ON LEVEL CROSSING IN (RANDOM) MATRIX PENCILS. INTRODUCTORY LECTURES

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Abstract. Below I give some basic information required for the research experience project at the Mathematics department of UIUC in summer 2015.

Contents

1. Introduction

Studies of the spectral behavior for linear pencils of the form

$$\mathcal{P} = A + \beta B \tag{1.1}$$

where A is an initial linear operator, B is a perturbing linear operator, and β is a small real or complex parameter, are scattered in many papers in mathematics and physics, see e.g. the fundamental treatise [Ka] and [ZVW]. In most of applications in physics A and B are self-adjoint and β is real which in many cases leads to the conclusion that the spectrum is real and simple for all real values of β. However since the late 60's, motivated by a number of fascinating observations of C. M. Bender and T. T. Wu [BW], physicists and mathematicians started considering the situations where A and B are, for example, self-adjoint while β is complex.

The basic motivating question for such study is whether it is possible to interchange two arbitrary real eigenvalues occurring for some fixed real value of β_0 by

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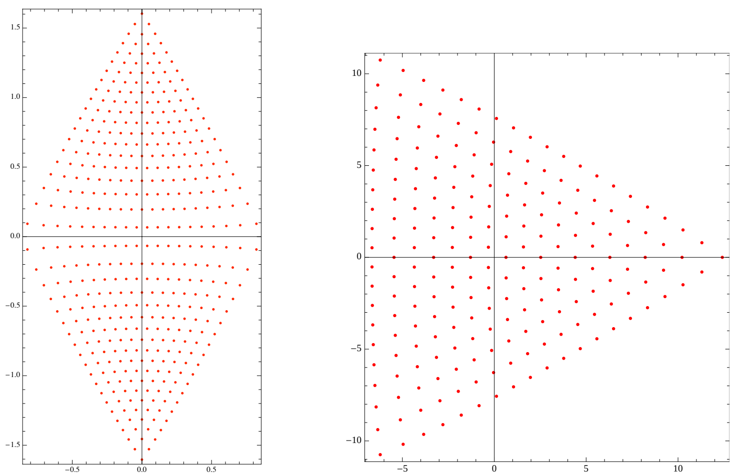

FIGURE 1. The set of branching points for the quasi-exactly solvable sextic and quartic. The value of the discrete parameter equals 20.

allowing β to move in the complex plane, or, in other words, whether the corresponding spectral surface $S_{\mathcal{P}}$ is connected. Here the spectral surface $S_{\mathcal{P}} \subset \mathbb{C}^2$ is the set of all pairs $(\lambda, \beta) \in \mathbb{C}^2$ where λ is an eigenvalue of $\mathcal{P}$ with a given value of parameter β. In several important situations it is proven that $S_{\mathcal{P}}$ is an analytic curve in $\mathbb{C}^2$ given as the zero locus of an appropriate entire function in two variables called the spectral determinant. Important results in this direction were recently obtained in e.g. [EG] and [AG]. (For matrix pencils $S_{\mathcal{P}}$ is an algebraic curve given by the obvious spectral equation (1.2).)

For large classes of linear pencils (1.1) there exists a discrete level crossing set $\mathcal{B}_{\mathcal{P}} \subset \mathbb{C}$ consisting of all values of α for which the spectrum of (1.1) is not simple. In other words, $\mathcal{B}_{\mathcal{P}}$ is the set of all branching points for the projection of the spectral curve $S_{\mathcal{P}}$ to the α-plane. In particular, for generic matrix pencils (1.1) of size n the branching set $\mathcal{B}_{\mathcal{P}}$ consists of $n(n-1)$ points counting multiplicities. (It is convenient to consider $\mathcal{B}_{\mathcal{P}}$ as a divisor in $\mathbb{CP}^1$.)

For many concrete pencils (1.1) of linear differential or matrix operators it is highly desirable to get the information about their level crossing sets $\mathcal{B}_{\mathcal{P}}$ as well as about the monodromy of eigenvalues if the parameter α runs over some loop in $\mathbb{C} \setminus \mathcal{B}_{\mathcal{P}}$, but the latter problem (especially its monodromy part) seems to be very hard, see examples below.

For matrix pencils with Hermitian A and B studies of the corresponding spectral surfaces and their branching points are related to the so-called Lax conjecture, see e.g. [?] and determinantal representations of polynomials [PlVi]. It turns out that one can explicitly characterize the class of real spectral determinants = real algebraic curves given by the equation

$$\Psi(\beta, \lambda) = \det(A + \beta B + \lambda I) = \det(\mathcal{P} + \lambda I) = 0, \tag{1.2}$$

with arbitrary Hermitian A and B of some size n. (Obviously (1.2) defines a plane algebraic curve of degree n.) For complex-valued square matrices A and B of a given size n it was already shown in 1902 by A. C. Dixon [Di] that any plane complex

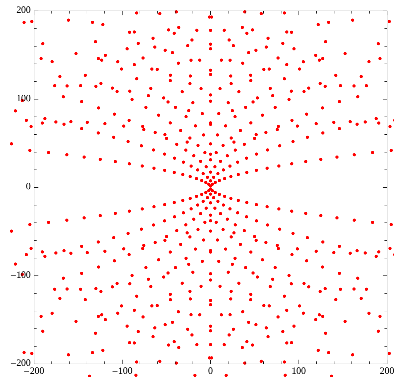

FIGURE 2. The set of branching points for the truncated Mathieu operator.

algebraic curve of degree n can be represented by (1.2) and he also found out how many different determinantal representations there exist for a generic plane curve of degree n.

Observe also that level crossing loci $\mathcal{B}_\mathcal{P}$ which can appear as the sets of branching points of complex plane curves of degree n (or, equivalently, of representations (1.1)) contain $n(n-1)$ points but depend only on $\binom{n+2}{2}-4=\frac{n^2+3n-6}{2}$ parameters. This means that starting with $n=4$ there exist some complicated relations among the branching points. The first non-trivial case $n=4$ there exists one relation on the 12 points in the level crossing set, i.e. they form a fascinating hypersurface in $\mathbb{CP}^{12}$ which iwas considered in [Va]. Observe that the degree of this hypersurface equals 3762.

2. Lecture 1. Meromorphic functions

In what follows by a Riemann surface we will always mean a compact connected orientable 2-dimensional manifold with a Riemann metric, see [Fo]. For any such metric there exists a unique complex structure adjusted to the Riemann metric (which means that at each point the operator of multiplication by $\sqrt{-1}$ coincides with the rotation by $\pi/2$ in the Riemann metric?) The genus $g(\mathcal{R})$ (=number of handles) of a Riemann surface $\mathcal{R}$ is defined as $g(\mathcal{R})=\frac{2-\chi(\mathcal{R})}{2}$, where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$, i.e. $\chi(\mathcal{R})=\sum_{j=0}^{2}(-1)^j\dim H^j(\mathcal{R},\mathbb{R})$, $H^j(\mathcal{R},\mathbb{R})$ being the j-th cohomology group of $\mathcal{R}$ with real coefficients.

Definition 1. *A* **meromorphic function** $f:\mathcal{R}\to\mathbb{C}P^1$ *is an arbitrary holomorphic map from a Riemann surface $\mathcal{R}$ to the projective line $\mathbb{C}P^1$. By the* **degree** *$d(f)$ of a meromorphic function f we mean the number of points in $f^{-1}(z)$ counted with multiplicities, where z is an arbitrary point in $\mathbb{C}P^1$ (which is the same as the number of points in $f^{-1}(z)$ for almost all but finitely many points in $\mathbb{C}P^1$). The degree can be also defined purely algebraically as the degree of extension of the fields $\mathbb{C}(\mathbb{C}P^1)\subset\mathbb{C}(\mathcal{R})$ where the field inclusion is induced by f. (Here $\mathbb{C}(\mathcal{R})$ is the field of all meromorphic functions on $\mathcal{R}$ and $\mathbb{C}(\mathbb{C}P^1)$ is the field of all meromorphic=rational functions on $\mathbb{C}P^1$.)*

Existence of meromorphic functions on an arbitrary Riemann surface is a deep analytic result going back to H.Weyl, see e.g. [Mi].

2.1. **Branching points.**

Definition 2. *Given $f:\mathcal{R}\to\mathbb{C}P^1$ of degree d, we say that $z\in\mathbb{C}P^1$ is a* **branching point** *of f, if $card(f^{-1}(z))<d$. The union $Br_f\subset\mathbb{C}P^1$ of all branching points of f is called its* **branching locus***.*

Observe that if f maps $p\in\mathcal{R}$ to $z\in\mathbb{C}P^1$, then in appropriate local charts near p and z respectively, f is given by $u\mapsto u^k$ for some positive integer k, called the **ramification index** of f at p. We denote the ramification index by $ind_f(p)$. Obviously, at all but finitely many points $p\in\mathcal{R}$, $ind_f(p)=1$.

Definition 3. *Given $f:\mathcal{R}\to\mathbb{C}P^1$ of degree d and its branching point $z\in\mathbb{C}P^1$, by a* **profile** *of z we mean the partition $\mu(z)\vdash d$ obtained as the collection of all ramification indices for points in $f^{-1}(z)\subset\mathcal{R}$. A branching point $z\in\mathbb{CP}^1$ of f is called* **simple** *if its profile is $(2,1,...,1)=(2,1^{d-2})$.*

The following well-known basic result is extremely useful.

Proposition 1 (Riemann-Hurwitz formula)**.** *For any holomorphic map $f:\mathcal{R}_1\to\mathcal{R}_2$ of degree d, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are two Riemann surfaces,*

$$\chi(\mathcal{R}_1)=d\chi(R_2)-\sum_{p\in\mathcal{R}_1}(ind_f(p)-1). \tag{2.1}$$

In particular, for a meromorphich function $f:\mathcal{R}\to\mathbb{C}P^1$ of degree d with only simple branching points, their number equals $2d+2g(\mathcal{R})-2$.

2.2. **Monodromy and monodromy representation.** Given a meromorphic function $f:\mathcal{R}\to\mathbb{C}P^1$ of degree d, consider a (base) point $b\in\mathbb{C}P^1$ which is not a branching point of f, i.e. $b\notin Br_f$. Then $f^{-1}(b)\subset\mathcal{R}$ has cardinality d. Now for any oriented loop γ in $\mathbb{C}P^1\setminus Br(f)$ based at b, (i.e. starting and ending at b,) we can consider its complete preimage $f^{-1}(\gamma)$ which consists of d disjoint (except for the endpoints) oriented segments in $\mathcal{R}$ starting and ending at points of $f^{-1}(b)$. In fact, we get a well-defined permutation perm_γ of points in $f^{-1}(b)$ (by assigning to the starting point of each segmant its endpoint) induced by g which is called the monodromy permutation of f along γ. In fact perm_γ depends only on the homotopy class of γ inside the fundamental group $\pi_1(\mathbb{C}P^1\setminus Br_f)$. (The latter group is a free group with card $(Br(f))-2$ generators.) Additionally if $\gamma=\gamma_1\times\gamma_2$ is the product of two loops in $\pi_1(\mathbb{C}P^1\setminus Br_f)$, then $\mathrm{perm}_\gamma=\mathrm{perm}_{\gamma_1}\times\mathrm{perm}_{\gamma_2}$. In other words, we get a homomorophism

$$\rho_f:\pi_1(\mathbb{C}P^1\setminus Br_f)\to Symm(f^{-1}(b)),$$

where $Symm(f^{-1}(b))$ is the group of all permutations of all d points in Br_f, i.e. in other words, $Symm(f^{-1}(b)$ is isomorphic to Symm_d which is the symmetric group on d elements. The homomorphism ρ is called the **monodromy representation** of the meromorphic function f and its image in $Symm(f^{-1}(b))$ is called the **monodromy group** of f.

The next result of B.Riemann is widely used.

Proposition 2 (Riemman's existence theorem)**.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two Riemann surfaces. Given a d-sheeted covering $f:\mathcal{R}_1\to\mathcal{R}_2$ with branching locus $Br_f=\{z_1,...,z_n\}\subset\mathcal{R}_2$, there is a homomorphism*

$$\rho:\pi_1(\mathcal{R}_2\setminus Br_f,.)\to Symm_d,$$

determined up to an inner automorphism (i.e. two homomorphism ρ_1, ρ_2 *are equivalent if there exists* $\sigma \in Symm_d$ *such that* $\rho_2(g) = \sigma\rho_1(g)\sigma^{-1}$ *for all* $g \in \pi_1(\mathcal{R}_2 \setminus Br_f, .)$*. Conversely, given a monodromy representation*

$$\rho : \pi_1(\mathcal{R}_2 \setminus Br, .) \to Symm_d,$$

there is a unique branched covering $f : \mathcal{R}_1 \to \mathcal{R}_2$ *with branched set contained in* Br*. Here* Br *is some finite subset of* $\mathcal{R}_2$*.*

More infomation and details about this topic can be found in e.g. [On] which is posted on the program's webpage.

3. Lecture 2. Plane algebraic curves

Definition 4. *A plane (affine complex) algebraic curve* $\Gamma \subset \mathbb{C}^2$ *is the zero locus of a non-constant bivariate polynomial* $P(x, y) = 0$*. (Here* (x, y) *are some chosen coordinates in* $\mathbb{C}^2$*.) The degree of* $P(x, y)$ *is called the degree of* Γ *and it coincides with the number of points (counted with multiplicities) in which a generic affine line intersects* Γ*. Analogously, a plane (projective complex) algebraic curve* $\Gamma \subset \mathbb{C}P^2$ *is the zero locus of a non-constant trivariate homogeneous polynomial* $P(X, Y, Z) = 0$ *considered projectively. (Here* $(X : Y : Z)$ *are some chosen homogeneous coordinates in* $\mathbb{C}P^2$*.) IF* $P(x, y)$ *(resp.* $P(X, Y, Z)$*) is a real-valued polynomial, the corresponding curve is called real. A plane curve is called reduced if it has no multiple components, i.e. its defining polynomial has no multiple factors.*

Numerous examples of plane algebraic curves are known since the time of ancient Greece. They have been studied for several centuries and are still one of the major topics in algebraic geometry. See e.g. [ACGH, BK, Mi, Wa].

Observe that for a generically chosen polynomial $P(x, y)$ (resp. $P(X, Y, Z)$), the corresponding plane algebraic curve is everywhere smooth, i.e. is diffeomorphic to a complex disk near any its point. On the other, hand there are also plenty of plane curves with singularities, i.e. having points where the gradient vector $\left(\frac{\partial P(x,y)}{\partial x}, \frac{\partial P(x,y)}{\partial y}\right)$ vanishes. The most important of such singular points are the nodes, at which a curve locally looks like a cross $xy = 0$.

Proposition 3. *The genus of a smooth plane projective curve of degree* d *equals* $\binom{d-1}{2}$*.*

To any (singular) plane projective curve $\Gamma \subset \mathbb{C}P^2$, one can associate its normalization $\widehat{\Gamma}$ which is a smooth compact Riemann surface birationally equivalent to Γ, i.e. there exists an isomorphism (biholomorphic map) between Γ and $\widetilde{\Gamma}$ with finitely many points removed. $\widehat{\Gamma}$ is uniquely defined as a Riemann surface. There exists a uniquely defined morphism $\mathfrak{n} : \widehat{\Gamma} \to \Gamma$. (Obviously, the normalization of a smooth plane curve is the curve itself.)

The geometric genus $g(\Gamma)$ of a (singular) projective curve Γ is by definition the (usual) genus of its normalization $\widehat{\Gamma}$.

Lemma 4. *The geometric genus of a (singular) plane curve of degree* d *can be any number from* 0 *to* $\binom{d-1}{2}$*.*

3.1. **Meromorphic functions induced by projections.** Given a point $p \in \mathbb{C}P^2$, consider the set $\mathbb{C}P^1_p$ of all lines in $\mathbb{C}P^2$ passing through p. This set is isomorphic to $\mathbb{C}P^1$. There is an obvious projection $\pi_p : \mathbb{C}P^2 \setminus p \to \mathbb{C}P^1_p$ obtained by assigning to each point $q \in \mathbb{C}P^2 \setminus p$ the unique line passing through p and q.

Given a point $p \in \mathbb{C}P^2$ and a reduced plane curve $\Gamma \subset \mathbb{C}P^2$ having no line component passing through p, we associate to this pair (Γ, p) the meromorphic function $pr_p : \widehat{\Gamma} \to \mathbb{C}P^1$ as follows. If $p \notin \Gamma$, then we restrict π_p to Γ and then (if Γ is singular) lift this projection to $\widehat{\Gamma}$ by composing it with $\mathfrak{n}$. If $p \in \Gamma$, then we restrict π_p to $\Gamma \setminus p$. Such a holomorphic map uniquely lifts to $\widehat{\Gamma} \setminus \mathfrak{n}^{-1}(p)$ and then uniquely extends to the whole $\widehat{\Gamma}$. We denote this meromorphic function by $\pi_p^\Gamma : \widetilde{\Gamma} \to \mathbb{C}P^1$.

Observe that if Γ has degree d and $p \notin \Gamma$, then $\deg \pi_p^\Gamma = d$. But if $p \in \Gamma$, then $\deg \pi_p^\Gamma = d - \mathrm{mult}\Gamma_p$, where $\mathrm{mult}\Gamma_p$ stands for the local multiplicity of Γ at p, i.e. the local multiplicity of the zeros of a univariate polynomial obtained by the restriction of the defining polynomials $P(x, y)$ to a generic line passing through p.

Lemma 5. *Given a smooth plane curve Γ of degree d and a generic point $p \notin \Gamma$, there exists $d(d-1)$ tangent lines to Γ passing through p. In other words, the induced meromorphic function π_p^Γ has $d(d-1)$ simple branching points.*

Proposition 6. *Any meromorphic function $f : \mathcal{R} \to \mathbb{C}P^1$ on a closed Riemann surface $\mathcal{R}$ can be represented as $f = \pi_p \circ \nu$ where $\nu : \mathcal{R} \to \mathbb{C}P^2$ is a birational mapping of $\mathcal{R}$ to its image and $\pi_p : \nu(\mathcal{R}) \to \mathbb{C}P^1$ is the projection of the image curve $\nu(\mathcal{R})$ from a point $p \in \mathbb{C}P^2$ to the pencil of lines through p.*

The following question is immediate.

Problem 1. *Given a meromorphic function $f : \mathcal{R} \to \mathbb{C}P^1$ can we always realize it as in Proposition 6 with $\deg \nu(\Gamma) = \deg f$?*

But already not all meromorphic functions of degree 3 on smooth cubics as well as not all meromorphic functions of degree 4 on smooth quartics can be realized in such a way, see [On]. However, the following holds.

Theorem 7. *Suppose that $\Gamma \subset \mathbb{C}P^2$ is a smooth projective plane curve of degree $d > 4$. Then any meromorphic function $f\mathcal{R} \to \mathbb{C}P^1$ of degree d can be realized as a linear projection $\pi_p\mathcal{R} \to \mathbb{C}P^1$.*

3.2. **Planarity stratification of the space of meromorphic functions.** The next definition introduced in [OnSh] seems very natural.

Definition 5. *The* **planarity defect** *pdef(f) of a meromorphic function $f : \mathcal{R} \to \mathbb{C}P^1$ equals*

$$pdef(f) := \min_\nu(\deg(\nu(\mathcal{R})) - \deg(f)$$

such that $f = \pi_p \circ \nu$ as above.

The small Hurwitz space of degree d functions of genus g curves is defined as:

$$\mathcal{H}_{g,d} = \{f : \mathcal{R} \to \mathbb{C}P^1 | f \text{ has only simple branched points}, \deg f = d \geq 2, \ \mathrm{gen}(\mathcal{R}) = g \geq 0\}.$$

Recall that $\dim \mathcal{H}_{g,d}$ equals the number of (simple) branching points of a function from $\mathcal{H}_{g,d}$ and is given by the formula

$$\dim \mathcal{H}_{g,d} = 2d + 2g - 2.$$

Small Hurwitz spaces were introduced and studied in substantial details by Clebsch [1] and Hurwitz [Hu] at the end of the 19-th century as a tool of investigation of the moduli space $\mathcal{M}_g$ of genus g curves.

Proposition 6 allows us to introduce the *planarity stratification* of $\mathcal{H}_{g,d}$:

$$\mathcal{H}_{g,d}^{m(g,d)} \subset \mathcal{H}_{g,d}^{m(g,d)+1} \subset \cdots \subset \mathcal{H}_{g,d}^{M(g,d)} = \mathcal{H}_{g,d}, \tag{3.1}$$

where $\mathcal{H}_{g,d}^{l}$ consists of all meromorphic functions in $\mathcal{H}_{g,d}$ whose planarity defect does not exceed l. Here $m(g,d)$ (resp. $M(g,d)$) is the minimal (resp. maximal) value of the planarity defect for degree d meromorphic functions with all simple branching points on curves of genus g.

We present some information about this stratification.

Proposition 8. *For any pair (g,d) where $g \geq 0$ and $d \geq 2$,*

$$m(g,d) = \min_{l \geq 0} \binom{d+l-1}{2} - \binom{l}{2} \geq g. \tag{3.2}$$

which gives

$$m(g,d) = \max\left(0, \left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil\right). \tag{3.3}$$

Moreover the following result holds.

Theorem 9. *In the above notation, given g, d and $l \geq m(g,d)$, the stratum $\mathcal{H}_{g,d}^{l}$ is irreducible and its dimension is given by:*

$$\dim \mathcal{H}_{g,d}^{l} = \min\,(3d + g + 2l - 4, 2d + 2g - 2). \tag{3.4}$$

Lemma 10. *Given g, d as above,*

$$M(g,d) = \max\left(0, \left\lceil \frac{g-d+2}{2} \right\rceil\right). \tag{3.5}$$

In particular, $m(g,d) = M(g,d) = 0$ if and only if $d \geq g+2$.

3.3. **Determinantal representations of plane curves.** In our project we consider (affine) plane curves whose defining equations are given by

$$P(\beta, \lambda) = \det(\lambda I + A + \beta B),$$

where A and B are complex, Hermitian or real-symmetric matrices, and $\beta \in \mathbb{C}$. One can obviously, make the situation homogeneous by considering homogeneous pencils

$$P(\lambda, \alpha, \beta) = \det(\lambda I + \alpha A + \beta B). \tag{3.6}$$

If matrices A and B have size n, then polynomial P (in both cases) will be of degree n.

Theorem 11. *Any plane complex curve $\Gamma \subset \mathbb{C}P^2$ of degree n has a determinantal presentation* (3.6). *The number of such presentations equals the theta-characteristics of* Γ.

3.4. **Which curves are representable by pencils of Hermitian matrices.** Obeserve that if A and B are Hermitian or real-symmetric then $P(\lambda,\alpha,\beta)$ is necessarily a real polynomial. If we instead of I allow any Hermitian or real-symmetric matrix, then any real trivariate homogeneous polynomial can be realized in such a way, see details in [Qu]. For (3.6) with Hermitian A an B, there is an obvious additional neccessary condition as follows.

Definition 6. *A real homogeneous polynomial $Q(x_1,\dots,x_n)$ is called hyperbolic with respect to a vector $\bar v\in\mathbb{R}^n\backslash 0$, if for any vector $\bar x\in\mathbb{R}^n$, the univariate poynomial of the form $Q_{\bar x}(t)=Q(\bar x+t\bar v)$ has a real zeros (counted with multiplicities).*

Since the spectrum of any Hermitian and real-symmetric matrix is real, the polynomial $P(\lambda,\alpha,\beta)$ given by (3.6) is hyperbolic w.r.t $(1,0,0)$.

The following conjecture by P. Lax was proven in [LPR] with a reference to a previous work of V. Vinnikov.

Conjecture 1 (P. Lax, 1958)**.** *A polynomial $p(x,y,z)$ is hyperbolic of degree d with respect to the vector $e=(1,0,0)$ and satisfies $p(e)=1$ if and only if there exist real-symmetric $d\times d$-matrices B,C such that p is given by*

$$p(x,y,z)=det(xI+yB+zC).$$

4. Lecture 3. Random matrices. Basic facts

5. Lecture 4. GE_n, GUE_n and GOE_n and other distributions.

The main goal of our project is to study the distribution of brancihng points and monodromy representation in case when A and B are random complex, Hermitian or real-symmetric martices. In the above two lectures i will recall some basic information about random matrices mainly following Ch. 3-5 of [Me] and some material of [AGZ]. Random matrices appeared first in nuclear physics, see very intreresting discussion in Ch.1 of [Me].

6. Projects

Project 1. Pencils of random matrices belonging to the Gaussian Ensemble.

Another natural distribution of matrices which might be easier to study in the (complex) Gaussian ensemble. In this case I can offer the following teo conjectures to study.

Conjecture 2. *For any given size $n\geq 2$, if A and B are independently chosen from GE_n then the density of the branching points in $\mathbb{C}$ is given by*

$$\mathcal{P}(\alpha,\bar\alpha)=\frac{3}{\pi(1+|\alpha|^2)^2}.$$

In other words, the distribution of the branching points is uniform of the Riemann sphere $\mathbb{CP}^1$ compactifying the α-plane.

Conjecture 3. *The conclusion of Conjecture 2 holds if A and B have entries independently distributed in the complex plane according to any radial law which has moments of all orders.*

Project 2. Pencils of Hermitian matrices.

Consider pencil (1.1) with Hermitian A and B of size n. We say that the pair (A,B) is *weakly generic* if $A+\alpha B$ has simple spectrum for all real α. One can easily prove that for any weakly generic pair (A,B) its level crossing set $\mathcal{B}_{\mathcal{P}}\subset\mathbb{C}$ consists of $n(n-1)$ points counting multiplicities are given as the zero set of a real polynomial in α, i.e., it consists of complex-conjugate pairs. Thus there are $\binom{n}{2}$ points of $\mathcal{B}_{\mathcal{P}}$ lying in the upper (lower) halfplane. We say that a weakly generic pair (A,B) is *strongly generic* if additionally the real parts of all level crossing points in the upper halfplane are distinct. Let us order them from left to right, i.e., $\mathcal{B}_{\mathcal{P}}=\left\{\alpha_1,\alpha_2,\dots,\alpha_{\binom{n}{2}}\right\}$. To each level crossing point α_j we can associate a transposition obtained as follows. Observe that if α is real the spectrum of $A+\alpha B$ is real and simple. Associate to α_j the following closed loop starting and ending at $\Re\alpha_j\in\mathbb{R}$. One moves along the vertical segment connecting $\Re\alpha_j$ and α_j, stops slightly below α_j, then runs counterclockwise over a small circle centered at α_j and finally returns back moving vertically down. As a result one gets a transposition of two points of spectrum corresponding to $\Re\alpha_j$. Since there is no monodromy of the spectrum over the whole real axis, we obtain an ordered sequence of $\binom{n}{2}$ transpositions $(\sigma_1,\sigma_2,\dots,\sigma_{\binom{n}{2}})$. One can observe that their product is the inverse permutation, i.e.,

$$\prod_{j=1}^{\binom{n}{2}}\sigma_j=(n,n-1,\dots,1). \tag{6.1}$$

Similar situation occurs in the study of sorting networks where additionally all transpositions are assumed to be simple, i.e., interchanging neighboring entries, see e.g., [AHRV]. In the above set-up however occurring transpositions are not supposed to be simple.

Natural questions one can ask in this situation are:

a) Is it possible to count the number of sequences of $\binom{n}{2}$ transpositions which generate S_n and satisfy (6.1)?

b) Can any such sequence be realized as the sequence associated with a pencil of Hermitian matrices?

c) Is it possible to count/enumerate connected components in the space of all strongly generic pairs (A,B)? If 'yes' what is the adjacency diagram of these components? How many components have the same given sequence $(\sigma_1,\sigma_2,\dots,\sigma_{\binom{n}{2}})$?

Remark. Some of these problems might be difficult, but there is a plenty of possible modifications of the latter questions.

While studying pencils (1.1), we use stochastic approach. In other words, we study the distribution of points in the level crossing set assuming that A and B are random matrices chosen independently from some known ensemble. (In a different setting a similar approach can be found in [GP].) The most natural from the physics point of view is to assume that A and B and indepedently chosen from the Gaussian Unitary ensemble of matrices of a given size n.

Problem 2. *For any given size $n\geq 2$, if A and B are independently chosen from GUE_n, what is the density of the branching points in $\mathbb{C}$?*

Problem 3. *For any given size $n \geq 2$, if A and B are independently chosen from GUE_n, what distribution one obtains on the sequences $(\sigma_1, \sigma_2, \dots, \sigma_{\binom{n}{2}})$ of transpositions?*

Project 3. Pencils of random symmetric matrices. Similar question to Project 2 but with GOE_n instead of GUE_n.

Project 4. Monodromy of random bivariate polynomials.

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