Symmetric Determinantal Representation of Polynomials

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Abstract

We give an elementary proof, only using linear algebra, of a result due to Helton, Mccullough and Vinnikov, which says that any polynomial can be written as the determinant of a symmetric affine linear pencil.

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Introduction

In [HMV, Theorem 14.1], it is stated that any polynomial $p(x_1, \ldots, x_n)$ in n variables with real coefficients, has a symmetric determinantal representation. Namely, there are symmetric matrices $A_0, A_1, \ldots, A_n$ with real coefficients such that

$$p(x) = \det \left(A_0 + \sum_{i=1}^{n} A_i x_i \right)$$

In fact, this result is established in a much more general setting, using the theory of systems realizations of noncommutative rational functions. Roughly speaking, the use of the theory of noncommutative rational series is sensible since there is a kind of homomorphism between the product of noncommutative rationnal series and the multiplication of matrices.

If $n = 2$, even much stronger results than the determinantal mentioned above has been obtained using tools of algebraic geometry (see [HV] for instance and confer to [HMV] for an exhaustive list of references on the subject). But these do not seem to generalize to higher dimension n.

In this paper, we inspire ourselves of several key steps performed in [HMV] to give a new proof of the result, dealing only with linear algebra and more precisely matrix computations. Moreover, the determinantal representation of polynomials we obtained in Theorem 3.4, can be realized by explicit formulas.

We also mention some variations : Theorem 4.1 for polynomial with coefficients over a ring, and Theorem 4.2 for noncommutative symmetric polynomials over the reals which can just be seen as a version of [HMV, Theorem 14.1].

1 Notations

Let R be a ring which shall be viewed as a ground ring of coefficients (it will often be the field of real numbers $\mathbb{R}$).

Denote by $R[x]$ the ring of all polynomials in n variables $(x) = (x_1, \ldots, x_n)$ with coefficients in R. A polynomial $p(x) \in R[x]$ can be written as a finite sum $p(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$, where $a_\alpha \in R$ and $x^{(\alpha_1,\ldots,\alpha_n)} = x_1^{\alpha_1} \times \ldots \times x_n^{\alpha_n}$. For any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we define the weight of α by $|\alpha| = \sum_{i=1}^n \alpha_i$. We also consider the lexicographic ordering on monomials, meaning that $x^\alpha > x^\beta$ if there is an integer $i_0 \in \{1, \ldots, n\}$ such that $\alpha_{i_0} > \beta_{i_0}$ and $\alpha_i = \beta_i$ for all $i = 1, \ldots, i_0 - 1$.

We denote by $\mathbb{S}R^{N\times N}$ the set of all symmetric matrices of size $N \times N$ with entries in R. The identity and the null matrix of size N will be repectively denoted by $\mathbf{Id}_N$, $\mathbf{0}_N$. A matrix J is called a *signature matrix* if it is a diagonal matrix with entries ± 1 onto the diagonal. Beware that there is a slight difference with [HMV], where a signature matrix J only satisfies $J = J^t$ and $J^t J = \mathbf{Id}$.

An $p \times q$ *linear pencil* L_M in the n indeterminates (x) is an expression of the form $L_M(x) = M_1 x_1 + \ldots + M_n x_n$ where $M_1, \ldots, M_n$ are $p \times q$ matrices with coefficients in R. Likewise, a $p \times q$ *affine linear* pencil L_M is an expression of the form $M_0 + L_M(x)$ where M_0 is an $p \times q$ matrix and L_M is a $p \times q$ linear pencil. Moreover, the linear pencil will be said *symmetric* if all the matrices $M_0, M_1, \ldots, M_n$ are symmetric.

Note also that, if we multiply the pencil L_M by two matrices P and Q, then we get a new pencil $PL_MQ = L_{PMQ}$.

We say that the polynomial $p(x)$ has a *linear description*, if there are a linear pencil L_A, a $N \times N$ signature matrix J, a $1 \times N$ line matrix L, a $1 \times N$ column matrix C, such that :

$$p(x) = L(J - L_A(x))^{-1} C$$

A linear description is called *symmetric* if L_A is symmetric and if $C = L^t$. It is called *unitary* if $J = \mathbf{Id}_N$.

2 From linear description to determinantal representation

2.1 Symmetrizable linear description

Roughly speaking, the idea of the proof of [HMV, Theorem 14.1] is the following. A classical theorem due to Schutzenberger [S] shows the existence, for any given polynomial $q(x)$, of a linear description $q(x) = L(\mathbf{Id} - L_A(x))^{-1}C$ which is *minimal* (we do not enter the detail of this minimality condition, confer to [HMV]). Then, one may derive another minimal linear description by transposition : $q(x) = C^t(J - L_{A^t}(x))^{-1}L^t$. By minimality, the result of Schutzenberger says that these two descriptions are *similar*, namely there is a unique invertible symmetric matrix S such that $SC = L^t$ and $SL_A = (L_A)^t\, S$. Note that $SL_A = (L_A)^t\, S$ is equivalent to $SA_i = A_i^t S$ for all $i = 1 \dots n$.

Then, by a formal process that we describe below, these properties allows one to derive a symmetric linear description of $q(x)$. It motivates our introduction of the notion of *symmetrizable* linear description.

Definition 2.1 *Let $q(x)$ be a polynomial together with a unitary linear description :*

$$q(x) = L(\mathbf{Id} - L_A(x))^{-1}C$$

where A has size $N \times N$, C has size $N \times 1$ and L has size $1 \times N$. Let also S be a symmetric invertible matrix of size $N \times N$ with entries in R.

Then, the linear description of $q(x)$ is said S-symmetrizable if $SL_A = (L_A)^t\, S$ and $SC = L^t$.

Under these assumptions, one may "symmetrize" the linear description of $q(x)$.

Proposition 2.2 *If a polynomial $q(x)$ has an S-symmetrizable linear description, for a given invertible and symmetric matrix S, then it has a symmetric linear description.*

Proof : We know that there is a matrix U and a signature matrix J, both of size $N \times N$, such that $S = UJU^t$. Then, we set $\widetilde{L} = L(U^{-1})^t$, and $L_{\widetilde{A}}(x) = JU^tL_A(x)(U^{-1})^t$.

It remains to check that

a) $q(x) = \widetilde{L}(J - L_{\widetilde{A}}(x))^{-1}(\widetilde{L})^t$

b) $L_{\widetilde{A}}(x)$ is a symmetric linear pencil.

Assertion a) comes from the identity

$$J - L_{\widetilde{A}}(X) = JU^t(\mathbf{Id} - L_A(x))(U^t)^{-1}$$

and hence

$$(\mathbf{Id} - L_A(x))^{-1} = (U^{-1})^t(J - L_{\widetilde{A}}(x))^{-1}JU^t$$

Indeed, it is enough to remark that

$$JU^tC = U^{-1}L^t = \widetilde{L}^t$$

Now, to check assertion b), it is enough to compute

$$(L_{\widetilde{A}})^t U^t = U^{-1}(L_A)^t S = U^{-1}SL_A = U^{-1}UJU^tL_A = U^{-1}UJU^tL_A = L_{\widetilde{A}}U^t$$

□

We end this section with an important remark for the following :

Remark 2.3 *With the notations of the previous proof, note that if the matrix L_A is nilpotent, then so is the matrix $JL_{\widetilde{A}}$.*

2.2 Schur complement and unipotent linear description

Again, we refer to [HMV] and more particularly to the proof of Theorem 14.1. One Technical key tool is Schur Complement, it appears in the "LDU decomposition" and the computation of the determinant.

We recall what will be needed in the following :

Proposition 2.4 Schur Complement

Let the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are matrices with entries in a general ring R and of size $(p \times p)$, $(p \times q)$, $(q \times p)$, $(q \times q)$ respectively. If A and D are invertible, then we have the identity :

$$\det(M) = \det(D)\det(A - BD^{-1}C) = \det(A)\det(D - CA^{-1}B)$$

Proof : Since D is invertible, the Schur Complement of M relative to the $(2, 2)$ entry is $A - BD^{-1}C$. We can write

$$ML = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ \mathbf{0}_p & \mathbf{Id}_q \end{pmatrix}$$

where

$$L = \begin{pmatrix} \mathbf{Id}_p & \mathbf{0}_q \\ -D^{-1}C & D^{-1} \end{pmatrix}$$

Then,

$$\det(M) = \det(D)\det(A - BD^{-1}C)$$

Symmetrically, if A is invertible, considering the Schur complement relative to the (1,1) entry, we get

$$\det(M) = \det(A)\det(D - CA^{-1}B)$$

□

Before stating the result, we introduce the notion of *unipotent* linear description, to deal with a new hypothesis needed in the following.

Definition 2.5 *The linear description* $q(x) = L(J - L_A(x))^{-1}C$ *is said to be unipotent if the matrix* JL_A *is nilpotent.*

In the proof of [HMV, Theorem 14.1.], the minimality condition of a linear description of $q(x)$ implies that L_A is nilpotent and hence $JL_{\widetilde{A}}$ is also nilpotent (cf. remark 2.3).

All our results about determinantal representation of polynomials are based on the following

Proposition 2.6 *Assume that the polynomial* $q(x)$ *admits a symmetric linear unipotent description. Namely,* $q(x) = C^t(J - L_A(x))^{-1}C$ *where* J *is a signature matrix and* A *is symmetric. Then, we have the identity*

$$1 - q(x) = \det(J)\det(J - CC^t - L_A(x))$$

Proof: Consider the following matrix of size $(N+1) \times (N+1)$:

$$G = \begin{pmatrix} J - L_A(x) & C \\ C^t & 1 \end{pmatrix}$$

The Schur complement relative to the entry $(1,1)$ gives

$$\det(G) = \det(1 - C^t(J - L_A(x))^{-1}C)\det(J - L_A(x))$$

The Schur complement relative to the entry $(2,2)$ gives

$$\det(G) = \det(J - CC^t - L_A(x))\det((1))$$

But

$$\det(J - L_A(x)) = \det(J)\det(\mathbf{Id} - L_{JA}(x)) = \det(J)$$

since JA is nilpotent. And hence, we deduce that

$$\det(1 - C^t(J - L_A(x))^{-1}C) = \det(J)\det(J - CC^t - L_A(x)) = 1 - q(x)$$

$\square$

3 Symmetric determinantal representation

Having in mind the results of the previous sections, we naturally focus on the existence of linear symmetrizable unipotent descriptions.

3.1 Naive linear description

Let $Q(x) = \sum_{|\alpha|=d} b_\alpha x^\alpha$ be a homogeneous polynomial of degree d. Let $m_{k,n}$ be the number of monomials of degree k in n variables : $m_{k,n} = \binom{n-1+k}{n-1}$. Sometimes we will forget the number of variables n and simply write m_k.

Let us define some linear pencils $L_{A_1}, \ldots, L_{A_d}$ given by :

$$L_{A_1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$L_{A_2} = \begin{pmatrix} x_1 \ \mathbf{Id}_n \\ x_2(\ \mathbf{0}_1 \mid \ \mathbf{Id}_{n-1}) \\ \vdots \\ x_n(\ \mathbf{0}_{n-1} \mid \ \mathbf{Id}_1)) \end{pmatrix},$$

and more generally for $k = 1 \ldots d$ and $i = 1 \ldots n$ we set

$$L_{A_k} = \begin{pmatrix} x_1(\ \mathbf{0}_{\alpha_{1,k}} \mid \ \mathbf{Id}_{\beta_{1,k}}) \\ \vdots \\ x_n(\ \mathbf{0}_{\alpha_{n,k}} \mid \ \mathbf{Id}_{\beta_{n,k}})) \end{pmatrix},$$

$$\text{where } \beta_{i,k} = m_{k-1,n-i+1} = \binom{n-i+k-1}{n-i} \text{ and } \alpha_{i,k} + \beta_{i,k} = m_{k-1,n}$$

Notice that the pencil L_{A_k} has size $m_k \times m_{k-1}$ and that the product $X_k = L_{A_k} \times L_{A_{k-1}} \ldots \times L_{A_1}$ is a $m_k \times 1$ matrix whose entries are all the monomials of degree k in n variables which appear ordered with respect to the lexicographic ordering.

Example 3.1 *For $n = 3$, we have*

$$L_{A_1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad L_{A_2} = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2 \\ 0 & 0 & x_3 \end{pmatrix}, \quad L_{A_2} L_{A_1} = \begin{pmatrix} x_1^2 \\ x_1 x_2 \\ x_1 x_3 \\ x_2^2 \\ x_2 x_3 \\ x_3^2 \end{pmatrix}.$$

In the following we will use some sub-diagonal (and hence nilpotent) linear pencils of the form :

$$
L_M(x) = \mathrm{SD}(L_{M_1},\dots,L_{M_d}) = \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ L_{M_1} & 0 & 0 & \vdots & 0 & 0 & \\ 0 & L_{M_2} & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & L_{M_3} & \vdots & 0 & 0 & 0 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & L_{M_{d-1}} & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & L_{M_d} & 0 \end{pmatrix}
$$

where the L_{M_i}'s are themselves linear pencils.

With the choice $L_{M_i} = L_{A_i}$, an elementary computation gives the following linear description for a given polynomial $q(x)$:

$$
q(x) = (\bar{a_0}, \bar{a_1} \dots, \bar{a_d})(\ \mathbf{Id} - L_A(x))^{-1}(1,0,\dots,0)^t
$$

where $\bar{a_i} = (a_\gamma)_{|\gamma|=i}$ is the list of coefficients of the homogeneous component of $Q(x)$ of degree i, ordered with respect to the lexicographic ordering on the variables $(x_1,\dots,x_n)$.

Then, copying the proof of Proposition 2.6, we get that any polynomial of degree d in n variables has a determinantal but *not symmetric* representation of size $1 + m_{1,n} + \dots + m_{d,n} = m_{d,n+1} = \binom{n+d}{n}$.

If $n = 1$ and $p(x) = \sum_{i=0}^{d} a_i x_1^i$, it yields the description

$$
p(x) = (a_0,\dots,a_d)\left(\mathbf{Id}_{d+1} - \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ x & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & x & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

Note that the matrix

$$
S = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}
$$

of size $(d+1)\times(d+1)$ is symmetric invertible and such that $SL_A = (L_A)^t\, S$. Despite the fact that the condition $SC = L^t$ is not satisfied, we may conjecture that our naive linear description of $p(x)$ is not far from being symmetrizable.

But, in the case of several variables ($n > 1$), for this naive linear description is not any more a symmetric invertible matrix S such that $SL_A = (L_A)^t\, S$.

For these reasons we change a bit this naive description to get a symmetrizable one. Our strategy will be to fix, *a priori*, particular matrices L_0, C_0 and S which fulfill some wanted conditions.

3.2 Symmetrizable unipotent linear description

Now and in all the following, we consider a polynomial $Q(x)$ *homogeneous* of odd degree $d = 2e+1$ and set $N = 2\sum_{k=1}^{e} m_{k,n} = 2\binom{n+e}{n}$.

Let $(L_0) = (0, \ldots, 0, 1)$ and $(C_0)^t = (1, 0, \ldots, 0)$ and

$$S_N = \begin{pmatrix} \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{Id}_{m_0} \\ \vdots & & & & & \iddots & \mathbf{Id}_{m_1} & \mathbf{0} \\ \vdots & & & & \iddots & \iddots & \iddots & \vdots \\ \vdots & & & \iddots & \mathbf{Id}_{m_k} & & & \vdots \\ \vdots & & \iddots & \mathbf{Id}_{m_k} & \iddots & & & \vdots \\ \vdots & \iddots & \iddots & \iddots & & & & \vdots \\ \mathbf{0} & \mathbf{Id}_{m_1} & \iddots & & & & & \vdots \\ \mathbf{Id}_{m_0} & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \end{pmatrix}$$

Let also $L_B(x) = \mathrm{SD}(B_1, \ldots, B_d)$ for some linear pencils $B_1, \ldots, B_d$ to be determined such that

$$Q(x) = L_0(\ \mathbf{Id} - L_B(x))^{-1} C_0$$

Such a linear description will be called of *type* $(L_0 S_N C_0)$. It obviously satisfies :

1) S_N is symmetric and invertible,

2) L_B is nilpotent,

3) $S_N C_0 = (L_0)^t$,

4) The condition $S_N L_B = (L_B)^t S_N$ is equivalent to $L_{B_{d-i+1}} = L_{B_i}^t$ for all $i = 1 \ldots d$.

In all the following by symmetrizable descriptions, we always mean S_N-symmetrizable, i.e. with respect to the fixed matrix S_N. A linear description of type $(L_0 S_N C_0)$ satisfying condition 4) is clearly unipotent and symmetrizable. The following proposition establishes the existence of such a description.

Proposition 3.2 *Any homogeneous polynomial $Q(x)$ of degree $d = 2e+1$ admits an S_N-symmetrizable unipotent linear description of type $(L_0 S_N C_0)$ and of size $N = 2\binom{n+e}{n}$.*

Proof : We set $L_{B_i} = L_{A_i}$ and $L_{B_{e+i+1}} = L_{A_i^t}$ for $i = 1 \ldots e$. It remains to define $L_{B_{e+1}}$ as a symmetric linear pencil satisfying $Q(x) = \sum_{|\alpha|=d} b_\alpha x^\alpha = (X_e)^t L_{B_{e+1}}(X_e)$

We index our matrices by the set of all n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| = k$ which we order with respect to the lexicographic ordering. Let $L_{B_{e+1}} = (\phi_{\alpha,\beta})_{|\alpha|=|\beta|=e}$, with $\phi_{\alpha,\beta} = \sum_{i=1}^{n} \lambda^{(i)}_{\alpha,\beta} b_{\alpha+\beta+\delta^{(i)}} x_i$, where $\delta^{(i)}$ is the n-tuple defined by $\delta^{(i)}_j = \delta_{i,j}$ and the $\lambda^{(i)}_{\alpha,\beta}$'s are scalars to be determined.

We compute

$$
\begin{array}{rcl}
(X_e)^t L_{B_{e+1}}(X_e) & = & \sum_{|\alpha|=e,|\beta|=e} x^\alpha \phi_{\alpha,\beta} x^\beta \\
& = & \sum_{|\gamma|=2e} x^\gamma \sum_{\alpha+\beta=\gamma} \phi_{\alpha,\beta} \\
& = & \sum_{|\gamma|=2e+1} \left(\sum_{i\in \mathrm{Supp}(\gamma)} \sum_{\alpha+\beta=\gamma-\delta^{(i)}} \lambda^{(i)}_{\alpha,\beta} \right) b_\gamma x^\gamma
\end{array}
$$

where $\mathrm{Supp}(\gamma)$ is the subset of the indexes $i \in \{1, \dots, n\}$ such that $\gamma_i \neq 0$.

Let $\Lambda^{(i)}_\gamma = \sum_{\alpha+\beta=\gamma} \lambda^{(i)}_{\alpha,\beta}$. We are reduced to find $\lambda^{(i)}_{\alpha,\beta}$'s such that, for all γ of weight $2e+1$, we have

$$
\sum_{i\in \mathrm{Supp}(\gamma)} \Lambda^{(i)}_{\gamma-\delta^{(i)}} = 1
$$

At this point, we shall say that there are a lot of possible choices for the linear pencil B_{e+1}. One solution can be obtained by setting, for each γ :

(i) If $i > \mathrm{Supp}(\gamma)$, then set $\Lambda^{(i)}_\gamma = 0$ with for instance $\lambda^{(i)}_{\alpha,\beta} = 0$ for all α, β's such that $\alpha + \beta = \gamma$.

(ii) If $i \leq \mathrm{Supp}(\gamma)$, let α_0 be the highest (for the lexicographic ordering) n-tuple such that there is β_0 with $\alpha_0 + \beta_0 = \gamma$. If $\alpha_0 = \beta_0$, then set $\lambda^{(i)}_{\alpha_0,\beta_0} = 1$, otherwise set $\lambda^{(i)}_{\alpha_0,\beta_0} = \lambda^{(i)}_{\beta_0,\alpha_0} = \frac{1}{2}$. And the other $\lambda^{(i)}_{\alpha,\beta}$'s are set equal to 0. Then by construction, we get $\Lambda^{(i)}_\gamma = 1$.

In conclusion, with this choice of $L_{B_{e+1}}$ we get a symmetrizable unipotent linear description $q(x) = L_0(\,\mathbf{Id} - L_B(x))^{-1} C_0$. ☐

We give here some examples :

Example 3.3

i) If $Q(x)$ is a homogeneous polynomial of degree $d = 2e + 1$ in $n = 2$ variables, then our choice of the $\lambda^{(i)}_{\alpha,\beta}$ leads to

$$
L_{B_{e+1}} = \begin{pmatrix}
b_{(2e,0)}x_1 & \frac{b_{(2e-1,1)}x_1}{2} & \dots & \frac{b_{(e+2,e-2)}x_1}{2} & \frac{b_{(e+1,e-1)}x_1}{2} \\
\frac{b_{(2e-1,1)}x_1}{2} & 0 & \dots & 0 & \frac{b_{(e,e)}x_1}{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{b_{(e+2,e-2)}x_1}{2} & 0 & \dots & 0 & \frac{b_{(2,2e-2)}x_1}{2} \\
\frac{b_{(e+1,e-1)}x_1}{2} & \frac{b_{(e,e)}x_1}{2} & \dots & \frac{b_{(2,2e-2)}x_1}{2} & b_{(1,2e-1)}x_1 + b_{(0,2e)}x_2
\end{pmatrix}
$$

But, for instance, another choice could also lead to the diagonal matrix

$$L'_{B_{e+1}} = \begin{pmatrix} b_{(d,0)}x_1 + b_{(d-1,1)}x_2 & 0 & \dots & 0 \\ 0 & b_{(d-2,2)}x_1 + b_{(d-3,3)}x_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & b_{(1,d-1)}x_1 + b_{(0,d)}x_2 \end{pmatrix}$$

ii) If $Q(x) = \sum_{|\beta|=3} b_\beta x^\beta$ is a homogeneous polynomial of degree 3 in 3 variables, then our construction gives

$$L_{B_2} = \begin{pmatrix} b_{(3,0,0)}x_1 & \frac{b_{(2,1,0)}x_1}{2} & \frac{b_{(2,0,1)}x_1}{2} \\ \frac{b_{(2,1,0)}x_1}{2} & b_{(1,2,0)}x_1 + b_{(0,3,0)}x_2 & \frac{b_{(1,1,1)}x_1+b_{(0,2,1)}x_2}{2} \\ \frac{b_{(2,0,1)}x_1}{2} & \frac{b_{(1,1,1)}x_1+b_{(0,2,1)}x_2}{2} & b_{(1,0,2)}x_1 + b_{(0,1,2)}x_2 + b_{(0,0,3)}x_3 \end{pmatrix}$$

Note that if our linear descriptions are obtained by explicit formulas, on the other hand their sizes are fixed a priori. To compare with the proof of [HMV, Theorem 14.1] which deals with *minimal* linear description (for symmetric non-commutative polynomials) although they are not given by explicit formulas.

3.3 Symmetric determinantal representation

Let us state the main result over the reals :

Theorem 3.4 *Let $p(x)$ be a polynomial of degree d in n variables over $\mathbb{R}$ such that $p(0) \neq 0$. Then, $p(x)$ admits a symmetric determinantal representation, namely there are a signature matrix $J \in \mathbb{SR}^{N\times N}$, and a $N \times N$ symmetric linear pencil $L_A(x)$ such that*

$$p(x) = p(0)\det(J)\det(J - L_A(x))$$

where $N = 2\binom{n+\lfloor \frac{d}{2}\rfloor}{n}$.

Proof: Set $q(x) = 1 - p(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha$. If the degree d of $p(x)$ is odd, then let $Q(x, x_{n+1}) = \sum_{|\alpha|\le d} a_\alpha x^\alpha x_{n+1}^{d-|\alpha|}$ be the homogenization of the polynomial $q(x)$ obtained by introducing the extra variable x_{n+1}. Then, $Q(x, x_{n+1})$ admits a symmetrizable unipotent linear description as shown in Proposition 3.2 :

$$Q(x, x_{n+1}) = L_0(\,\mathbf{Id} - L_B(x))^{-1}C_0$$

where $S_N C_0 = L_0^t$ and $S_N L_B = (L_B)^t S_N$.

Set $(\widetilde{x}) = (x, x_{n+1})$. By symmetrization, we get the linear unipotent symmetric description

$$Q(\widetilde{x}) = \widetilde{C}^t (J - L_{\widetilde{A}})^{-1}\widetilde{C}$$

where J is a signature matrix, $A_1, \ldots, A_{n+1}$ are symmetric matrices and $L_{\tilde{A}}(x) = \sum_{i=1}^{n+1} A_i x_i = L_A(x) + A_{n+1} x_{n+1}$. Then, we deduce by Proposition 2.6 the following determinantal representation

$$1 - Q(\widetilde{x}) = \det(J) \det(J - \widetilde{C}\widetilde{C}^t - L_{\widetilde{A}}(\widetilde{x}))$$

If do the substitution $x_{n+1} = 1$, we get

$$p(x) = 1 - q(x) = 1 - Q(x, 1) = \det(J) \det(J - \widetilde{C}\widetilde{C}^t - A_{n+1} - L_A(x))$$

Since $p(0) \neq 0$, the matrix $J - \widetilde{C}\widetilde{C}^t - A_{n+1}$ is symmetric invertible, so there is another signature matrix J' and a symmetric invertible matrix V such that

$$J - \widetilde{C}\widetilde{C}^t - A_{n+1} = V^{-1} J' (V^{-1})^t$$

If we set $A' = VAV^t$, we get $\det(J - \widetilde{C}\widetilde{C}^t - L_A(x)) = \det(V)^{-2} \det(J' - L_{A'}(x))$ and we obtain the wanted identity

$$p(x) = \det(J) \det(V)^{-2} \det(J' - L_{A'}(x))$$

Now, if the degree d of $p(x)$ is even, then let $Q(x, x_{n+1}) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha x_{n+1}^{d-|\alpha|+1}$ which is the homogenization of the polynomial $q(x)$ times the extra variable x_{n+1}. Then, $Q(x, x_{n+1})$ is a homogeneous polynomial of odd degree $d+1$ such that $q(x) = Q(x, 1)$. Thus, we reduce to the proof of the previous case. $\square$

We shall emphasis that this proof gives explicit determinantal formulas for *families* of polynomials of given degree. Here is an example of such formulas when $n = 2$ and $d = 3$:

Example 3.5

Let $p(x_1, x_2) = \sum_{|\alpha| \leq 3} a_\alpha x^\alpha$ be a generic polynomial of degree 3 in 2 variables. Let $Q(x_1, x_2, x_3) = \sum_{|\beta|=3} x^\beta$ be the homogenization of $1 - p(x_1, x_2)$. We construct the unipotent symmetrizable linear description of Q given in example 3.3 ii). We obtain a linear pencil L_B which we specialize by setting $b_{(i,j,0)} = -a(i, j)$ for all $(i, j) \neq (0, 0)$ and $b_{(0,0,3)} = -a(0, 0) + 1$. Then, the polynomial $p(x_1, x_2) = 1 - Q(x_1, x_2, 1)$ has a symmetric determinantal representation

$$p(x_1, x_2) = \det(A_0 + L_A(x))$$

where $L_A(x)$ is the matrix

$$\begin{pmatrix}
0 & 0 & -\frac{x_2}{2} & -\frac{x_1}{2} & \frac{x_1}{2} & \frac{x_2}{2} & 0 & 0 \\
0 & \frac{a_{(1,0)}x_1+a_{(0,1)}x_2}{2} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{a_{(2,0)}x_1}{4} & \frac{a_{(2,0)}x_1}{4} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{a_{(1,0)}x_1+a_{(0,1)}x_2}{2} & 0 \\
-\frac{x_2}{2} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{a_{(1,2)}x_1+a_{(0,3)}x_2}{2} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(1,2)}x_1+a_{(0,3)}x_2}{2} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & -\frac{x_2}{2} \\
-\frac{x_1}{2} & \frac{a_{(2,0)}x_1}{4} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(3,0)}x_1}{2} & \frac{a_{(3,0)}x_1}{2} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(2,0)}x_1}{4} & -\frac{x_1}{2} \\
\frac{x_1}{2} & \frac{a_{(2,0)}x_1}{4} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(3,0)}x_1}{2} & \frac{a_{(3,0)}x_1}{2} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(2,0)}x_1}{4} & \frac{x_1}{2} \\
\frac{x_2}{2} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{a_{(1,2)}x_1+a_{(0,3)}x_2}{2} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(2,1)}x_1}{4} & \frac{a_{(1,2)}x_1+a_{(0,3)}x_2}{2} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{x_2}{2} \\
0 & \frac{a_{(1,0)}x_1+a_{(0,1)}x_2}{2} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{a_{(2,0)}x_1}{4} & \frac{a_{(2,0)}x_1}{4} & \frac{a_{(1,1)}x_1+a_{(0,2)}x_2}{4} & \frac{a_{(1,0)}x_1+a_{(0,1)}x_2}{2} & 0 \\
0 & 0 & -\frac{x_2}{2} & -\frac{x_1}{2} & \frac{x_1}{2} & \frac{x_2}{2} & 0 & 0
\end{pmatrix}$$

and

$$A_0 = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}+\frac{a_{(0,0)}}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2}+\frac{a_{(0,0)}}{2} & -\frac{1}{2} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2}+\frac{a_{(0,0)}}{2} & 0 & 0 & 0 & 0 & -\frac{3}{2}+\frac{a_{(0,0)}}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{3}{2}
\end{pmatrix}$$

To get a determinantal representation as given in Theorem 3.4, it remains to compute a decomposition $A_0 = V^{-1}J(V^{-1})$ where J is a signature matrix. Of course, this decomposition (and for instance the eigenvalues of A_0) depends on the value of $a_{(0,0)}$.

4 Extensions

4.0.1 Over a ring

First, we construct a slightly different linear description as in section 3.2, which will be more convenient to handle point 2) of the forthcoming Theorem 4.1.

We still consider a polynomial $Q(x)$, homogeneous of degree $d = 2e + 1$, and set $L_B(x) = \mathrm{SD}(L_{B_1}, \ldots, L_{B_d})$ and S_N exactly as in section 3.2. The change is that we now set $(D_0)^t = (1, 0, \ldots, 0, 1)$. We obviously have $S_N D_0 = D_0$.

With the choice of the L_{B_i}'s as in Proposition 3.2, we still get a symmetrizable unipotent linear description, but for the polynomial $Q(x) + 2$:

$$Q(x) + 2 = \sum_{|\alpha|=d} b_\alpha x^\alpha = (X_e)^t L_{B_{e+1}}(X_e) + 2 = D_0^t(\,\mathbf{Id} - L_B(x))^{-1} D_0$$

Such a linear description will be said of *type* $(D_0 S_N D_0)$.

Theorem 4.1 *Let $p(x)$ be a polynomial of degree d in n variables over a ring R of characteristic different from 2. Let $N = 2\binom{n+\lfloor \frac{d}{2} \rfloor}{n}$. Then,*

1) *There is a symmetric $N \times N$ affine linear pencil $A_0 + L_A(x)$ with entries in R such that*

$$p(x) = \det(A_0 + L_A(x)).$$

2) *If $p(x) = P(x) + p(0)$ where $P(x)$ is a homogeneous polynomial of odd degree and $p(0)$ is invertible in R, then there is a symmetric determinantal representation as in Theorem 3.4. Namely, there are a signature matrix $J \in \mathbb{SR}^{N \times N}$, and a $N \times N$ symmetric linear pencil $L_A(x)$ with coefficients in R such that*

$$p(x) = p(0) \det(J) \det(J - L_A(x))$$

Proof: We follow the proof of Theorem 3.4. We just have to check that we are dealing with matrices with coefficients in the ring R.

First, we assume that the degree d of $p(x)$ is odd : $d = 2e + 1$.

Set $q(x) = (1 - p(x)) - 2 = -1 - p(x)$ and let $Q(x, x_{n+1})$ be the homogenization of $q(x)$.

The polynomial $Q(x, x_{n+1})$ admits a linear description of type $(D_0 S_N D_0)$:

$$Q(x, x_{n+1}) = D_0^t(\,\mathbf{Id} - L_B(x))^{-1} D_0$$

We must be careful at the symmetrization step. Indeed, over the reals, the existence of a matrix U and a signature matrix J such that $S_N = U J U^t$ is given by the diagonalization theorem for real symmetric matrices. For instance, if $N = 2$ we have the following identity over the reals

$$S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

In order to work over R, we will prefer to write the following

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

To perform this slight transformation for general N, we need to introduce some new matrices :

- The signature matrix

$$J = \begin{pmatrix} \mathbf{Id}_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{Id}_{n_k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{Id}_{n_k} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{Id}_{n_0} \end{pmatrix}$$

- The anti-diagonal matrix of size $k \times k$

$$\mathbf{Ad}_k = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

- The permutation matrix

$$P = \begin{pmatrix} \mathbf{Ad}_{m_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{Ad}_{m_k} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{Id}_{m_k} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{Id}_{m_0} \end{pmatrix}$$

- And the matrix

$$Y = \begin{pmatrix} \mathbf{Id}_{m_0} & 0 & 0 & 0 & 0 & \mathbf{Ad}_{m_0} \\ 0 & \ddots & 0 & 0 & \iddots & 0 \\ 0 & 0 & \mathbf{Id}_{m_k} & \mathbf{Ad}_{m_k} & 0 & 0 \\ 0 & 0 & \mathbf{Ad}_{m_k} & -\mathbf{Id}_{m_k} & 0 & 0 \\ 0 & \iddots & 0 & 0 & \ddots & 0 \\ \mathbf{Ad}_{m_0} & 0 & 0 & 0 & 0 & -\mathbf{Id}_{m_0} \end{pmatrix}$$

Then, we only check that $S_N = \frac{1}{2} WJW^t$ where $W = PY$.

All the considered matrices have entries in R, and this property hold also for the following matrices

$$W^{-1} = \frac{1}{2}(JW^t S_N) = \frac{1}{2} W^t$$

and

$$L_{\widetilde{A}} = \frac{1}{2} JW^t \, L_A (W^{-1})^t$$

Now, if we set formally $\widetilde{D_0}(\widetilde{D_0})^t = 2W^{-1}D_0D_0^t(W^{-1})^t$ (the right side of the equality has entries in R although $\widetilde{D_0}$ has not), then we are able to deduce from Proposition 2.6 the following determinantal representation with coefficients in R :

$$1-(Q(\widetilde{x})+2) = \det(J)\det(J-\widetilde{D_0}\widetilde{D_0}^t - L_{\widetilde{A}}(\widetilde{x}))$$

If do the substitution $x_{n+1}=1$, we get

$$p(x) = 1-(q(x)+2) = 1-(Q(x,1)+2) = \det(J)\det(J-\widetilde{D_0}\widetilde{D_0}^t - A_{n+1} - L_A(x))$$

This conclude the proof of the first point when the degree of $p(x)$ is odd. We do the same trick as in the proof of Theorem 3.4 if the degree of $p(x)$ is even.

Now, to prove point 2), we first observe that $p(x) = -p(0)\left(1-\left(\frac{P(x)}{p(0)}+2\right)\right)$, which has sense since $p(0)$ is invertible. Next, we copy the proof of point 1) with the polynomial $Q(x) = \frac{P(x)}{p(0)}$, except that we do not need to add any extra-variable x_{n+1}. In fact we have :

$$1-(Q(x)+2) = \det(J)\det(J-\widetilde{D_0}\widetilde{D_0}^t - L_{\widetilde{A}}(x))$$

And we compute

$$\widetilde{D_0}(\widetilde{D_0})^t = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

Hence,

$$J-\widetilde{D_0}\widetilde{D_0}^t = \begin{pmatrix} -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{Id}_{m_1+\ldots+m_e} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{Id}_{m_0+m_1+\ldots+m_e} \end{pmatrix}$$

which is a signature matrix, and we are done.

Note that linear descriptions of type $(D_0S_ND_0)$ play a crucial role in order to find a signature matrix at this step. □

4.1 Noncommutative symmetric polynomials

The aim of this section is to adapt the construction of section 3.2 to the setting of noncommutative polynomials (in short NC-polynomials).

We denote by Γ_n the free semi-group on the n symbols $\{\xi_1,\ldots,\xi_n\}$. Let $x_1,\ldots,x_n$ be n noncommutating formal variables and for a word $\alpha = \xi_{i_1}\ldots\xi_{i_k} \in \Gamma_n$, we define $x^\alpha = x_{i_1}\ldots x_{i_k}$. For instance we have the identity $x^\alpha x^\beta = x^{\alpha\beta}$.

Then, a general NC-polynomials is a finite sum of the form $\sum_{\alpha\in\Gamma_n} a_\alpha x^\alpha$, with $a_\alpha \in \mathbb{R}$. We write $\mathbb{R}[x] = \mathbb{R}[x_1,\ldots,x_n]$ for the ring of all NC-polynomials over the reals.

On $\mathbb{R}[x]$, we define a transposition involution ^{t}. It is $\mathbb{R}$-linear and such that, if $x^\alpha = x_{i_1}x_{i_2}\ldots x_{i_k}$, then $(x^\alpha)^t = x_{i_k}\ldots x_{i_2}x_{i_1}$.

Then, a NC-polynomial $p(x)$ will be called *symmetric* if $p(x)^t = p(x)$ (we implicitly assumed that the variables x_i are symmetric).

In this setting, we still may define the weight of a monomial x^ω, as the weight of the word ω. Furthermore, we will still consider the lexicographic ordering on monomials as the lexicographic ordering on the words corresponding to the exponents. So, it appears possible to adapt the construction to NC-polynomial, and we get :

Theorem 4.2 *Let $p(x)$ be a symmetric NC-polynomial of degree d in n variables over $\mathbb{R}$ such that $p(0) \neq 0$. Then, $p(x)$ admits a symmetric determinantal representation, namely there are a signature matrix $J \in \mathbb{SR}^{N\times N}$, and a $N \times N$ symmetric linear pencil $L_A(x)$ such that*

$$p(x) = p(0)\det(J)\det(J - L_A(x))$$

where $N = 2\left(\frac{n^{\lfloor \frac{d}{2} \rfloor}-1}{n-1}\right)$.

Proof: First of all, as in Proposition 3.2, we show the existence of a unipotent symmetric linear description for $Q(x)$, a given NC homogeneous symmetric polynomial of odd degree $d = 2e+1$. We write $Q(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$.

Let $p_k = p_k(n) = n^k$ be the number of NC-monomials of degree k in the n variables $(x_1, \dots, x_n)$ and denote by X_k the column matrix of all NC-monomials of degree k, ordered with the lexicographic ordering.

We still consider a linear pencil of the form $L_B(x) = \mathrm{SD}(L_{B_1}, \dots, L_{B_d})$ where the L_{B_i}'s are given as follows. For $i = 1 \dots e$, let

$$L_{B_i} = (x_1 \,\mathbf{Id}_{p_{i-1}}, x_2 \,\mathbf{Id}_{p_{i-1}}, \dots, x_n \,\mathbf{Id}_{p_{i-1}})^t \quad \text{and} \quad L_{B_{e+i+1}} = (L_{B_i})^t$$

We shall note that $B_i = B_{i-1} \bigotimes \mathbf{Id}_{p_{i-1}}$ for $i = 1 \dots e$, and also that $X_e = L_{B_e} \times L_{B_{e-1}} \times \dots \times L_{B_1}$.

Then, it remains to define $L_{B_{e+1}}$, which appears to be even more canonical than in the commutative setting. Indeed, we may simply set $L_{B_{e+1}} = (\phi_{\alpha,\beta})_{|\alpha|=e,|\beta|=e}$ where $\phi_{\alpha,\beta} = \sum_{i=1}^n a_{(\alpha,i,\beta)} x_i$.

Since $Q(x)$ is symmetric we have relations $a_\alpha = a_{\alpha^t}$, which lead to the equality $\phi_{\alpha,\beta} = \phi_{\beta,\alpha}$. Thus, the matrix $L_{B_{e+1}}$ is symmetric and such that

$$Q(x) = \sum_{|\alpha|=2e+1} b_\alpha x^\alpha = (X_e)^t L_{B_{e+1}} (X_e)$$

It corresponds to an unipotent symmetrizable linear description of $Q(x)$ of size $N = 2\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} p_k$:

$$Q(x) = L_0(\,\mathbf{Id} - L_B(x))^{-1} C_0$$

with $L_0 = (0, \dots, 0, 1)$ and $C_0 = (1, 0, \dots, 0)^t$.

Then, the only thing we have to change in the proof of Theorem 3.4, is that we shall consider the symmetric-homogenization $Q(x, x_{n+1})$ of our polynomial $q(x)$. Namely

$$Q(x, x_{n+1}) = \sum_{|\alpha| \leq d} \frac{a_\alpha}{2} \left(x^\alpha x_{n+1}^{d-|\alpha|} + x_{n+1}^{d-|\alpha|} x^\alpha \right)$$

□

5 Some questions

- It would be very interesting to find polynomials which have a *unitary* determinantal representation (i.e. $A_0 = \mathbf{Id}$ in the formulation of Theorem 4.1). Indeed, this polynomials are of great interest for instance in optimization (see [HV] for properties of such polynomials and references on the subject).

  Unfortunately, our construction (with the choices of S_N, L_0, C_0 and $L_{B_{e+1}}$) gives representations which has no chance to be unitary. Indeed, the signature of the matrix S_N has as many $+1$ than -1.

- For a given NC-polynomial, there are several criterions to say if a linear description is *minimal*. For instance, there is an interesting one by the rank of the so-called Hankel matrix (see [BR, Theorem II.1.5]).

  In fact, we may note that the integer $N = 2\sum_{k=0}^{e} p_k$ which appears in Theorem 4.2 is equal to the rank of the Hankel matrix associated to a *generic* symmetric homogeneous NC-polynomial of degree $2e + 1$.

  In the commutative setting, if $p(x)$ is a given polynomial, we may consider a NC-symmetric lifting of $p(x)$ and compute the rank of its Hankel matrix. So, the minimal size of a linear description of a given commutative polynomial can be obviously bounded by the minimum size of all linear descriptions associated to all possible NC-symmetric lifting of the polynomial. Although, it is not clear how to get a criterion.

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