3 pts on the boundary



but not 4.

$w = f(z)$ $\quad C^1$ homeo

the tangent map (derivative)

can be written as

$$\begin{cases} du = u_x dx + u_y dy \\ dv = v_x dx + v_y dy \end{cases}$$

Complex notation

$$dw = f_z \, dz + f_{\bar{z}} \, d\bar{z}$$

$$f_z = \frac{1}{2}(f_x - i f_y)$$

$$f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$$

The Riemannian distance is

$$du^2 + dv^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2$$

$$E = u_x^2 + v_x^2$$

$$F = u_x u_y + v_x v_y$$

$$G = u_y^2 + v_y^2$$

Eigenvalues: $\begin{vmatrix} E-\lambda & F \\ F & G-\lambda \end{vmatrix} = 0$

Infinitesimal ellipses are maped to cirkles.

$$f_z = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y)$$

$$f_{\bar z} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

$$|f_z|^2 - |f_{\bar z}|^2 = u_x v_y - u_y u_x = J$$

(Jacobian)

We will consider the sense preserving case when $J > 0$, $|f_{\bar z}| < |f_z|$

Now $dw = f_z\,dz + f_{\bar z}\,d\bar z$

gives

$$(|f_z| - |f_{\bar z}|)\,|dz| \le |dw| \le (|f_z| + |f_{\bar z}|)\,|dz|$$

Ratio of major axis to minor axis is

$$D_f = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \ge 1$$

D_f Dilatation

Also $$d_f = \frac{|f_{\bar z}|}{|f_z|} < 1$$

$$D_f = \frac{1+d_f}{1-d_f}, \quad d_f$$

Introduce complex dilatation as

$$\mu_f = \frac{f_{\bar z}}{f_z}, \quad |\mu_f| = d_f$$

Max correspond to

$$\arg dz = \alpha = \frac{1}{2} \arg \mu$$

Min to

$$\arg dz = \alpha \pm \frac{\pi}{2}$$

In w plane dir of major axis is

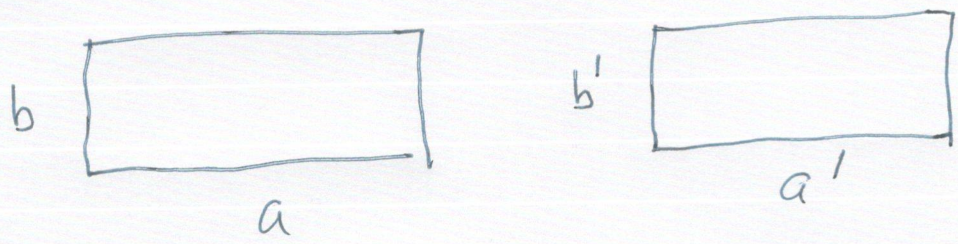
$$\arg dw = \beta, \quad \beta - \alpha = \arg f_z$$

Solution to Grötzsch problem

Let R be a rectangle with sides a, b

R' " " " a', b'

Can assume $\frac{a}{b} \leq \frac{a'}{b'}$



$$a' \leq \int_0^a |df(x+iy)| \leq \int_0^a (|f_z| + |f_{\bar z}|)\,dx$$

$$a'b \leq \int_0^a \int_0^b (|f_z| + |f_{\bar z}|)\,dx\,dy$$

$$a'^2 b^2 \leq \int_0^a \int_0^b \left(\frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \right) dx\,dy \int_0^a \int_0^b (|f_z|^2 - |f_{\bar z}$$

$$= a'b' \int_0^a \int_0^b D_f \,dx\,dy$$

$$\frac{\frac{a'}{b'}}{\frac{a}{b}} \leq \sup D_f$$

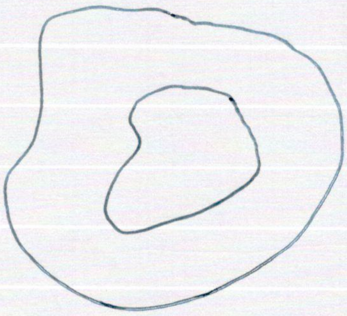
Min attained for

$$f(z) = \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) z + \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right)$$

Similar problem

Equivalence of annuli

An annular domain



can always be mapped to a circular annular domain $r_1 < |z| < r_2$

but two annuli

$$A_1 = \{ r_1 < |z| < r_2 \}$$

$$A_2 = \{ R_1 < |w| < R_2 \}$$

are conformally equivalent iff $\dfrac{r_2}{r_1} = \dfrac{R_2}{R_1}$

There is a "Grötzsch" problem mapping

$$\{ 1 < |z| < R_1 \} \to \{ 1 < |w| < R_2 \}.$$

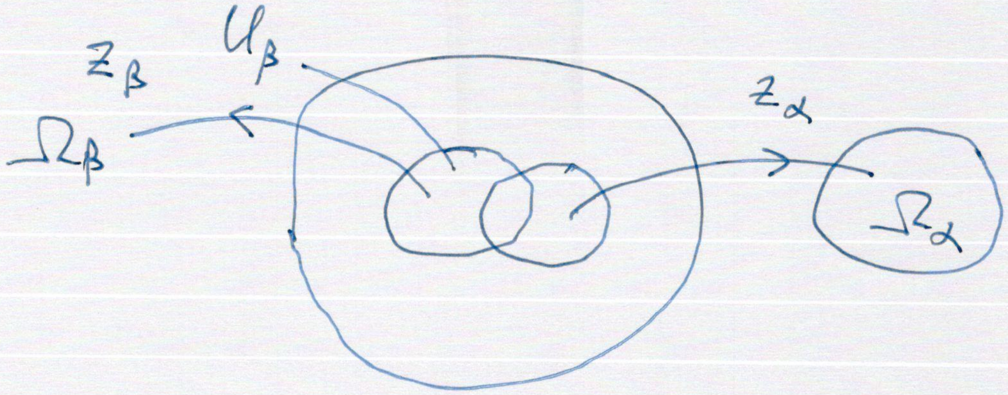
The extremal map is

$$re^{i\varphi} \to \exp\{ B \log r \} e^{i\varphi}$$

$$B = \frac{\log R_2}{\log R_1}$$

Riemann Surfaces.

One-dimensional Complex manifold



$\exists$ atlas $z_\alpha : U_\alpha \to \Omega_\alpha$

$$z_\alpha \circ z_\beta^{-1} \Big|_{z_\beta(U_\alpha \cap U_\beta)}$$

holomorphic

What is the structure of Riemann surfaces?

Fundamental theorem. (Koebe)

There are only 3 conformally inequivalent simply connected Riemann surfaces

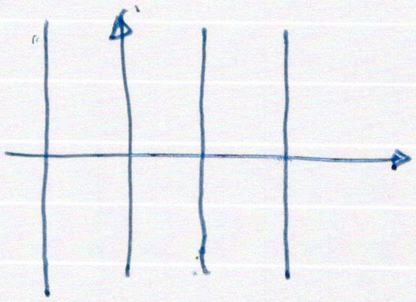
1) $\hat{\mathbb{C}} = \hat{\mathbb{C}} \cup \{\infty\}$ (elliptic)
2) $\mathbb{C}$ (parabolic)
3) Unit disk $\mathbb{D} \simeq \mathbb{H}$, Upper halfplane (hyperbolic)

The only elliptic Riemann surface is $\hat{\mathbb{C}}$

There are three parabolic RS

1) $\mathbb{C}$ itself

2) $\mathbb{C}/\mathbb{Z}$



3) $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, $\operatorname{Im}\tau>0$

Note $\mathbb{C}/(\tau_1\mathbb{Z}+\tau_2\mathbb{Z})$

and $\mathbb{C}/(\tau_1'\mathbb{Z}+\tau_2'\mathbb{Z})$

are equivalent if $\dfrac{\tau_2}{\tau_1}=\dfrac{\tau_2'}{\tau_2'}$

The upper halfplane

$$\mathbb{H}=\{\tau:\operatorname{Im}\tau>0\}$$

is the parameter space for parabolic Riemann surfaces top. equiv to a torus

This is the "first" exaple of a Teichmüller space

3) Hyperbolic Riemann surfaces X

This is isomorphic to $\mathbb{H}/\Gamma$

Γ discrete subgroup of Moebius transformations of self maps of $\mathbb{H}$. ($PSL_2(\mathbb{R})$).

Γ discrete if $\{g^n z_0\}_{n=-\infty}^{\infty}$ discrete

Special case X is compact.

Then the topological type is determined by its genus g, $g \geq 2$.

The Teichmüller space is complex $3g-3$ dimensional (real $6g-6$ dim).

<u>Royden's theorem</u>. This Teichmüller space has the structure

Isotermal coordinates

Suppose X is a Riemannian manifold

It's Riemannian metric is given by

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2$$

In complex coordinates $z = x+iy$

it takes the form

$$ds^2 = \lambda \,|dz + \mu\, d\bar{z}|^2$$

$$\lambda,\ \mu \text{ smooth} \quad \|\mu\|_\infty < 1$$

$$\lambda = \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right)$$

$$\mu = (E - G + 2iF)/\lambda$$

In isotermal coordinates $w = u+iv$

the metric takes the form

$$ds^2 = \rho\,(du^2 + dv^2) \quad , \ \rho > 0 \text{ smooth}$$

The complex coordinate $w = u+iv$ satisfies

$$\rho\,|dw|^2 = \rho\,|w_z|^2 \left|dz + \frac{w_{\bar{z}}}{w_z}\, d\bar{z}\right|^2$$

$w = u+iv$ isotermal if

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}$$

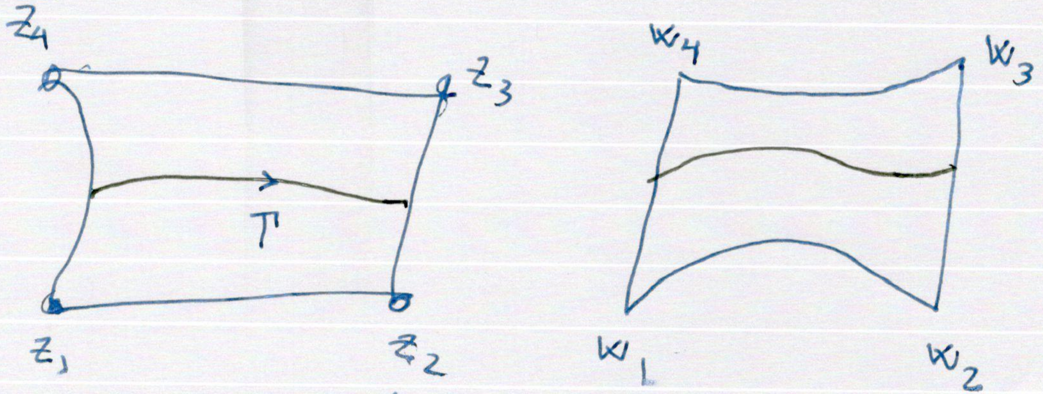
In the real analytic case

existence of solutions to the Beltrami equation was proved by Gauss

If $\|\mu\|_\infty < 1$ Morrey proved the existence

Ahlfors & Bers proved the existence for $\|\mu\|_\infty < 1$ and also analytic dependence of the solution as a function of μ

Modules of path families



Q quadrillateral

Γ path family (joining (z_1, z_4) to (z_2, z_3))

P family meas non-negative Borel-measurable functions

$\rho \in P$ is admissable if

$$\int_\gamma \rho \, |dz| \ge 1 \qquad \forall \gamma \in \Gamma$$

$$M(\Gamma) = \inf_{\rho \in P} \iint \rho^2 \, dx \, dy$$

Def. Given a domain A, $f : A \to A'$ sence preserving homeomorphism

Consider all Quadrlaterals in A

$$K = \sup_Q \frac{M\big(f(Q)(f(z_1), f(z_2), f(z_3), f(z_4))\big)}{M\big(Q(z_1, z_2, z_3, z_4)\big)}$$

(max dilatation)

If $K < \infty$ f is K-quasiconformal

A quasiconformal map has L^2_{loc} derivatives (actually L^p_{loc} for some $p > 2$)

Thm. A sense preserving homeomorphism is K-quasiconformal if

1. f is ACL (absolute cont. on lines)
2. $\max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)|$

Thm (Ahlfors - Bers)

Suppose μ is bounded on $\mathbb{C}$ and

$\|\mu\|_\infty < 1$

Then there is a homeomorphism f satisfying

(1) $$f_{\bar z} = \mu f_z \quad \text{for a.e. } z$$

If f, g are quasiconformal maps with complex dilatations μ_f and μ_g

Then

$$\mu_{f \circ g^{-1}}(\xi) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\frac{\partial g(z)}{|\partial g(z)|} \right)^2$$

$$\xi = g(z)$$

Uniqueness theorem. Suppose f, g are quasiconformal so that $\mu_f = \mu_g$ then $f \circ g^{-1}$ is conformal

Cor. Two solutions of (1) are equivalent up to a Moebius transformation

The compactly supported case

Theorem. Suppose $\mu \in L^\infty$ and compactly supported. Suppose $\|\mu\|_\infty \leq k < 1$

Then there is $p = p(k) > 2$ so that there is a unique f normalized so that

$$\begin{cases} f(z) = z + O\left(\frac{1}{z}\right) \\ f(0) = 0 \end{cases}$$

satisfying $f_{\bar{z}} = \mu f_z$

so that $f_{\bar{z}}$ and $f_z - 1 \in L^p$

f is a homeomorphism analytic outside $\mathrm{supp}(\mu)$.

Cauchy-Green formula. Suppose D C^1 Jordan domain and $f \in C^1$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint_D \frac{f_{\bar{z}}(\zeta)}{\zeta - z} dA$$

$(dA = d\xi d\eta)$

For $\varphi \in C_0^1$ we can define

$$S\varphi(z) = -\frac{1}{\pi} \int \frac{\varphi(\zeta)}{\zeta - z} d\xi d\eta$$

Thm. Suppose that $\mu \in L^\infty$ and compactly supported. Suppose $\|\mu\|_\infty \le k < 1$

Then there is $p = p(k) > 2$ so that there is f normalized so that

$$\begin{cases} f(z) = z + O\left(\frac{1}{z}\right) \\ f(0) = 0 \end{cases}$$

satisfying $f_{\bar z} = \mu f_z$

so that $f_{\bar z}$ and $f_z - 1 \in L^p$

f is a homeomorphism, analytic outside $\mathrm{supp}(\mu)$

Pf. Suppose $f \in QC'(K, R)$ (Quasiconformal & C')

Cauchy-Green formula is

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint \frac{f_{\bar z}(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

We look for a solution $f(z) = z + O\left(\frac{1}{z}\right)$

Replace $f(z) \to z$ in first integral

$$f(z) - z = -\frac{1}{\pi} \iint_{|\zeta| < R} \frac{f_{\bar z}(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

$$\equiv \mathcal{S} f_{\bar z}$$

$f = S\varphi$ is a solution of

$$\bar\partial f = f_{\bar z} = \varphi$$

Pf, Suppose $f \in QC'(K, R)$, (K-qc. & C^1 and anal. in $|z|>R$)

Cauchy - Green's formula becomes

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint \frac{f_{\bar z}(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

for $r > R$

We look for a solution to the Betrami equation with, $f(z) = z + O\left(\frac{1}{z}\right)$ at ∞

Can replace $f(z)$ by z in first integral

$$f(z) - z = -\frac{1}{\pi} \iint_{|\zeta|<R} \frac{f_{\bar z}(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

$$= S f_{\bar z}$$

For $\varphi \in C_0^1$, $S\varphi(z) = -\frac{1}{\pi} \iint \frac{\varphi(\zeta)}{\zeta - z}\, d\xi\, d\eta$ is a solution of $\bar\partial$

$$\iint \psi_{\bar z}(z)\,(S\varphi)(z)\,dx\,dy = -\iint \psi(\zeta)\varphi(\zeta)\,d\xi\,d\eta$$

By formally differentiating

$$S\varphi(z) = -\frac{1}{\pi}\iint \frac{\varphi(\xi)}{\xi - z}\, d\xi\, d\eta$$

we get

$$T\varphi(z) = (S\varphi)_z(z) = \frac{1}{\pi}\iint \frac{\varphi(\xi)}{(\xi-z)^2}\, d\xi\, d\eta$$

One can prove that for $\varphi \in C_0^1$

$$T\varphi = \lim_{\varepsilon\to 0} \frac{1}{\pi}\iint_{|\xi - z|>\varepsilon} \frac{\varphi(\xi)}{(\xi-z)^2}\, d\xi\, d\eta$$

exists by Cauchy-Green's formula

Then $T(f_{\bar z}) = (S f_{\bar z})_z = f_z$

T maps $f_{\bar z}$ to f_z

T can be extended to L^p, $1<p<\infty$

Important fact

1) $\|T\varphi\|_2 = \|\varphi\|_2$

T is a fourier multiplier

$$\mathcal{F}: \frac{1}{z^2} \to \frac{w^2}{|w|^2}$$

$$\|T\varphi\|_p \le C_p \|\varphi\|_p \quad , \; 1<p<\infty$$

$C_p \to 1$ for as $p \to 2$

Differentiate

$$f(z) - z = -\frac{1}{\pi} \iint_{|\zeta|<R} \frac{f_{\bar z}(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

with respect to z

We get

$$f_z - 1 = T(f_{\bar z})$$

Introduce $\begin{cases} f_{\bar z} = \mu f_z \\ g = f_z - 1 \end{cases}$

We get an integral equation for g

$$g = T(\mu(1+g)) = T(\mu g) + T\mu$$

Itroduce the operator

$$\mathcal{U}_\mu = T(\mu g)$$

Then

$$(I - \mathcal{U}_\mu) g = T\mu$$

$$\|\mathcal{U}_\mu g\|_p \le \underbrace{C_p k}_{<1} \|g\|_p$$

$$g = (I - \mathcal{U}_\mu)^{-1}(T\mu)$$

$$(I - \mathcal{U}_\mu)^{-1} = \sum_{n=0}^{\infty} \mathcal{U}_\mu^n$$

$$\| (I - \mathcal{U}_\mu)^{-1} \| \leq \sum_{n=0}^{\infty} (k C_p)^n = \frac{1}{1 - k C_p}$$

$$g = (I - \mathcal{U}_\mu)^{-1} T(\mu) \in L^p$$

Now go back to

$$f(z) - z = S f_{\bar{z}}$$

Define f by

$$f(z) = z + S(\underbrace{\mu g + \mu}_{\mu(1+g) = \mu f_z})$$

$$f_{\bar{z}} = \mu g + \mu \quad = \mu(1+g)$$

$$f_z = 1 + T(\mu g + \mu) = 1 + g$$

g solves integal eq

Hence $\quad f_{\bar{z}} = \mu f_z$

Uniqueness

Suppose F is another solution of

$$F_{\bar z} = \mu F_z$$

with $F_{\bar z}, F_z - 1 \in L^p$

Define $G = F_z - 1$

If h has distr. der in L^p

then $h_z = T(h_{\bar z})$

Then $G = T(F_{\bar z})$

$$G = (I - U_\mu)^{-1} T(\mu)$$

Then $G = g$, $F_z = f_z$

$F_{\bar z} = f_{\bar z}$

Hence $F = f + c$

$c = 0$ because of normalization at ∞.

We also have to prove that φ is a homeomorphism

In C^1-case this follows since

$J_f > 0$

and f 1-1 near ∞

The general case

Write $\mu = \mu_1 + \mu_2$, $\mu_1 = 0$ near ∞

$\mu_2 = 0$ near 0

Wish to find λ so that

$$f^\lambda \circ f^{\mu_2} = f^\mu$$

$$f^\lambda = f^\mu \circ f^{-\mu_2}$$

$$\lambda = \left[\left(\frac{\mu - \mu_2}{1 - \mu \bar{\mu}_2} \right) \left(\frac{f_z^{\mu_2}}{\bar{f}_{\bar{z}}^{\mu_2}} \right) \right] \circ \left(f^{\mu_2} \right)^{-1}$$