Traces of Hecke operators on Drinfeld modular forms

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The crystal of cusp forms

Let $\Theta\colon \mathfrak{M}_2^A \to \mathrm{Spec}(A)$ denote the moduli stack of rank 2 Drinfeld A-modules.

Any $(E/S, \varphi) \in \mathfrak{M}_2^A(S)$ has an associated τ-sheaf $\underline{\mathcal{M}}(\varphi) \in \mathsf{Coh}_\tau(S, A)$, given by $\mathcal{H}\mathrm{om}_{S\text{-}\mathsf{Grp}}^{\mathbb{F}_q}(E, \mathbb{G}_a)$. This induces a morphism of stacks

$$\underline{\mathcal{M}} : \mathfrak{M}_2^A \longrightarrow \mathsf{Crys}(-, A)^{\mathsf{op}}.$$

Write $\underline{\mathbb{V}} := \underline{\mathcal{M}}(\varphi_{\mathsf{univ}}) \in \mathsf{Crys}(\mathfrak{M}_2^A, A)$.

For $x\colon \mathrm{Spec}(\mathbb{F}_{\mathfrak{p}^n}) \to \mathfrak{M}_2^A$, we have

$$x^*\underline{\mathbb{V}} = \underline{\mathcal{M}}(\varphi_x) = (\mathbb{F}_{\mathfrak{p}^n}\{\tau\}, \tau\cdot) \text{ with } A\text{-action via } \varphi_x.$$

The crystal of cusp forms

For $k, l \in \mathbb{Z}$, define

$$\underline{\mathbb{V}}_{k,l} := \mathrm{Sym}^{k-2}(\underline{\mathbb{V}}) \otimes \det(\underline{\mathbb{V}})^{\otimes l-k+1}.$$

The *crystal of cusp forms of weight k and type l* is the crystal

$$\underline{\mathcal{S}}_{k,l} := R^1\Theta_! \underline{\mathbb{V}}_{k,l} \in \mathsf{Crys}(\mathrm{Spec}(A), A).$$

For each $0 \neq \mathfrak{p} \trianglelefteq A$, there is a Hecke correspondence $\mathbf{T}_{\mathfrak{p}}$ acting on $\underline{\mathcal{S}}_{k,l}$.

Since $\underline{\mathcal{S}}_{k,l}$ is a crystal, there is also a natural map $\tau\colon \sigma^* \underline{\mathcal{S}}_{k,l} \to \underline{\mathcal{S}}_{k,l}$.

Böckle-Eichler-Shimura

$$\mathbf{T}_{\mathfrak{p}} \circlearrowright \underline{\mathcal{S}}_{k,l} \circlearrowleft \tau$$

$$\mathrm{Spec}(\mathbb{F}_{\mathfrak{p}}) \xrightarrow{i_{\mathfrak{p}}} \mathrm{Spec}(A) \xleftarrow{i_{K_\infty}} \mathrm{Spec}(K_\infty)$$

Theorem (Böckle '02, dV '24)

There is a Hecke-equivariant Eichler-Shimura isomorphism

$$i_{K_\infty}^* \underline{\mathcal{S}}_{k,l}^{rig} \cong \underline{\mathbb{1}}_{K_\infty} \otimes_A \mathrm{S}_{k,l}^{\vee}\,.$$

Moreover, the Hecke operator $i_{\mathfrak{p}}^* \mathbf{T}_{\mathfrak{p}}$ *coincides with*

$$\tau^{\deg(\mathfrak{p})} \in \mathrm{End}_{Crys}(i_{\mathfrak{p}}^* \underline{\mathcal{S}}_{k,l}).$$

A trace formula for Hecke operators

Let $s\colon \mathfrak{X} \to \operatorname{Spec}(\mathbb{F}_q)$ and $\underline{\mathcal{F}} \in \mathsf{Crys}(\mathfrak{X}, A)$. By the Lefschetz trace formula,

$$\sum_{x \in [\mathfrak{X}(\mathbb{F}_{q^n})]} \frac{\operatorname{Tr}_{\mathbb{F}_{q^n} \otimes A}(\tau^n \mid x^* \underline{\mathcal{F}})}{\# \operatorname{Aut}(x)} = \operatorname{Tr}_A(\tau^n \mid Rs_! \underline{\mathcal{F}}).$$

Setting $\mathfrak{X} = \mathfrak{M}^A_{2,\mathfrak{p}}$ and $\underline{\mathcal{F}} = i_{\mathfrak{p}}^* \underline{\mathbb{V}}_{k,l}$, we obtain

$$\sum_{[\varphi]/\mathbb{F}_{\mathfrak{p}^n}} \operatorname{Tr}_{k-2}(\pi_\varphi) \cdot (\pi_\varphi \bar{\pi}_\varphi)^{l-k+1} = \operatorname{Tr}_{\mathbb{C}_\infty}(\mathbf{T}_{\mathfrak{p}}^n \mid \mathrm{S}_{k,l}),$$

where the sum is over isomorphism classes of Drinfeld modules over $\mathbb{F}_{\mathfrak{p}^n}$, $\pi_\varphi \in \bar{K}$ denotes the Frobenius endomorphism of φ, and $\operatorname{Tr}_{k-2}(\pi_\varphi) = \sum_{i=0}^{k-2} \pi_\varphi^i \bar{\pi}_\varphi^{k-2-i}$.

Ramanujan bound

If $d = \deg(\mathfrak{p})$ and $\varphi \in \mathfrak{M}_2^A(\mathbb{F}_{\mathfrak{p}^n})$, then $|\pi_\varphi|_\infty = q^{nd/2}$. Hence

$$\begin{aligned} |\operatorname{Tr}_{\mathbb{C}_\infty}(\mathbf{T}_{\mathfrak{p}}^n \mid \mathrm{S}_{k,l})|_\infty &= |\sum_{[\varphi]/\mathbb{F}_{\mathfrak{p}^n}} \operatorname{Tr}_{k-2}(\pi_\varphi) \cdot (\pi_\varphi \bar{\pi}_\varphi)^{l-k+1}|_\infty \\ &\leq \max_\varphi |\operatorname{Tr}_{k-2}(\pi_\varphi) \cdot (\pi_\varphi \bar{\pi}_\varphi)^{l-k+1}|_\infty \\ &\leq q^{nd(k-2)/2} \cdot q^{nd(l-k+1)}. \end{aligned}$$

If $A = \mathbb{F}_q[T]$, this becomes

$$\deg \operatorname{Tr}(\mathbf{T}_{\mathfrak{p}}^n \mid \mathrm{S}_{k,l}) \leq \frac{nd(k-2)}{2}.$$

$A = \mathbb{F}_q[T]$: Ramanujan bound

The Ramanujan bound also holds for $\mathrm{S}_{k,l}(\Gamma)$. At level 1, the bound is **not** sharp (for any q, $\mathfrak{p}$, n, k and l).

This implies the decomposition

$$\mathrm{S}_{k,l}(\Gamma_0(\mathfrak{p})) = \mathrm{S}^{\mathrm{old}}_{k,l}(\Gamma_0(\mathfrak{p})) \oplus \mathrm{S}^{\mathrm{new}}_{k,l}(\Gamma_0(\mathfrak{p})),$$

under the condition that Hecke eigenvalues are not repeated p times.

Conjecture (Strong Ramanujan bound)

$$\deg \mathrm{Tr}(\mathbf{T}_{\mathfrak{p}}^n \mid \mathrm{S}_{k,l}) \leq \frac{nd(k-(q+1))}{2}.$$

The strong Ramanujan bound holds if $nd = 1$.

$A = \mathbb{F}_q[T]$: Primes of degree 1

Let $\mathfrak{p} = (T) \trianglelefteq \mathbb{F}_q[T]$. The trace formula simplifies to

Theorem

$$\mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k+2,l}) = \begin{cases} \sum\limits_{\substack{0 \leq j < k/2 \\ j \equiv l-1 (mod\ q-1)}} (-1)^j \binom{k-j}{j} T^j & \textit{if } k+2 \equiv_{q-1} 2l; \\ 0 & \textit{otherwise.} \end{cases}$$

Example (Type 2)

Let $1 \leq n \leq q$. Then the cusp form $E^{n-1}h^2$ spans the one-dimensional space $\mathrm{S}_{q+3+n(q-1),2}$, and it has $\mathbf{T}_T$-eigenvalue

$$nT - (n-1)T^q.$$

Traces

The trace formula can be turned into an algorithm. Some examples:

q	$\mathfrak{p}$	k	l	$\mathrm{Tr}(\mathbf{T}_{\mathfrak{p}} \mid \mathrm{S}_{k,l})$
3	T	30	0	$2T^{11} + T^9 + 2T^5 + T^3$
3	T	30	1	$2T^{12} + T^2 + 1$
3	$T^2 + 1$	24	0	$T^{18} + T^{12} + 2T^{10} + 2T^8 + T^6 + 2T^2$
5	T	74	3	$3T^{30} + T^{26} + T^{10} + 3T^6$

Difficult to understand the precise patterns. Can we at least understand the degree ("complex norm")?

In what follows, logarithms will be taken to base q.

Traces

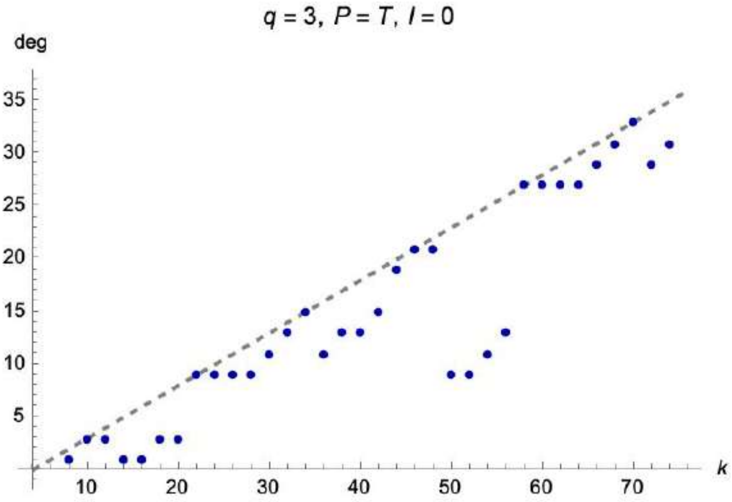

Figure 1: $\deg \mathrm{Tr}(\mathbf{T}_T \,|\, \mathrm{S}_{k,0})$ for $q = 3$ and $2 \leq k \leq 74$ with the strong Ramanujan bound.

Traces

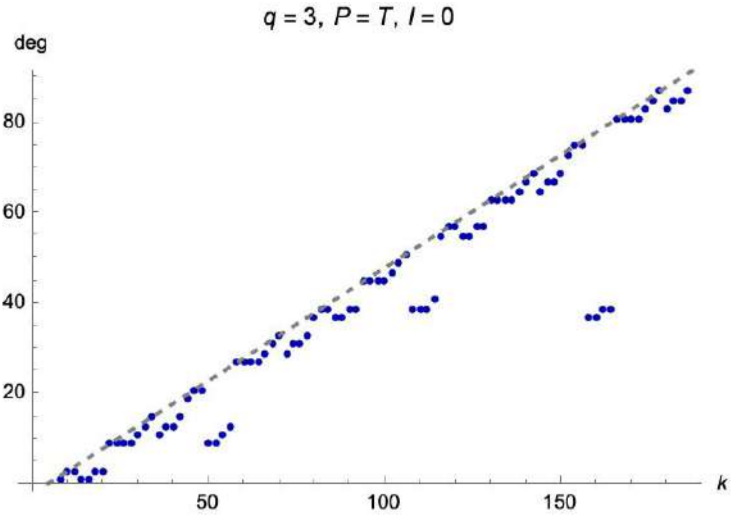

Figure 2: $\deg \operatorname{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ for $q = 3$ and $2 \leq k \leq 186$.

Traces

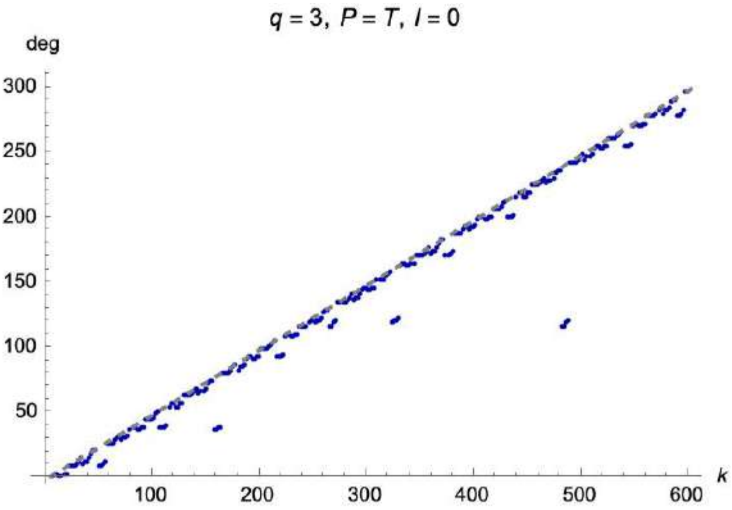

Figure 3: $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ for $q = 3$ and $2 \leq k \leq 600$.

Traces

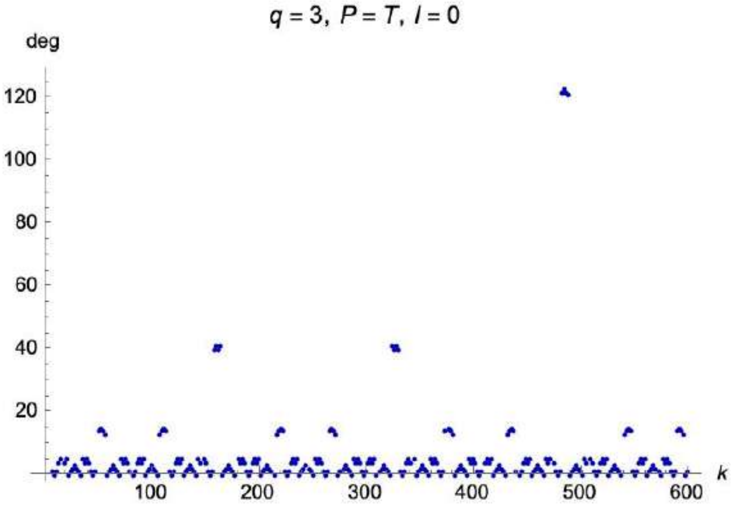

Figure 4: Distance of $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ to the Ramanujan bound.

Traces

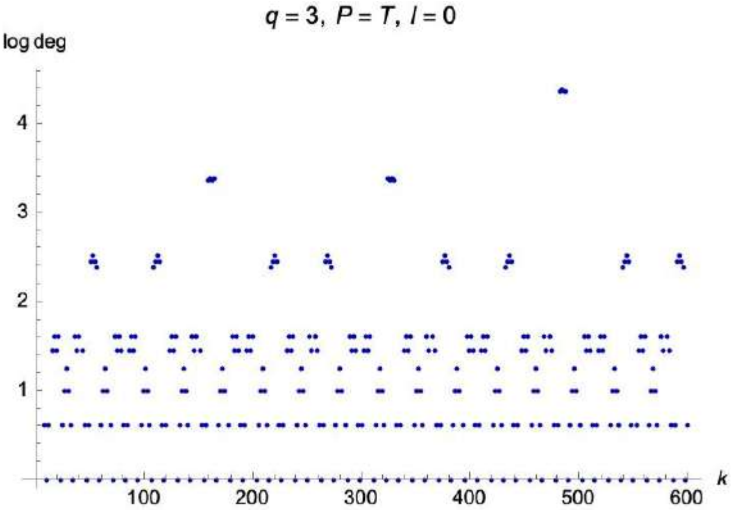

Figure 5: Logarithmic distance of $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ to the Ramanujan bound.

Traces

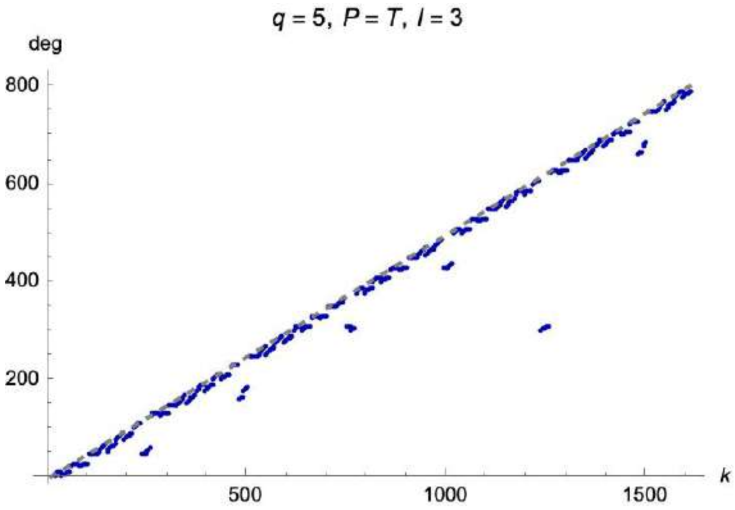

Figure 6: $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ for $q = 5$ and $2 \leq k \leq 1614$.

Traces

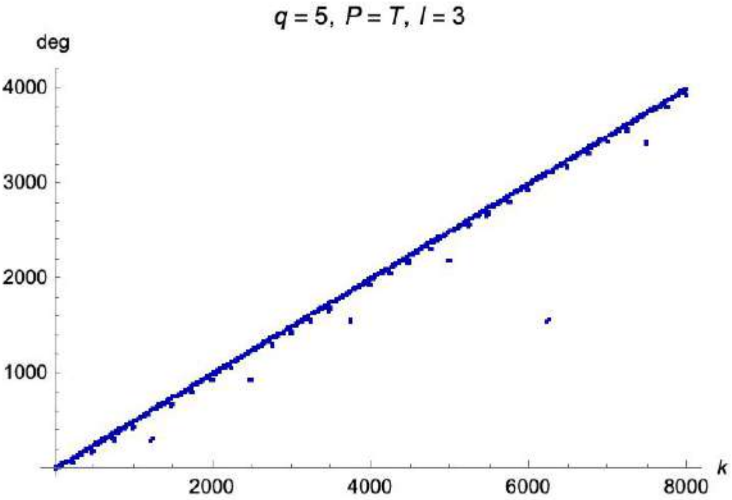

Figure 7: $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ for $q = 5$ and $2 \leq k \leq 8014$.

Traces

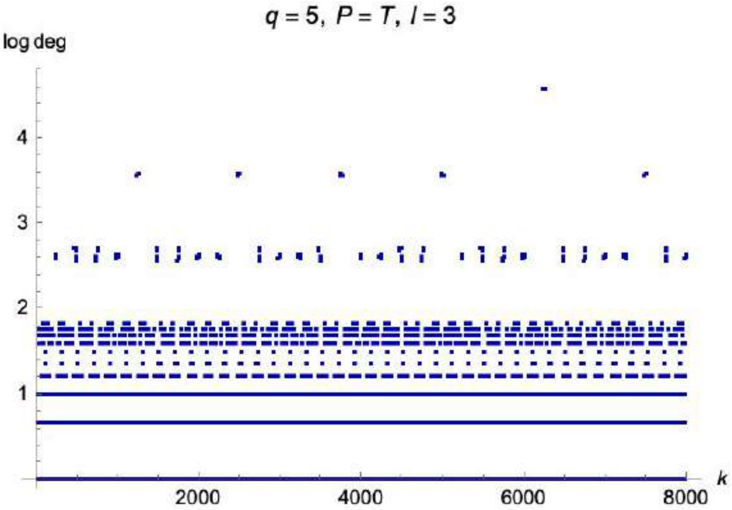

Figure 8: Logarithmic distance of $\deg \operatorname{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ to the Ramanujan bound.

Traces

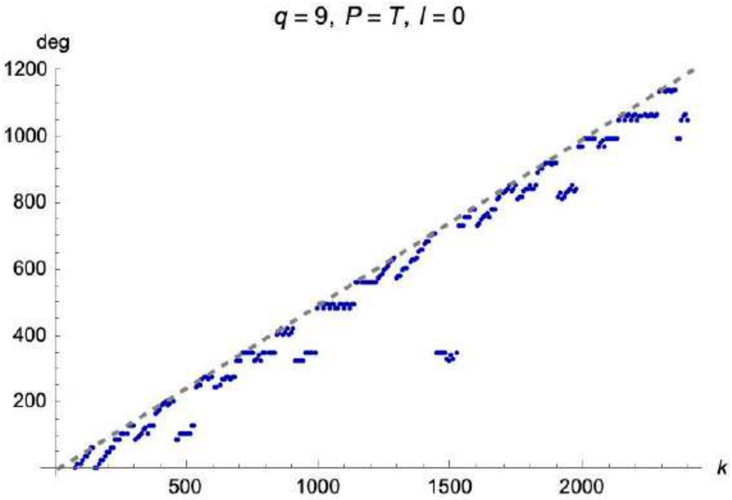

Figure 9: $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ for $q = 9$ and $2 \leq k \leq 2400$.

Traces

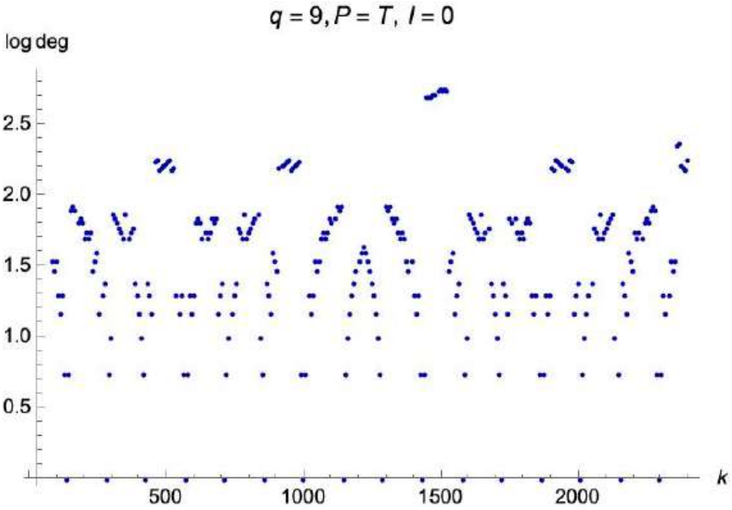

Figure 10: Logarithmic distance of $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ to the Ramanujan bound

Traces

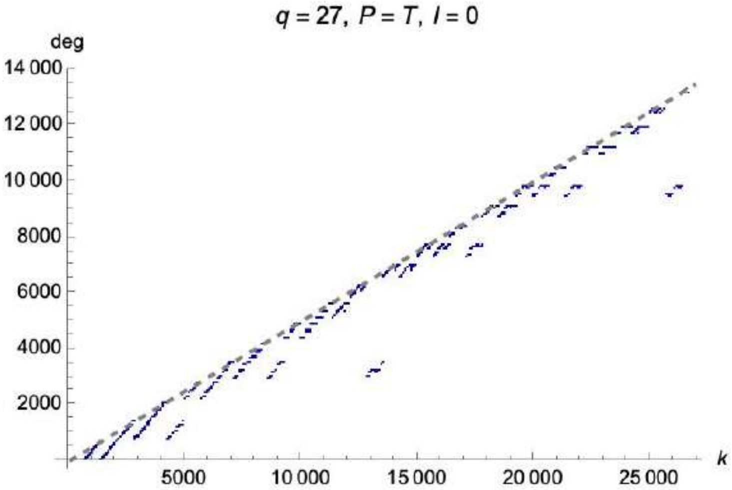

Figure 11: $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ for $q = 27$ and $2 \leq k \leq 26702$.

Traces

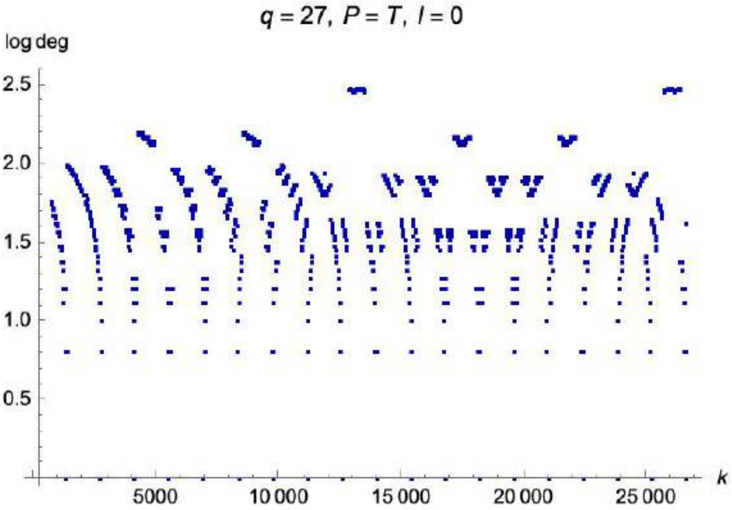

Figure 12: Logarithmic distance of $\deg \mathrm{Tr}(\mathbf{T}_T \mid \mathrm{S}_{k,0})$ to the Ramanujan bound

Traces

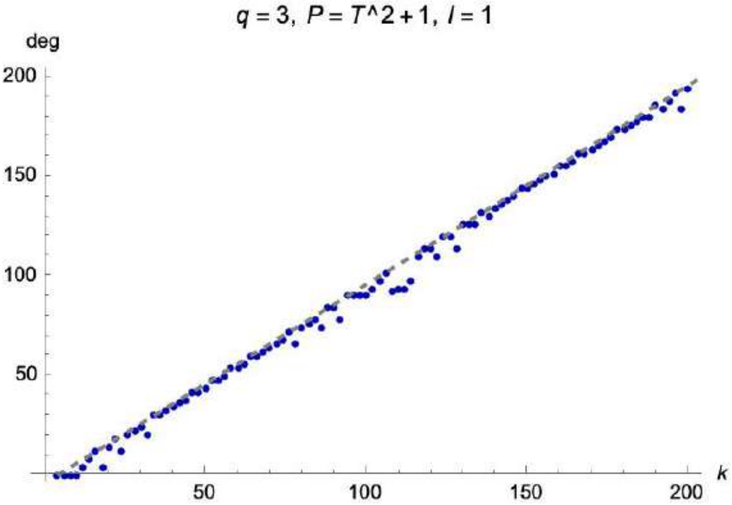

Figure 13: $\deg \mathrm{Tr}(\mathbf{T}_{T^2+1} \mid \mathrm{S}_{k,1})$ for $q = 3$ and $2 \leq k \leq 200$.

Traces

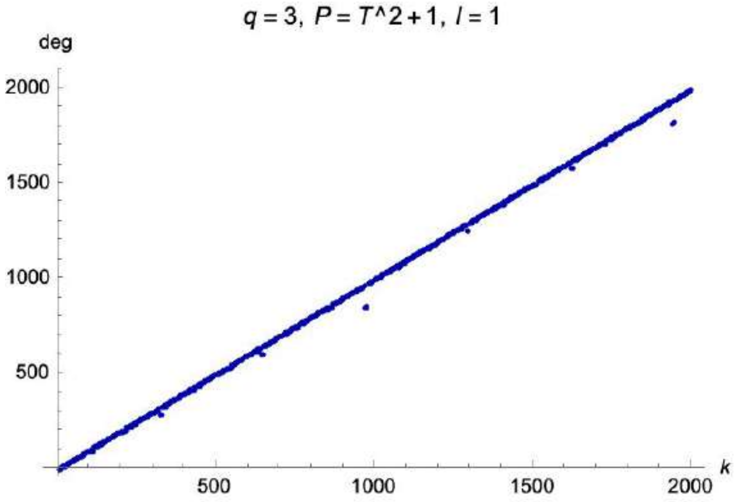

Figure 14: $\deg \mathrm{Tr}(\mathbf{T}_{T^2+1} \mid \mathrm{S}_{k,1})$ for $q = 3$ and $2 \leq k \leq 2000$.

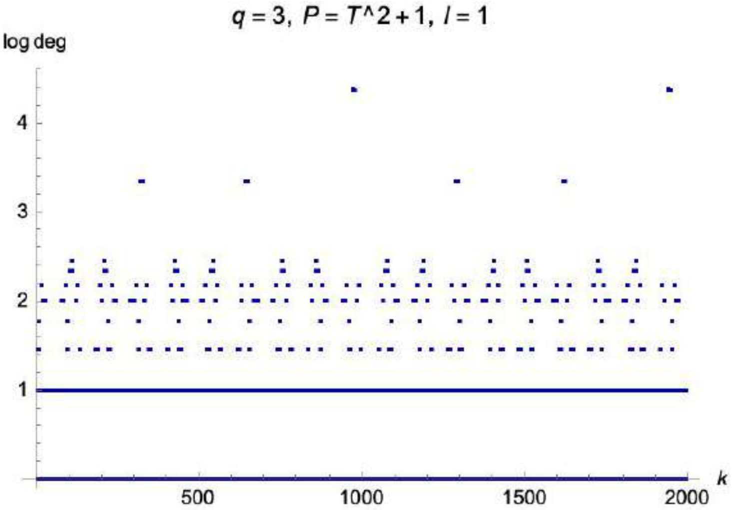

Figure 15: Logarithmic distance of $\deg \mathrm{Tr}(\mathbf{T}_{T^2+1} \mid \mathrm{S}_{k,1})$ to the Ramanujan bound

Traces

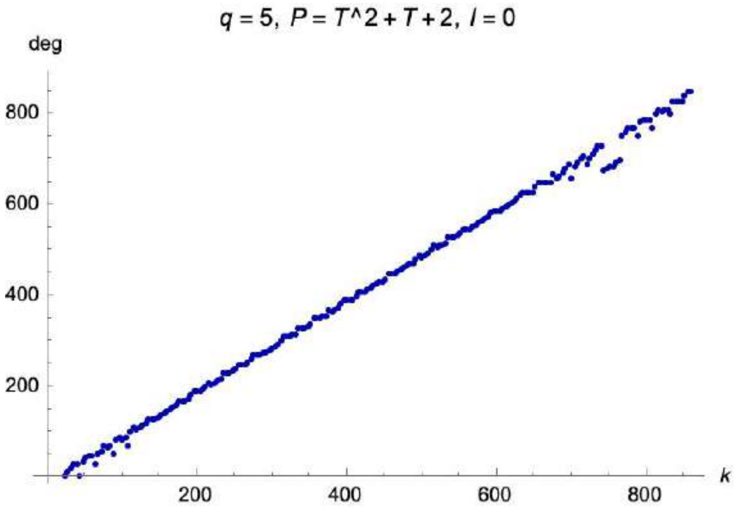

Figure 16: $\deg \mathrm{Tr}(\mathbf{T}_{T^2+T+2} \,|\, \mathrm{S}_{k,0})$ for $q = 5$ and $2 \leq k \leq 860$.

Traces

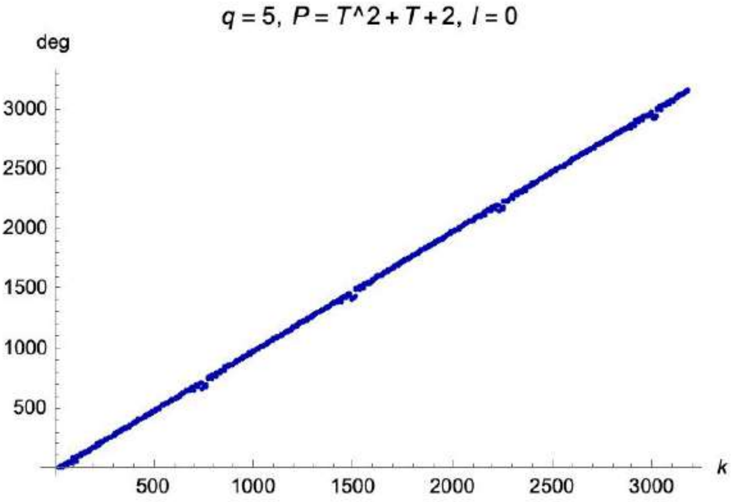

Figure 17: $\deg \mathrm{Tr}(\mathbf{T}_{T^2+T+2} \,|\, \mathrm{S}_{k,0})$ for $q = 5$ and $2 \leq k \leq 3180$.

Traces

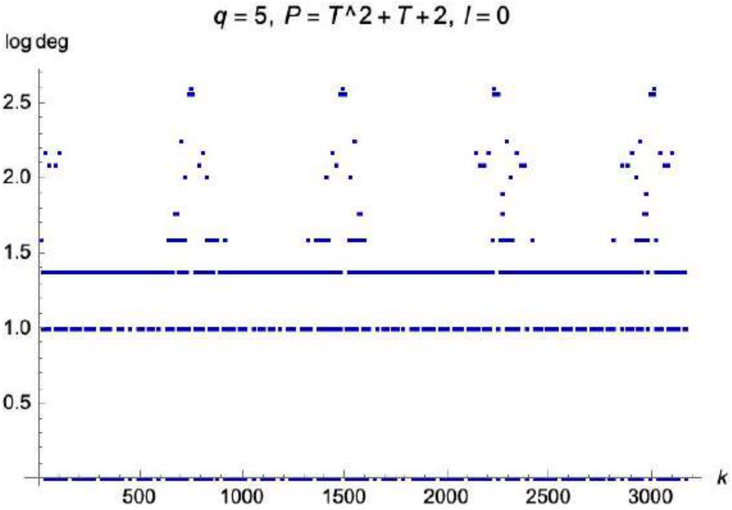

Figure 18: Logarithmic distance of $\deg \operatorname{Tr}(\mathbf{T}_{T^2+T+2} \mid \mathrm{S}_{k,0})$ to the Ramanujan bound.

Traces

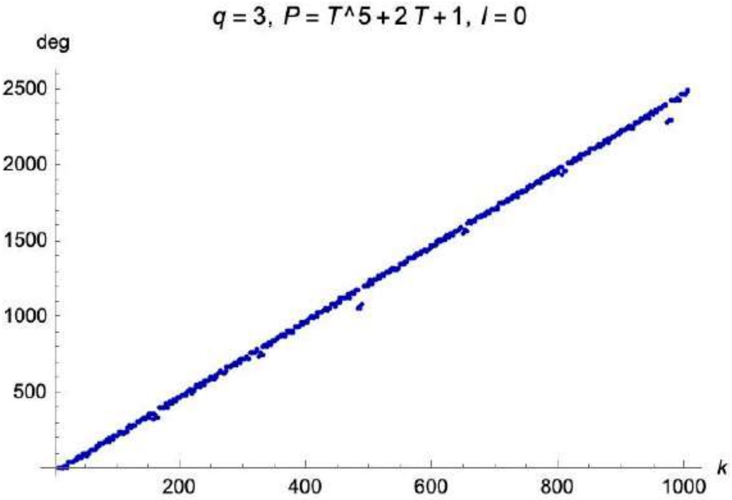

Figure 19: $\deg \mathrm{Tr}(\mathbf{T}_{T^5+2T+1} \,|\, \mathrm{S}_{k,0})$ for $q = 3$ and $2 \leq k \leq 1006$.

Traces

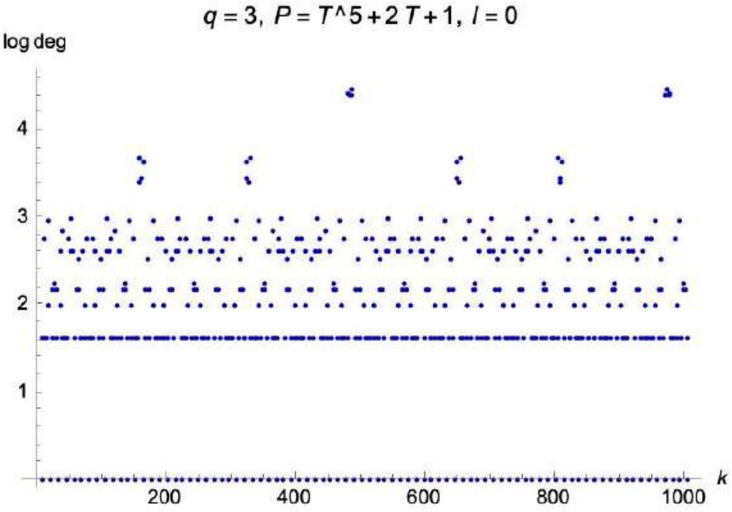

Figure 20: Logarithmic distance of $\deg \mathrm{Tr}(\mathbf{T}_{T^5+2T+1} \,|\, \mathrm{S}_{k,0})$ to the Ramanujan bound.

Slopes

Definition

Let v be a place of K. A *v-adic slope of weight k and type l* is a v-adic valuation of an eigenvalue of $\mathbf{T}_{\mathfrak{p}} \circlearrowright \mathrm{S}_{k,l}$.

Given $(\mathrm{Tr}(\mathbf{T}_{\mathfrak{p}}^n \mid \mathrm{S}_{k,l}))_{n \geq 1}$, one can compute the eigenvalues of $\mathbf{T}_{\mathfrak{p}}$...
...**under the condition** that no eigenvalue is repeated p times.

In practice, one needs $\mathrm{Tr}(\mathbf{T}_{\mathfrak{p}}^n \mid \mathrm{S}_{k,l})$ for $n = 1, \ldots, d + \lfloor \frac{d-1}{p-1} \rfloor$, where $d = \dim \mathrm{S}_{k,l}$. This gets computationally expensive as d grows.

Slopes

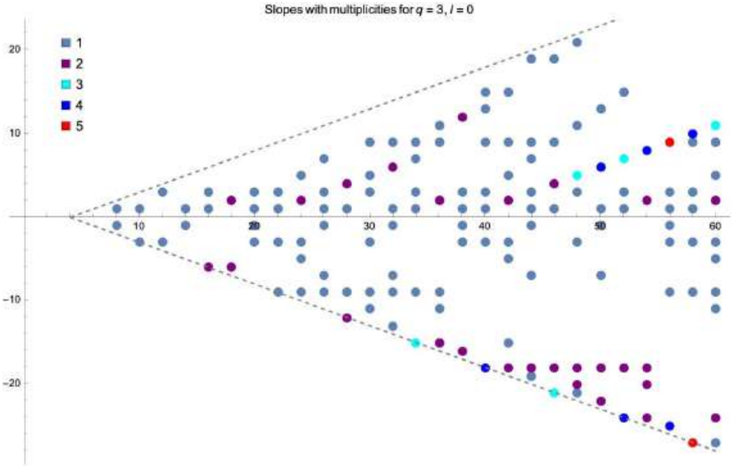

Figure 21: T-adic and ∞-adic slopes of $\mathbf{T}_T$ acting on $\mathrm{S}_{k,0}$ for $q = 3$.

Slopes

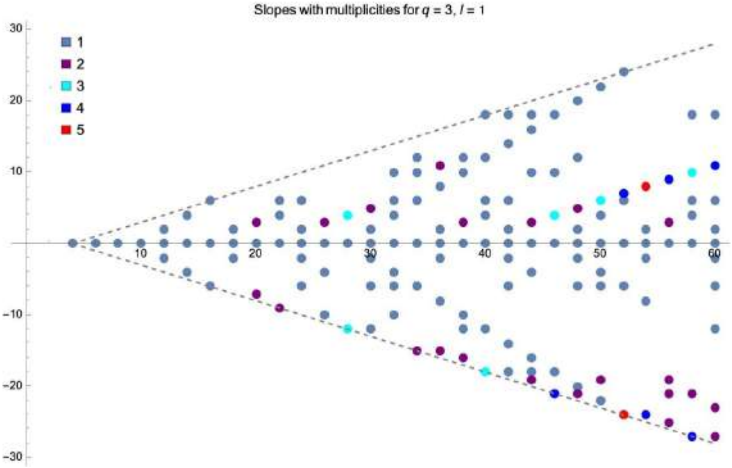

Figure 22: T-adic and ∞-adic slopes of $\mathbf{T}_T$ acting on $\mathrm{S}_{k,1}$ for $q = 3$.

Slopes

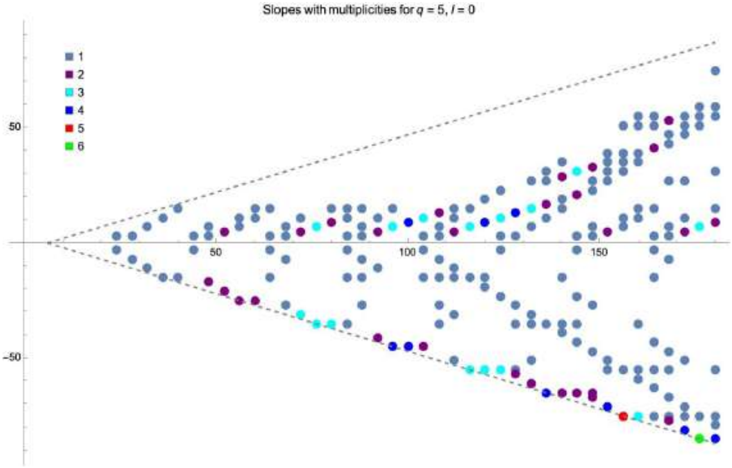

Figure 23: T-adic and ∞-adic slopes of $\mathbf{T}_T$ acting on $\mathrm{S}_{k,0}$ for $q = 5$.

Slopes

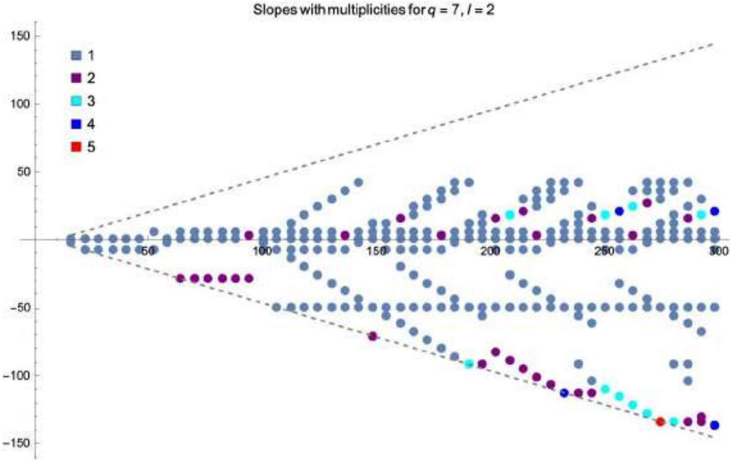

Figure 24: T-adic and ∞-adic slopes of $\mathbf{T}_T$ acting on $\mathrm{S}_{k,2}$ for $q = 7$.

Conclusion

- Böckle-Eichler-Shimura theory + Lefschetz trace formula $\implies$ trace formula for Hecke operators
- One deduces a Ramanujan bound, which is not sharp in level 1
- There is a conjectural strong Ramanujan bound
- The distance of the trace to the bound exhibits interesting patterns as k varies, but this needs further study
- The trace formula yields an efficient way to compute traces (but not so much slopes)

The algorithms to compute traces and slopes are available on Github:
`https://github.com/Sjoerd-deVries/DMF_Trace_Formula`

References

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