3 / Gaussian Ensembles. The Joint Probability Density Function for the Eigenvalues

In this chapter we will derive the joint probability density function for the eigenvalues of H implied by the Gaussian densities for its matrix elements. Finally an argument based on information theory is given. This argument rationalizes any of the Gaussian probability densities for the matrix elements; it even allows one to define an ensemble having a preassigned eigenvalue density.

3.1. Orthogonal Ensemble

The joint probability density function (abbreviated j.p.d.f. later in the chapter) for the eigenvalues $\theta_1, \theta_2, ..., \theta_N$ can be obtained from Eq. (2.6.18) by expressing the various components of H in terms of the N eigenvalues θ_j and other mutually independent variables p_μ, say, which together with the θ_j form a complete set. In an $(N \times N)$ real symmetric matrix the number of independent real parameters which determine all H_{kj} is $N(N+1)/2$. We may take these as H_{kj} with $k \le j$. The number of extra parameters p_μ needed is therefore

$$\ell = \frac{1}{2}N(N+1) - N = \frac{1}{2}N(N-1). \tag{3.1.1}$$

Because

$$\text{tr } H^2 = \sum_1^N \theta_j^2, \qquad \text{tr } H = \sum_1^N \theta_j, \tag{3.1.2}$$

the probability density that the N roots and $N(N-1)/2$ parameters will occur around the values $\theta_1, ..., \theta_N$ and $p_1, p_2, ..., p_\ell$ is, according to

Eq. (2.6.18)

$$P(\theta_1, ..., \theta_N; p_1, ..., p_\ell) = \exp\left(-a\sum_1^N \theta_j^2 + b\sum_1^N \theta_j + c\right) J(\theta, p), \tag{3.1.3}$$

where J is the Jacobian

$$J(\theta, p) = \left|\frac{\partial (H_{11}, H_{12}, ..., H_{NN})}{\partial (\theta_1, ..., \theta_N, p_1, ..., p_\ell)}\right|. \tag{3.1.4}$$

Hence, the j.p.d.f. of the eigenvalues θ_j can be obtained by integrating Eq. (3.1.3) over the parameters $p_1, ..., p_\ell$. It is usually possible to choose these parameters so that the Jacobian (3.1.4) becomes a product of a function f of the θ_j and a function g of the p_μ. If this is the case, the integration provides the required j.p.d.f. as a product of the exponential in Eq. (3.1.3), the function f of the θ_j, and a constant. The constant can then be absorbed in c in the exponential.

To define the parameters p_μ (Wigner, 1962) we recall that any real symmetric matrix H can be diagonalized by a real orthogonal matrix (*cf.* Appendix A.3):

$$H = U\Theta U^{-1} \tag{3.1.5}$$

$$= U\Theta U^T, \tag{3.1.5'}$$

where Θ is the diagonal matrix with diagonal elements $\theta_1, \theta_2, ..., \theta_N$ arranged in some order, say, $\theta_1 \le \theta_2 \le \cdots \le \theta_N$, and U is a real orthogonal matrix

$$UU^T = U^TU = 1, \tag{3.1.6}$$

whose columns are the normalized eigenvectors of H. These eigenvectors are, or may be chosen to be, mutually orthogonal. To define U completely we must in some way fix the phases of the eigenvectors, for instance by requiring that the first nonvanishing component be positive. Thus U depends on $N(N-1)/2$ real parameters and may be chosen to be U_{kj}, $k > j$. If H has multiple eigenvalues, further conditions are needed to fix U completely. It is not necessary to specify them, for they apply only in regions of lower dimensionality which are irrelevant to the probability density function. At any rate, enough appropriate conditions are imposed on U so that it is uniquely characterized by the $N(N-1)/2$ parameters

p_μ. Once this is done, the matrix H, which completely determines the Θ and the U subject to the preceding conditions, also determines the θ_j and the p_μ uniquely. Conversely, the θ_j and p_μ completely determine the U and Θ, and hence by Eq. (3.1.5) all the matrix elements of H.

Differentiating Eq. (3.1.6), we get

$$\frac{\partial U^T}{\partial p_\mu} U + U^T \frac{\partial U}{\partial p_\mu} = 0, \tag{3.1.7}$$

and because the two terms in Eq. (3.1.7) are the Hermitian conjugates of each other,

$$S^{(\mu)} = U^T \frac{\partial U}{\partial p_\mu} = -\frac{\partial U^T}{\partial p_\mu} U \tag{3.1.8}$$

is an antisymmetric matrix.

Also from Eq. (3.1.5) we have

$$\frac{\partial H}{\partial p_\mu} = \frac{\partial U}{\partial p_\mu} \Theta U^T + U \Theta \frac{\partial U^T}{\partial p_\mu}. \tag{3.1.9}$$

On multiplying Eq. (3.1.9) by U^T on the left and by U on the right, we get

$$U^T \frac{\partial H}{\partial p_\mu} U = S^{(\mu)} \Theta - \Theta S^{(\mu)}. \tag{3.1.10}$$

In terms of its components, Eq. (3.1.10) reads

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial p_\mu} U_{j\alpha} U_{k\beta} = S^{(\mu)}_{\alpha\beta} \left(\theta_\beta - \theta_\alpha\right). \tag{3.1.11}$$

In a similar way, by differentiating Eq. (3.1.5) with respect to θ_γ,

$$\sum_{j,k} \frac{\partial H_{jk}}{\partial \theta_\gamma} U_{j\alpha} U_{k\beta} = \frac{\partial \Theta_{\alpha\beta}}{\partial \theta_\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}. \tag{3.1.12}$$

The matrix of the Jacobian in Eq. (3.1.4) can be written in partitioned form as

$$[J(\theta, p)] = \begin{bmatrix} \dfrac{\partial H_{jj}}{\partial \theta_\gamma} & \dfrac{\partial H_{jk}}{\partial \theta_\gamma} \\ \dfrac{\partial H_{jj}}{\partial p_\mu} & \dfrac{\partial H_{jk}}{\partial p_\mu} \end{bmatrix}. \tag{3.1.13}$$

The two columns in Eq. (3.1.13) correspond to N and $N(N-1)/2$ actual columns; $1 \le j < k \le N$. The two rows in Eq. (3.1.13) correspond again to N and $N(N-1)/2$ actual rows: $\gamma = 1, 2, ..., N$; $\mu = 1, 2, ..., N(N-1)/2$. If we multiply the $[J]$ in Eq. (3.1.13) on the right by the $N(N+1)/2 \times N(N+1)/2$ matrix written in partitioned form as

$$[V] = \begin{bmatrix} (U_{j\alpha}U_{j\beta}) \\ (2U_{j\alpha}U_{k\beta}) \end{bmatrix}, \tag{3.1.14}$$

in which the two rows correspond to N and $N(N-1)/2$ actual rows, $1 \le j < k \le N$, and the column corresponds to $N(N+1)/2$ actual columns, $1 \le \alpha \le \beta \le N$, we get by using Eqs. (3.1.11) and (3.1.12)

$$[J]\,[V] = \begin{bmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} \\ S^{(\mu)}_{\alpha\beta}\,(\theta_\beta - \theta_\alpha) \end{bmatrix}. \tag{3.1.15}$$

The two rows on the right-hand side correspond to N and $N(N-1)/2$ actual rows and the column corresponds to $N(N+1)/2$ actual columns. Taking the determinant on both sides of Eq. (3.1.15), we have

$$J(\theta, p)\,\det V = \prod_{\alpha<\beta} (\theta_\beta - \theta_\alpha) \det \begin{bmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} \\ S^{\mu}_{\alpha\beta} \end{bmatrix}$$

or

$$J(\theta, p) = \prod_{\alpha<\beta} |\theta_\beta - \theta_\alpha| f(p), \tag{3.1.16}$$

where $f(p)$ is independent of the θ_j and depends only on the parameters p_μ.

By inserting this result into Eq. (3.1.3) and integrating over the variables p_μ we get the j.p.d.f. for the eigenvalues of the matrices of an orthogonal ensemble

$$P(\theta_1, ..., \theta_N) = \exp\left[-\sum_1^N \left(a\theta_j^2 - b\theta_j - c\right)\right] \prod_{j<k} |\theta_k - \theta_j|, \tag{3.1.17}$$

where c is some new constant. Moreover, if we shift the origin of the θ to $b/2a$ and change the energy scale everywhere by a constant factor

$\sqrt{2a}$, we may replace θ_j with $\left(1/\sqrt{2a}\right) x_j + b/2a$. By this formal change (3.1.17) takes the simpler form

$$P_{N1}(x_1, ..., x_N) = C_{N1} \exp\left(-\frac{1}{2}\sum_1^N x_j^2\right) \prod_{j<k} |x_j - x_k|, \qquad (3.1.18)$$

where C_{N1} is a constant. (Subscript 1 is to remind of the power of the product of differences.)

3.2. Symplectic Ensemble

As the analysis is almost identical in all three invariant cases, we have presented the details for one particular ensemble, the orthogonal one. Here and in Section 3.3 and Section 3.4 we indicate briefly the modifications necessary to arrive at the required j.p.d.f. in the other cases.

Corresponding to the result that a real symmetric matrix can be diagonalized by a real orthogonal matrix, we have the following:

Theorem 3.2.1. *Given a quaternion-real, self-dual matrix H, there exists a symplectic matrix U such that*

$$H = U\Theta U^{-1} = U\Theta U^R, \qquad (3.2.1)$$

where Θ is diagonal, real, and scalar (cf. Appendix A.3).

The fact that Θ is scalar means that it consists of N blocks of the form

$$\begin{bmatrix} \theta_j & 0 \\ 0 & \theta_j \end{bmatrix} \qquad (3.2.2)$$

along the main diagonal. Thus, the eigenvalues of H consist of N equal pairs. The Hamiltonian of any system that is invariant under time reversal, has odd spin, and has no rotational symmetry satisfies the conditions of Theorem 3.2.1. All energy levels of such a system will be doubly degenerate. This is Kramer's degeneracy (Kramer, 1930), and Theorem 3.2.1 shows how it appears naturally in the quaternion language.

Apart from the N eigenvalues θ_j, the number of real independent parameters p_μ needed to characterize an $N \times N$ quaternion-real, self-dual matrix H is

$$\ell = 4 \times \frac{1}{2} N(N-1) = 2N(N-1). \tag{3.2.3}$$

Equations (3.1.2) and (3.1.3) are replaced, respectively, by

$$\operatorname{tr} H^2 = 2 \sum_{j=1}^{N} \theta_j^2, \qquad \operatorname{tr} H = 2 \sum_{j=1}^{N} \theta_j, \tag{3.2.4}$$

and

$$P(\theta_1, \ldots, \theta_N; p_1, \ldots, p_\ell) = \exp\left[-\sum_{j=1}^{N} \left(2a\theta_j^2 - 2b\theta_j - c \right) \right] J(\theta, p), \tag{3.2.5}$$

where $J(\theta, p)$ is now given by

$$J(\theta, p) = \left| \frac{\partial \left(H_{11}^{(0)}, \ldots, H_{NN}^{(0)}, H_{12}^{(0)}, \ldots, H_{12}^{(3)}, \ldots, H_{N-1,N}^{(0)}, \ldots, H_{N-1,N}^{(3)} \right)}{\partial \left(\theta_1, \ldots, \theta_N, p_1, \ldots, p_{2N(N-1)} \right)} \right|. \tag{3.2.6}$$

Equation (3.1.5) is replaced by Eq. (3.2.1); Eqs. (3.1.6)–(3.1.10) are valid if U^T is replaced by U^R. Note that these equations are now in the quaternion language, and we need to separate the four quaternion parts of the modified Eq. (3.2.1). For this we let

$$H_{jk} = H_{jk}^{(0)} + H_{jk}^{(1)} e_1 + H_{jk}^{(2)} e_2 + H_{jk}^{(3)} e_3, \tag{3.2.7}$$

$$S_{\alpha\beta}^{(\mu)} = S_{\alpha\beta}^{(0\mu)} + S_{\alpha\beta}^{(1\mu)} e_1 + S_{\alpha\beta}^{(2\mu)} e_2 + S_{\alpha\beta}^{(3\mu)} e_3, \tag{3.2.8}$$

and write Eq. (3.1.10) and the equation corresponding to Eq. (3.1.12) in the form of partitioned matrices:

$$\begin{bmatrix} \dfrac{\partial H_{jj}^{(0)}}{\partial \theta_\gamma} & \dfrac{\partial H_{jk}^{(0)}}{\partial \theta_\gamma} & \dfrac{\partial H_{jk}^{(1)}}{\partial \theta_\gamma} & \cdots & \dfrac{\partial H_{jk}^{(3)}}{\partial \theta_\gamma} \\[2ex] \dfrac{\partial H_{jj}^{(0)}}{\partial p_\mu} & \dfrac{\partial H_{jk}^{(0)}}{\partial p_\mu} & \dfrac{\partial H_{jk}^{(1)}}{\partial p_\mu} & \cdots & \dfrac{\partial H_{jk}^{(3)}}{\partial p_\mu} \end{bmatrix} \begin{bmatrix} v & w \\ A^{(0)} & B^{(0)} \\ \cdots & \cdots \\ A^{(3)} & B^{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} \rho_{\gamma,\alpha} & \sigma^{(0)}_{\gamma,\alpha\beta} & \cdots & \sigma^{(3)}_{\gamma,\alpha\beta} \\ \varepsilon^{(\mu)}_{\alpha} & S^{(0\mu)}_{\alpha\beta}(\theta_\beta - \theta_\alpha) & \cdots & S^{(3\mu)}_{\alpha\beta}(\theta_\beta - \theta_\alpha) \end{bmatrix} \tag{3.2.9}$$

$$1 \le j < k \le N, \quad 1 \le \alpha < \beta \le N, \quad 1 \le \gamma \le N, \quad 1 \le \mu \le 2N(N-1),$$

where the matrices $\partial H^{(0)}_{jj}/\partial\theta_\gamma$, v, and ρ are $N \times N$; the matrices $\partial H^{(\lambda)}_{jk}/\partial\theta_\gamma$ and $\sigma^{(\lambda)}_{\gamma,\alpha\beta}$, with $\lambda = 0, 1, 2, 3$, are $N \times N(N-1)/2$; the $A^{(\lambda)}$ are all $N(N-1)/2 \times N$; the $\partial H^{(0)}_{jj}/\partial p_\mu$ and the $\varepsilon^{(\mu)}_\alpha$ are $2N(N-1) \times N$; the w is $N \times 2N(N-1)$; the $\partial H^{(\lambda)}_{jk}/\partial p_\mu$ and the $S^{(\lambda\mu)}_{\alpha\beta}$ are $2N(N-1) \times N(N-1)/2$; and the matrices $B^{(\lambda)}$ are $N(N-1)/2 \times 2N(N-1)$. The matrices ρ and the σ appear as we separate the result of differentiation of Eq. (3.2.1) with respect to θ_γ into quaternion components. Because Θ is diagonal and scalar, the $\sigma^{(\lambda)}$ are all zero matrices. Moreover, the matrix ρ does not depend on θ_γ, for Θ depends linearly on the θ_γ. The computation of the matrices $v, w, A^{(\lambda)}$, and $B^{(\lambda)}$ is straightforward, but we do not require them. All we need is to note that they are formed of the various components of U, and hence do not depend on θ_γ.

Now we take the determinant on both sides of Eq. (3.2.9). The determinant of the first matrix on the left is the Jacobian (3.2.6). Because the $\sigma^{(\lambda)}$ are all zero, the determinant of the right-hand side breaks into a product of two determinants:

$$\det\left[\rho_{\gamma,\alpha}\right] \det\left[S^{(\lambda\mu)}_{\alpha\beta}(\theta_\beta - \theta_\alpha)\right], \tag{3.2.10}$$

the first one being independent of the θ_γ, whereas the second is

$$\prod_{\alpha<\beta} (\theta_\beta - \theta_\alpha)^4 \det\left[S^{(\lambda\mu)}_{\alpha\beta}\right]. \tag{3.2.11}$$

Thus

$$J(\theta, p) = \prod_{\alpha<\beta} (\theta_\beta - \theta_\alpha)^4 f(p), \tag{3.2.12}$$

which corresponds to Eq. (3.1.16).

By inserting Eq. (3.2.12) into Eq. (3.2.5) and integrating over the parameters, we obtain the j.p.d.f.

$$P(\theta_1, ..., \theta_N) = \exp\left(-2a\sum_{j=1}^{N}\theta_j^2 + 2b\sum_{j=1}^{N}\theta_j + c\right)\prod_{j<k}(\theta_j - \theta_k)^4. \tag{3.2.13}$$

As before, we may shift the origin to make $b = 0$ and change the scale of energy to make $a = 1$. Thus, the j.p.d.f. for the eigenvalues of matrices in the symplectic ensemble in its simple form is

$$P_{N4}(x_1, ..., x_N) = C_{N4}\exp\left(-2\sum_{j=1}^{N}x_j^2\right)\prod_{j<k}(x_j - x_k)^4, \tag{3.2.14}$$

where C_{N4} is a constant. (Subscript 4 to remind again of the power of the product of differences.)

3.3. Unitary Ensemble

In addition to the real eigenvalues, the number of real independent parameters p_μ needed to specify an arbitrary Hermitian matrix H completely is $N(N-1)$. Equations (3.1.2) and (3.1.3) remain unchanged, but Eq. (3.1.4) is replaced by

$$J(\theta, p) = \frac{\partial\left(H_{11}^{(0)}, ..., H_{NN}^{(0)}, H_{12}^{(0)}, H_{12}^{(1)}, ..., H_{N-1,N}^{(0)}, H_{N-1,N}^{(1)}\right)}{\partial\left(\theta_1, ..., \theta_N, p_1, ..., p_{N(N-1)}\right)}, \tag{3.3.1}$$

where $H_{jk}^{(0)}$ and $H_{jk}^{(1)}$ are the real and imaginary parts of H_{jk}. Equations (3.1.5) to (3.1.10) are valid if U^T replaced by $U^\dagger$. Instead of Eqs. (3.1.11) and (3.1.12) we now have

$$\sum_{j,k}\frac{\partial H_{jk}}{\partial p_\mu}U_{j\alpha}^{*}U_{k\beta} = S_{\alpha\beta}^{(\mu)}(\theta_\beta - \theta_\alpha), \tag{3.3.2}$$

$$\sum_{j,k}\frac{\partial H_{jk}}{\partial p_\gamma}U_{j\alpha}^{*}U_{k\beta} = \frac{\partial\Theta_{\alpha\beta}}{\partial\theta_\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}. \tag{3.3.3}$$

By separating the real and imaginary parts we may write these equations in partitioned matrix notation as

$$\begin{bmatrix} \dfrac{\partial H_{jj}^{(0)}}{\partial \theta_\gamma} & \dfrac{\partial H_{jk}^{(0)}}{\partial \theta_\gamma} & \dfrac{\partial H_{jk}^{(1)}}{\partial \theta_\gamma} \\ \dfrac{\partial H_{jj}^{(0)}}{\partial p_\mu} & \dfrac{\partial H_{jk}^{(0)}}{\partial p_\mu} & \dfrac{\partial H_{jk}^{(1)}}{\partial p_\mu} \end{bmatrix} \begin{bmatrix} v & w \\ A^{(0)} & B^{(0)} \\ A^{(1)} & B^{(1)} \end{bmatrix}$$
$$= \begin{bmatrix} \rho_{\gamma,\alpha} & \sigma_{\gamma,\alpha\beta}^{(0)} & \sigma_{\gamma,\alpha\beta}^{(1)} \\ \varepsilon_\alpha^{(\mu)} & S_{\alpha\beta}^{(0\mu)}(\theta_\beta - \theta_\alpha) & S_{\alpha\beta}^{(1\mu)}(\theta_\beta - \theta_\alpha) \end{bmatrix}, \quad (3.3.4)$$

$$1 \le j < k \le N, \quad 1 \le \alpha < \beta \le N,$$

$$1 \le \mu \le N(N-1), \quad 1 \le \gamma \le N.$$

where $S_{\alpha\beta}^{(0\mu)}$ and $S_{\alpha\beta}^{(1\mu)}$ are the real and imaginary parts of $S_{\alpha\beta}^{(\mu)}$. The matrices $\partial H_{jj}^{(0)}/\partial\theta_\gamma, v$, and ρ are $N \times N$; the $\partial H_{jk}^{(\lambda)}/\partial\theta_\gamma$ and the $\sigma_{\gamma,\alpha\beta}^{(\lambda)}$ are $N \times N(N-1)/2$; the $A^{(\lambda)}$ are $N(N-1)/2 \times N$; the $\partial H_{jk}^{(\lambda)}/\partial p_\mu$ and $S_{\alpha\beta}^{(\lambda\mu)}$ are $N(N-1) \times N(N-1)/2$; the $B^{(\lambda)}$ are $N(N-1)/2 \times N(N-1)$; the $\partial H_{jj}^{(0)}/\partial p_\mu$ and the $\varepsilon_\alpha^{(\mu)}$ are $N(N-1) \times N$; and the matrix w is $N \times N(N-1)$. To compute $v, w, A^{(\lambda)}, \rho, \varepsilon, \sigma^{(\lambda)}$, etc., is again straightforward, but we do not need them explicitly. What we want to emphasize is that they are either constructed from the components of U or arise from the differentiation of Θ with respect to θ_j and consequently are all independent of the eigenvalues θ_j. Similarly, $S^{(\mu)}$ is independent of θ_j. One more bit of information we need is that $\sigma^{(0)}$ and $\sigma^{(1)}$ are zero matrices, which can easily be verified.

Thus, by taking the determinants on both sides of Eq. (3.3.4) and removing the factors $(\theta_\beta - \theta_\alpha)$ we have

$$J(\theta, p) = \prod_{\alpha<\beta} (\theta_\beta - \theta_\alpha)^2 f(p), \quad (3.3.5)$$

where $f(p)$ is some function of the p_μ.

By inserting Eq. (3.3.5) into Eq. (3.1.3) and integrating over the parameters p_μ we get the j.p.d.f. for the eigenvalues of matrices in the unitary ensemble

$$P(\theta_1, ..., \theta_N) = \exp\left(-a\sum_1^N \theta_j^2 + b\sum_1^N \theta_j + c\right)\prod_{j<k}(\theta_j - \theta_k)^2, \quad (3.3.6)$$

and, as before, by a proper choice of the origin and the scale of energy we have

$$P_{N2}(x_1, ..., x_N) = C_{N2}\exp\left(-\sum_1^N x_j^2\right)\prod_{j<k}(x_j - x_k)^2. \quad (3.3.7)$$

We record Eqs. (3.1.18), (3.2.14), and (3.3.7) as a theorem.

Theorem 3.3.1. *The joint probability density function for the eigenvalues of matrices from a Gaussian orthogonal, Gaussian symplectic, or Gaussian unitary ensemble is given by*

$$P_{N\beta}(x_1, ..., x_N) = C_{N\beta}\exp\left(-\frac{1}{2}\beta\sum_1^N x_j^2\right)\prod_{j<k}|x_j - x_k|^\beta, \quad (3.3.8)$$

where $\beta = 1$ *if the ensemble is orthogonal,* $\beta = 4$ *if it is symplectic, and* $\beta = 2$ *if it is unitary. The constant* $C_{N\beta}$ *is chosen in such a way that the* $P_{N\beta}$ *is normalized to unity:*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} P_{N\beta}(x_1, ..., x_N)\, dx_1 \cdots dx_N = 1. \quad (3.3.9)$$

According to Selberg the normalization constant $C_{N\beta}$ is given by (see Chapter 17)

$$C_{N\beta}^{-1} = (2\pi)^{N/2}\beta^{-N/2-\beta N(N-1)/4}\left[\Gamma(1+\beta/2)\right]^{-N}\prod_{j=1}^N \Gamma(1+\beta j/2). \quad (3.3.10)$$

For the physically interesting cases $\beta = 1, 2$, and 4 we will recalculate this value in a different way later (see Sections 5.2, 6.4 and 7.2).

It is possible to understand the different powers β that appear in Eq. (3.3.8) by a simple mathematical argument based on counting dimensions. The dimension of the space T_{1G} is $N(N+1)/2$, whereas the dimension of the subspace T'_{1G}, composed of the matrices in T_{1G} with two equal eigenvalues, is $N(N+1)/2-2$. Because of the single restriction, the equality of two eigenvalues, the dimension should normally have decreased by one; as it is decreased by two, it indicates a factor in Eq. (3.3.8) linear in $(x_j - x_k)$. Similarly, when $\beta = 2$, the dimension of T_{2G} is N^2, whereas that of T'_{2G} is $N^2 - 3$. When $\beta = 4$, the dimension of T_{4G} is $N(2N-1)$, whereas that T'_{4G} is $N(2N-1)-5$ (see Appendix A.4).

3.4. Ensemble of Antisymmetric Hermitian Matrices

The eigenvalues and eigenvectors of anti-symmetric Hermitian matrices come in pairs; if θ is an eigenvalue with the eigenvector V_θ, then $-\theta$ is an eigenvalue with the eigenvector V_θ^*. The vectors V_θ and V_θ^* can be normalized, and if $\theta \neq 0$ they are orthogonal. Thus if $V_\theta = \xi^{(\theta)} + i\eta^{(\theta)}$, $\xi^{(\theta)}$ and $\eta^{(\theta)}$ are real, then

$$(V_\theta, V_\theta^*) = \sum_j \left(\xi_j^{(\theta)} + i\eta_j^{(\theta)}\right)^2 = 0, \tag{3.4.1}$$

$$(V_\theta, V_\theta) = \sum_j \left|\xi_j^{(\theta)} + i\eta_j^{(\theta)}\right|^2 = 2. \tag{3.4.2}$$

These two equations are equivalent to

$$\sum_j \left(\xi_j^{(\theta)}\right)^2 = \sum_j \left(\eta_j^{(\theta)}\right)^2 = 1, \quad \sum_j \xi_j^{(\theta)}\eta_j^{(\theta)} = 0. \tag{3.4.3}$$

Therefore, if $\theta \neq 0$, the real and imaginary parts of V_θ have the same length and are orthogonal to each other. Moreover, the real and imaginary parts of V_θ are each orthogonal to the real and imaginary parts of V_λ if $\theta \neq \lambda$. If the matrices are of odd order, $2N+1$, then zero is an additional eigenvalue with an essentially real eigenvector V_0 orthogonal to all the ξ and η. As before, it is not necessary to consider the case in which two or more eigenvalues coincide.

With this information on the matrix diagonalizing H, we can derive the j.p.d.f. for the eigenvalues. Denoting the positive eigenvalues of H by $\theta_1, ..., \theta_n$, one has

$$\text{tr } H^2 = \sum_{j=1}^{n} 2\theta_j^2.$$

The Jacobian will again be a product of the differences of all pairs of eigenvalues and of a function independent of them. Thus the j.p.d.f. for the eigenvalues will be proportional to

$$\prod_{1\leq j<k\leq n} \left(\theta_j^2 - \theta_k^2\right)^2 \exp\left(-2\sum_{j=1}^{n} \theta_j^2\right), \tag{3.4.4}$$

if the order N of H is even, $N = 2n$, and

$$\prod_{j=1}^{n} \theta_j^2 \prod_{1\leq j<k\leq n} \left(\theta_j^2 - \theta_k^2\right)^2 \exp\left(-2\sum_{j=1}^{n} \theta_j^2\right), \tag{3.4.5}$$

if $N = 2n + 1$ is odd. The constants of normalization will be calculated in chapter 13.

3.5. Another Gaussian Ensemble of Hermitian Matrices

To derive the j.p.d.f. for the eigenvalues is a little tricky in this case. This is so because the matrix element probability densities depend on the eigenvalues and the angular variables characterizing the eigenvectors; and one has to integrate over these angular variables. When either $c_1 = 0$ or $c_2 = 0$ or $c_1 = c_2$, the matrix element probability densities depend only on the eigenvalues. Also the Jacobian separates into a product of two functions, one involving only the eigenvalues and the other only the eigenvectors; therefore the integral over the eigenvectors, giving only a constant need not be calculated. For arbitrary c_1 and c_2 this simplification is not there. In view of these difficulties we will come back to this question in Chapter 14, when we are better prepared with the method of integration over alternate variables (Section 6.5), with quaternion determinants (Section 6.2) and the integral over the unitary group

$$\int \exp \left(\mathrm{tr}\,(A - U^{\dagger}BU)^2\right) dU$$
$$= \text{const} \times \det\left(\exp\,(a_j - b_k)^2\right) \prod_{j<k} \left[(a_j - a_k)\,(b_j - b_k)\right]^{-1}, \tag{3.5.1}$$

valid for arbitrary Hermitian matrices A and B having eigenvalues $a_1, ..., a_n$ and $b_1, ..., b_n$, respectively (see Appendix A.5).

3.6. Random Matrices and Information Theory

A reasonable specification of the probability density $P(H)$ for the random matrix H can be supplied from a different point of view, which is more satisfactory in some ways. Let us define the amount of information $\mathcal{I}(P(H))$ carried by the probability density $P(H)$. For discrete events $1, ..., m$, with probabilities $p_1, ..., p_m$, additivity and continuity of the information fixes it uniquely, apart from a constant multiplicative factor, to be (Shanon 1948; Khinchin 1957)

$$\mathcal{I} = -\sum_j p_j \ln p_j. \tag{3.6.1}$$

For continuous variables entering H it is reasonable to write

$$\mathcal{I}(P(H)) = -\int dH\; P(H) \ln P(H), \tag{3.6.2}$$

where dH is given by Eq. (2.5.1). One may now adopt the point of view that H should be as random as possible and compatible with the constraints it must satisfy. In other words, among all possible probabilities $P(H)$ of a matrix H constrained to satisfy some given properties, we must choose the one which minimizes the information $\mathcal{I}(P(H))$; $P(H)$ must not carry more information than what is required by the constraints. The constraints may, for example, be the fixed expectation values k_i of some functions $f_i(H)$,

$$\langle f_i(H)\rangle \equiv \int dH\ P(H)\ f_i(H) = k_i. \tag{3.6.3}$$

To minimize the information in Eq. (3.6.2) subject to Eq. (3.6.3) one may use Lagrange multipliers. This gives us for an arbitrary variation $\delta P(H)$ of $P(H)$,

$$\int dH\ \delta P(H) \left(1 + \ln\, P(H) - \sum_i \lambda_i f_i(H)\right) = 0 \tag{3.6.4}$$

or

$$P(H) \propto \exp\left(\sum_i \lambda_i f_i(H)\right), \tag{3.6.5}$$

and the Lagrange multipliers λ_i are then determined from Eqs. (3.6.3) and (3.6.5).

For example, requiring H Hermitian, $H = H_1 + iH_2$, H_1, H_2 real, and

$$\left\langle \text{tr}\ H_1^2 \right\rangle = k_1, \quad \left\langle \text{tr}\ H_2^2 \right\rangle = k_2,$$

will give us Eq. (2.7.1) for $P(H)$.

Another example is to require the level density to be a given function $\sigma(x)$

$$\langle \text{tr}\ \delta(H - x)\rangle = \sigma(x). \tag{3.6.6}$$

The Dirac delta function $\delta(H - x)$ is defined through the diagonalization of H, $\delta(H - x) = U^\dagger \delta(\theta - x) U$, if $H = U^\dagger \theta U$, U unitary, θ [or $\delta(\theta - x)$] diagonal real with diagonal elements θ_i [or $\delta(\theta_i - x)$]. This gives then

$$P(H) \propto \exp\left(\int \mathrm{d}x\ \lambda(x)\ \text{tr}\ \delta(H - x)\right) = \exp[\text{tr}\ \lambda(H)] = \det\left[\mu(H)\right]; \tag{3.6.7}$$

the Lagrange multipliers $\lambda(x) \equiv \ln\ \mu(x)$ are then determined by Eq. (3.6.6). Thus one may determine a $P(H)$ giving a preassigned level density. For more details see Balian (1968).

It is important to note that if $P(H)$ depends only on the traces of powers of H, then the joint probability density of the eigenvalues will contain the factor $\prod |\theta_j - \theta_k|^\beta$, $\beta = 1$, 2, or 4; coming from the Jacobian of the elements of H with respect to its eigenvalues. And the local fluctuation properties are mostly governed by this factor.

Summary of Chapter 3

For the random Hermitian matrices H considered in Chapter 2, the joint probability density of its real eigenvalues $x_1, x_2, ..., x_N$ is derived. It turns out to be proportional to

$$\prod_{1 \le j < k \le N} |x_j - x_k|^\beta \exp\left(-\beta \sum_{j=1}^{N} x_j^2/2\right), \tag{3.3.8}$$

where $\beta = 1,\ 2$, or 4 according as the ensemble is Gaussian orthogonal, Gaussian unitary, or Gaussian symplectic.

For the Gaussian ensemble of Hermitian antisymmetric random matrices the joint probability density of the eigenvalues $\pm x_1, ..., \pm x_m$, with $x_j \ge 0$, is proportional to

$$(x_1 \cdots x_m)^\gamma \prod_{1 \le j < k \le m} \left(x_j^2 - x_k^2\right)^2 \exp\left(-\sum_{j=1}^{m} x_j^2\right),$$

where $\gamma = 0$ if the order N of the matrices is even, $N = 2m$, and $\gamma = 2$ if this order is odd, $N = 2m + 1$. In the $N = 2m + 1$, odd case, zero is always an eigenvalue.

The noninvariant ensemble of Hermitian random matrices being more complicated is taken up later in Chapter 14.

An additional justification of our choices of the ensembles is supplied from an argument of the information theory.