QUESTIONS AND CONJECTURES POSED AT THE PROBLEM-SOLVING SEMINAR IN COMMUTATIVE ALGEBRA, STOCKHOLM UNIVERSITY

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Abstract. This is a large collection of problems related to the Waring problem for polynomials, ABC-conjecture for polynomials, superFermat problem over functional fields, Fröberg's conjecture etc. All of them were formulated by the participants of the problem-solving seminar at Stockholm university or their collaborators. If you want to use this material in your own research, please, contact the participants of the seminar at the e-mail address: problem-solving@math.su.se for the permission.

1. The Waring problem for polynomials

1.1. **Generic rank.** The celebrated Theorem of Alexander-Hirschowitz completely describes the generic rank $r_{gen}(d,n)$ of degree d forms in $n+1$ variables. i.e., the minimal number of terms necessary to represent a general form of degree d as a sum of $r_{gen}(d,n)$ d-th powers of linear forms in $n+1$ variables.

A somewhat simplified version of this result is as follows.

Theorem 1. *For all pairs (d,n), except for the series $(2,n)$, and 4 special exceptions: $d=4,\ n=2,3,4$; and $d=3, n=4$,*

$$r_{gen}(d,n)=\left\lceil \frac{\binom{n+d}{n}}{n+1}\right\rceil, \tag{1}$$

In all exceptional cases $r_{gen}(d,n)$ equals the r.h.s of (1) increased by 1.

In 2012 together with G. Ottaviani, we obtained first important results about the general version of the Waring problem for polynomials which is closer to the original Waring problem over $\mathbb{Z}$, see [8]. Namely, the natural general set-up is as follows. Given $k\geq 2$, $d\geq 1$, and $n\geq 1$, define the generic rank $r_{gen}(k,d,n)$ as the minimal number of terms such that a generic form of degree kd in $n+1$ variables can be represented as the sum of $r_{gen}(k,d,n)$ k-th powers of forms of degree k. In this notation, $r_{gen}(d,n)=r_{gen}(d,1,n)$. The main result of [8] states that, for any triple (k,d,n) as above,

$$r_{gen}(k,d,n)\leq k^n. \tag{2}$$

Moreover, for any fixed $n\geq 1$ and $k\geq 2$, there exists a positive integer $d_{k,n}$ such that $r_{gen}(k,d,n)=k^n$, for all $d\geq d_n$.

Recently G. Ottaviani has formulated a beautiful conjecture about $r_{gen}(k,d,n)$ (oral communication).

Conjecture 1.

$$r_{gen}(k,d,n) = \begin{cases} \min\{s \geq 1 | s\binom{n+d}{n} - \binom{s}{2} \geq \binom{n+2d}{n}, \; \textit{for } k=2; \\ \min\{s \geq 1 | s\binom{n+d}{n} \geq \binom{n+kd}{n}, \; \textit{for } k \geq 3. \end{cases} \tag{3}$$

Observe that, for $k \geq 3$,

$$r_{gen}(k,d,n) = \left\lceil \frac{\binom{n+kd}{n}}{\binom{n+d}{d}} \right\rceil.$$

One can show that $r_{gen}(k,d,n) \leq$ r.h.s of (3). Additionally (3) gives the correct answer in all special cases when $r_{gen}(k,d,n)$ is known explicitly, see [19]. Observe that Conjecture 1 will follow from a certain special case of Fröberg's conjecture given below, see [9].

The following addiional problem is inspired by [8].

1. Determine $d_{k,n}$ such that k^n is the correct answer if $d \geq d_{k,n}$?

1.2. Maximum rank.

Definition 1. Given a triple (k,d,n) as above, denote by $r_{max}(k,d,n)$ as the minimal number of terms such that any form of degree kd in $n+1$ variables can be represented as the sum of at most $r_{max}(k,d,n)$ k-th powers of forms of degree k. $r_{max}(k,d,n)$ is called the *maximum rank.*

The best known results about maximum rank in different situations are collected in [2]. In particular, they show that $r_{max}(k,d,n) \leq 2r_{gen}(k,d,n)$. Our conjectures below concern only the case of binary forms. The most famous result on binary forms was proven by J. J. Sylvester in 1851 and states the following.

Theorem 2 (Sylvester's Theorem [22])**.** (i) A general binary form p of odd degree $k=2s-1$ with complex coefficients can be written as

$$p(x,y) = \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^k.$$

(ii) A general binary form p of even degree $k=2s$ with complex coefficients can be written as

$$p(x,y) = \lambda x^k + \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^k$$

for some $\lambda \in \mathbb{C}$.

We can rephrase Sylvester's Theorem in terms of Waring rank.

Definition 2. Given a binary form p of degree k and complex coefficients, the *Waring rank* of p is the minimal number s of linear forms $l_1, \ldots, l_s$ such that we can write $p = l_1^k + \ldots + l_s^k$. We denote it as $\mathrm{rk}(p)$.

Thus, Sylvester's Theorem states that the Waring rank of a general binary form of degree $k = 2s-1$ equals s, while the Waring rank of a general binary form of degree $k = 2s$ equals $s+1$.

Additionally Theorem 5.4. of [20] (which might have been known earlier) by B. Reznick claims that for $k \geq 3$, the maximal Waring rank of binary polynomials of degree k equals k and this maximal value is attained exactly on binary forms representable as $l_1 l_2^{k-1}$, where l_1 and l_2 are distinct linear forms.

Our next goal is to extend Sylvester's theorem to sums of powers of forms of higher degree. The question under consideration is as follows.

Definition 3. Given a binary form p of degree kd, with $k \geq 2,\ d \geq 1$, and complex coefficients, the k^{th}- *Waring rank* of p is the minimal number s of degree d forms $g_1, \ldots, g_s$ such that we can write $p = g_1^k + \ldots + g_s^k$. We denote it as $\mathrm{rk}_k(p)$.

Conjecture 2. *For any $k \geq 2$ any d, and any binary form p of degree kd, $rk_k(p) \leq k$. Additionally, $rk_k(xy^{kd-1}) = k$.*

1.3. **Degree of the Waring map.** Here again we will concentrate on the case $n = 1$ (i.e., binary forms) in which it is proven that $r_{gen}(k,d,1) = \left\lceil \frac{\dim S^{kd}(1)}{\dim S^d(1)} \right\rceil = \left\lceil \frac{kd+1}{d+1} \right\rceil$, see [20].

Definition 4. We say that the pair (k,d) is perfect if $\frac{kd+1}{d+1}$ is an integer.

It turns out that all perfect pairs are easy to describe.

Lemma 1. *The set of all pair (k,d) for which $\frac{kd+1}{d+1}$ is an integer splits into a series of disjoint sequences $E_j,\ j = 1, 2, \ldots.$ Here E_j consists of all pairs of the form $(jd+j+1, d),\ d = 1, 2, \ldots.$ The corresponding value of s equals $s(j,d) = jd+1$.*

For perfect pairs (k,d), the following question is of substantial interest and difficulty. Given a perfect pair (k,d), denote by $s := \frac{kd+1}{d+1}$. Consider the Waring map

$$W_{k,d} : \oplus_{j=1}^{s} S_j^d \to S^{kd},$$

sending an s-tuple of binary forms of degree d to the sum of their k-th powers. Here S^m stands for the linear space of binary forms of degree m. By the above result on the generic rank, $W_{k,d}$ is a generically finite map of complex linear spaces of the same dimension. Call be its *degree* the cardinality of the inverse image of a generic form in S^{kd}.

Problem 1. Calculate the degree of $W_{k,d}$ for perfect pairs (k,d).

This problem is apparently related to the topic of superFermat equations discussed below.

2. Fröberg's conjecture

In [7] R. Fröberg formulated the following general conjecture.

Conjecture 3. *Let J be a homogeneous ideal in $S = \mathbb{C}[x_0, \dots, x_{n+1}]$ generated by generic forms $f_1, \dots, f_s$ of degrees $d_1, \dots, d_s$ resp. The the Hilbert series of the quotient ring S/J is given by:*

$$HS_{S/J}(t) = \left\lceil \frac{\prod_{j=1}^{s}(1 - t^{d_j})}{(1-t)^{n+1}} \right\rceil, \tag{4}$$

where $\lceil \sum_j a_j t^j \rceil := \sum_j b_j t^j$ where $b_j := a_j$, if $a_i > 0$ for all $i \leq j$, and $b_j := 0$ otherwise.

Observe that [7] contains the proof of the important inequality:

$$HS_{S/J}(t) \geq \left\lceil \frac{\prod_{j=1}^{s}(1 - t^{d_j})}{(1-t)^{n+1}} \right\rceil,$$

which should be understood lexicographically. We say a homegeneous ideal J is *Hilbert-generic* if the Hilbert series of S/J satisfies (4).

Below we formulate a large number questions related to Fröberg's conjecture.

Question 1. What is the Hilbert series of $\mathbb{C}[x_1, \dots, x_n]/I$, I generated by generic forms of degree d? In other words, try to settle Fröberg's conjecture in case when all forms have the same degree.

The main question is if there are Hilbert-generic ideals for all triples (d, n, s). (Equivalently, is the generic ideal Hilbert-generic?) It suffices to get one example of a Hilbert-generic ideal for fixed (d, n, s) to prove Fröberg's conjecture for these values. The best results so far is Anick's ($n = 3$ any d and s), Stanley's (any d, $s = n + 1$), and a very technical result by Aubry. Chandler, Miro Roig, and Iarrobino (among others) have written about this. Iarrobino also discusses when an ideal generated by powers of linear forms can be Hilbert-generic. Iarrobino-Fröberg conjecture.

Special case. Computer calculations show that the ideal generated by $(x_1 \pm x_2 \pm x_3)^d$ is a new example for $n = 3$. In all examples we have tried, also $(l_1^d, \dots, l_s)^d)$, $s = 2^{n-1}$, where the l_i's are linear forms which are sums of an odd number of variables, are Hilbert-genereic.

Question 2. Do k-th powers of generic forms of degree $d > 1$ have the same Hilbert series as generic forms of degree kd?

Reference: [18]

Question 3. Is it true that $\mathbb{C}[x_0, \dots, x_n]/(l_1^d, \dots, l_k^d)$ (l_i generic linear forms) has the same Hilbert series as $\mathbb{C}[x_0, \dots, x_n]/(f_1, \dots, f_k)$ (f_i generic forms of degree d) if $d >> 0$? (Are both Hilbert-generic?) Can we give a bound?

Question 4. There are some exceptions for the equality of Hilbert series in Question 3. In other words, for some d, d-th powers of linear forms are not generic enough. If d is the degree, $n+1$ the number of variables, and s the number of generators we "know" that the following are exceptions: $(d, n+1, s) = (2, 5, 7), (2, 7, 9), (2, 8, 10), (2, 9, 11), (2, 10, 12), (2, 11, 13), (2, 11, 14), (3, 3, 5), (3, 4, 9), (3, 5, 7), (3, 5, 14), (3, 6, 8), (3, 6, 9), (3, 7, 9), (3, 8, 10)$. Is there a geometric explanation of these exceptions?

Question 5. Is the set of exceptions finite ? Is it finite for each n? Is it finite for each d? Are they the same if $d - n >> 0$?

Question 6. In a few cases, the Hilbert series differ already in degree $d+1$ (there are linear syzygies). There a geometric explanation (Alessandro). Is there a finite set of counterexamples in degree $d+2$?

Symbolic powers ideal. DEFINITION! Let I be an ideal of points. The symbolic power $I^{(k)}$ is a very natural geometric construction. If consists of the polynomials which vanish up to order $k-1$ in the points. On the other hand I^k is a very natural algebraic construction. We have $I^k \subseteq I^{(k)}$. (In fact, if P is a prime, then $P^{(k)}$ is the smallest P-primary ideal containing P^k.)

Question 7. For which ideals, the symbolic power coincides with the usual power? This is a difficult problem, see e.g. [14] and references therein.

Question 8. Consider Question 7 for ideals generated by powers of linear forms.

Question 9. It is proved that symbolic powers coincide with the usual powers for ideals generated by $(x_1 \pm x_2 \pm \cdots \pm x_n)^d$ (and even with the k^n forms with k'th roots of unity as coefficients) [1] but probably not for power of linear forms generated by sums of an odd number of variables if $n > 3$. Could this be proved? For which sets of points are they equal?

Question 10. Take $n+1$ generic forms of degree d in n variables.

Conjecture. a) The highest nonzero degree in the factor ring is $\left[\frac{(n+1)(d-1)}{2}\right]$.

b) The total degree is (for odd n) equals the largest coefficient in $1+t+t^2+\cdots+t^{d-1}$ (conjecture).

Question 11. Calculate the Hilbert series for our PNAS system for $k=2$ and for $k>2$. What about its symbolic powers?

Question 12. Is the polynomial $\left\lceil \frac{(1-z^d)^s}{(1-z)^n} \right\rceil$ unimodal?

Question 13. If the conjectured Hilbert series for generic forms is correct for s forms of degree d in $n+1$ variables, is it correct for $s-1$ forms of degree d in $n+1$ variables? For s forms of degree d in n variables?

3. ABC-conjecture for polynomials and superFermat problem over functional fields

3.1. **ABC-conjecture.** The famous ABC-conjecture in number theory is stated as follows.

Conjecture 4. *Let $a, b, c \in \mathbb{N}$ be relatively prime natural numbers, and $a + b = c$. Let d denote the product of all prime factors of abc. Then, usually $c < d$. More precisely, for every $\epsilon > 0$, there exist at most finitely many $(a, b, c) \in \mathbb{N}^3$ such that*

$$c > d^{1+\epsilon}.$$

Although the original ABC-conjecture is at present widely open, it has a reasonable solution in the case of polynomials in one variable, see [**?**] and [**?**].

Theorem 3 (Mason-Stothers theorem)**.** *Let K be an algebraically closed field of characteristic zero. Let $e, f, g \in K[X]$ be relatively prime polynomials satisfying*

$$e(X) + f(X) = g(X). \tag{5}$$

Then,

$$h \leq \deg rad(efg) - 1,$$

where $h = \max(\deg e, \deg f, \deg g)$. Moreover, equality occurs precisely when f/g is a Belyi map.

Here $\mathrm{rad}(p)$ means the radical of a univariate polynomial p, i.e. the polynomial containing each distinct root of p with multiplicity 1.

There are some recently generalizations of Mason-Stothers theorem to the case when one adds more polynomials in the left-hand side of (5) and also consider multivariate polynomials instead of univariate, see e.g., [**?**], [2], [3]. But the obtained results seem to be quite far from optimal.

3.2. **superFermat Problem.** See [11]. Let $\mathcal{C}$ be a semi-ring and let k be an integer satisfying $k \geq 2$. We let $F_{\mathcal{C}}(k)$ denote the smallest positive integer l, such that we have a non-trivial representation

$$y_1^k + y_2^k + \cdots + y_l^k = y^k, \text{where } y, y_1, \ldots, y_l \in \mathcal{C}. \tag{6}$$

The notion "non-trivial" should be clarified for each specific $\mathcal{C}$, see example below.

Next let $W\mathcal{C}(k)$ be the smallest number l, such that every $y \in \mathcal{C}$ can be expressed in the form,

$$y = y_1^k + y_2^k + \ldots + y_l^k. \tag{7}$$

It is of some interest to discuss $F\mathcal{C}(k)$ and $W\mathcal{C}(k)$ for some classes of non-constant analytic functions in the plane. Let us denote by M, R, E, P the rings of meromorphic, rational, entire functions in the complex plane and univariate polynomials, respectively. The corresponding questions were treated in [10],[12] and the results were more satisfactory than those for the more difficult problem of Z. We quote Theorem 4.1, p.439, [10].

Theorem 4. *If $k \geq 2$,*

$$F_M(k) \geq \sqrt{k+1}, \text{ and } F_R(k) > \sqrt{k+1},$$

$$F_E(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}} \text{ and } F_P(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}.$$

For W, it is in all cases enough to represent the identity function z, see [10] [p.442] The results for W are almost identical to those of F.

Theorem 5. *If $n \geq 2$,*

$$W_M(k) \geq \sqrt{(k+1)}, \text{ and } W_R(k) > \sqrt{(k+1)},$$

$$W_E(k) \geq \frac{1}{2} + \sqrt{\left(k + \frac{1}{4}\right)}.$$

Further, if $k \geq 3$,

$$W_P(k) > \frac{1}{2} + \sqrt{\left(k + \frac{1}{4}\right)}.$$

We also have $WP(2) = 2$, since

$$\left(\frac{1}{2}(z+1)\right)^2 + \left(\frac{1}{2}i(z-1)\right)^2 = z.$$

Observe that $F_P(2) = 3$. It turns out that for $\mathcal{C} = E, R$ or M we have

$$F_{\mathcal{C}}(k) \leq W_{\mathcal{C}}(k), \quad k \geq 2,$$

see Lemma 5.2, p 442, [10].

It is an open question, whether

$$F_P(k) \leq W_P(k), \quad k > 2.$$

What about upper bounds?

By [17],

$$F_P(k) \leq \sqrt{(4k+1)},$$

with the same bound for the larger classes R, E and M.

Reference: [11], [10], [12] , [17].

3.3. **SuperFermat problem for multivariate forms.** Consider the superFermat equation

$$y_1^k + y_2^k + \cdots + y_l^k = 0 \tag{8}$$

for which we look for solutions in homogeneous forms of a given degree d in $n+1$ homogeneous variables $(x_0, x_1, \ldots, x_n)$. We say that a solution $(p_1, \ldots, p_l)$ of (8) in homogeneous forms is *very trivial* if all forms p_j, $j = 1, \ldots, l$ are proportional to one and the same non-zero form. The set of all very trivial solutions in forms of degree d is a product of the projective space of all non-zero forms of degree d (considered up to a scalar factor) times the hypersurface of all solutions of (8) in complex numbers. We say that a solution $(p_1, \ldots, p_l)$ of (8) in homogeneous forms is *trivial* if it can be partitioned in the union of very trivial solutions for smaller values of l. The remaining solutions of (8) are called *non-trivial*.

The next lemma describes the dimension of the set of trivial solutions.

Lemma 2. *Except for some trivial cases,*

$$\dim(ST) = \begin{cases} \frac{l}{2}\binom{n+d}{n}, & \text{for even } l; \\ \frac{l-1}{2}\binom{n+d}{n} + 1, & \text{for odd } l. \end{cases}$$

Conjecture 5. *Non-trivial solutions of* (8) *in forms of degree d in $n+1$ homogeneous variables exists if and only if $l\binom{n+d}{n} - \dim ST > \binom{n+kd}{n}$.*

For $l = 3$, $n = 1$ and any $k \geq 2$, Conjecture 5 follows from a result of Liouville, see [21], p. 263.

4. Miscellanea

4.1. **Hilbert series of numerical semigroup rings.** Let $S = \langle s_1, \dots, s_k \rangle$ be a numerical semigroup, i.e. S consists of all linear combinations with non-negative integer coefficients of the positive integers s_i, and let $k[x^{s_1}, \dots, x^{s_k}] = k[S]$ be the semigroup ring. The Hilbert series of $k[S]$ is of the form $p(t)/q(t)$, p, q polynomials with integer coefficients. A semigroup is called cyclotomic if the polynomial $p(t)$ has all its roots in the unit circle (then in fact on the unit circle).

Conjecture 6. *S is cyclotomic if and only if $k[S]$ is a complete intersection.*

Reference: [6]

4.2. **Positive semidefinite real forms.** By positive semidefinite real forms we mean non-negative forms.

Let $P_{n,m}$ be the set of positive semidefinite real forms of (an even) degree m in n variables, $\Sigma_{n.m} \subseteq P_{n,m}$ those which are sums of squares of real forms of half the degree. $\Delta_{n,m} = P_{n,m} \setminus \Sigma_{m,n}$. Let $\mathcal{Z}(p)$ denote the real zero set of p. Let $B_{n,m}$ (resp. $B'_{n,m}$) be $\sup|\mathcal{Z}(p)|$, $|\mathcal{Z}(p)| < \infty$, $p \in P_{n,m}$ (resp. $p \in \Sigma_{n,m}$).

Question 1. Is $B_{n,m} < \infty$? Is $B'_{n,m} < \infty$? (True for $(n,m) = (3,m)$ and for (4,4).)
Question 2. What is $\lim_{m\to\infty} B_{3,m}/m^2$? (The limit exists.)

Question 3. Is $B_{4,4} = 10$ or 11?

Question 4. Is $B'_{4,4} = 8$ or 9 or 10?

Reference: [5]

4.3. **Perp ideals and inverse systems.** For a given form $F \in \mathbb{C}[x_0, \dots, x_n]$, denote by $F^\perp$ is an ideal in $\mathbb{C}[y_0, \dots, y_n]$ consisting of all forms $\Phi[y_0, \dots, y_n]$ such that

$$\Phi\left[\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n}\right] \circ F$$

vanishes. $F^\perp$ is called the *perp ideal* of F. DEFINITION OF THE INVERSE SYSTEM

Question 1. For which F is $F^\perp$ a complete intersection?

Question 2. For which F is $F^\perp$ generated in degree 2?

Question 3. What is the perp ideal of a generic form?

Question 4. Are there, besides monomial complete intersections and ideals of fat points, any class of ideals for which the inverse system can be easily described?

4.4. **Lefschetz properties of graded algebras.** A graded algebra A has the weak (resp. strong) Lefschetz property if $\times l : A_i \to A_{i+1}$ (resp. $\times l^k : A_i \to A_{i+k}$) is of maximal rank (i.e. injective or surjective) for a generic linear form l and all i (resp. for all i and k). Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of d (so $\sum_{i=1}^k \mu_i = d$). An algebra has the μ-Lefschetz property if $\times \mathbf{l}^\mu : A_i \to A_{i+d}$ has maximal rank for all i, where $\mathbf{l}^\mu = l_1^{\mu_1} \cdots l_k^{\mu_k}$, l_i generic linear forms.

Question 1. Has, for any partition $\mu \neq (d)$ the ideal $(\mathbf{l}_1^\mu, \dots, \mathbf{l}_k^\mu)$ the same Hilbert series as k generic forms of degree k?

Question 2. Does a general ideal satisfy the μ-Lefschetz property for all μ?

Question 3. Is there, for each $\mu \neq (d)$ an algebra which satisfy μ-Lefschetz but not weak Lefschetz? (Alessandro Oneto has an example for $\mu = (1,1,1)$.)

References: [15], [4].

4.5. **Exterior algebras.** Let f be a generic form of odd degree $d > 3$ in the exterior algebra E_n with n generators. Then $(f) \subseteq \mathrm{Ann} f$. Let $h(t)$ be the difference between the Hilbert series of $\mathrm{Ann}(f)$ and (f).

Problem 1. Let $n - d = 0, 1, 3, 4, 5$. Is $h(t) = 0$?

Problem 2. Let $n - d = 2$. Is $h(t) = t$?

Problem 3. Let $n - d = 6$. Is $h(t) = t^3$ if $d = 1 \pmod 4$ and $h(t) = 0$ otherwise?

Problem 4. Is, for fixed d, $\mathrm{Ann} f = (f)$ if $n >> 0$?

Let f_1, f_2 be generic of degree 2 in E_n. Let $[(1-t^2)^2(1+t)^n]$ be the "expected" Hilbert series for $E_n/(f_1, f_2)$. (The brackets means "take only terms as long as they are positive".)

Problem 5. Is the correct series equal to the expected in degrees $< [(n+1)/2]$?
For these problems see [16]

4.6. **Symbolic powers.** If P is a prime ideal in a Noetherian ring R, then P^n is not necessarily primary.

Definition The n'th symbolic power of P is $P^{(n)} = PR_P \cap R$ (the P-primary component of P^n). For any ideal I, $I^{(n)} = \cap_{P \in \mathrm{Ass}(I)} (I^n R_P \cap R) = \cap_{P \in \mathrm{Min}(I)} (I^n R_P \cap R)$.

It is clear that $I^n \subseteq I^{(n)}$.

Theorem (Ein-Lazarsfeld-Smith) and (Hochster-Huneke). If I is homogeneous in $\mathbb{C}[x_0, \dots, x_N]$, then $I^{(rN)} \subseteq I^r$.

From now on, let I be an ideal of points $Z = \{p_1, \dots, p_s\} \subset P^N(\mathbb{C})$, and let $S = \mathbb{C}[x_0, \dots, x_N]$. Thus $I(Z) = \cap_{i=1}^s I(p_i)$, where $I(p_i)$ is the prime ideal generated by N

linear forms. Then $(I(Z))^{(n)} = \cap_{i=1}^{s} I(p_i)^n$, so $(I(Z)^{(n)}$ consists of the polynomials for which all derivatives up to order $n-1$ vanishes in all points.

Theorem $I^{(2)} \subseteq (x_0, \dots, x_N)I$.
Proof If $F \in I^{(2)}$, then $\partial F/\partial x_i \in I$, so $\deg F \cdot F = \sum_{i=1}^{N} x_i \partial F/\partial x_i \in (x_0, \dots, x_N)I$.

Example Consider the set of points $Z = \{p_1 = (1:1), p_2 = (1:-1)\} \subset P^1$. Then $I(p_1) = (x_0 - x_1)$ and $I(p_2) = (x_0 + x_1)$, so $I(Z) = (x_0 - x_1) \cap (x_0 - x_1) = (x_0^2 - x_1^2)$. Then $(I(Z))^n = ((x_0^2 - x_1^2)^n)$ and $(I(Z))^{(n)} = (x_0 - x_1)^n \cap (x_0 + x_1)^n$, so $(I(Z))^n = (I(Z))^{(n)}$.

Theorem If $I(Z) = I$ is a complete intersection, then $I^n = I^{(n)}$ for all n.
Proof Hochster has proved that if S/I is a complete intersection, then S/I^n is Cohen-Macaulay. I^n equals $I^{(n)}$ or $I^{(n)} \cap Q$, where Q is $(x_0, \dots, x_N)$-primary. But in the second case all elements of positive degree in S/I^n are zerodivisors, so S/I^n is not CM.

Question If $(I(Z))^n = (I(Z))^{(n)}$ for all n, is $I(Z)$ a complete intersection?

Huneke asked if $I^{(3)} \subseteq I^2$ for points in P^2. Harbourne conjectured that $I^{(m)} \subseteq I^r$ for points in P^N if $m \geq rN - (N-1)$. There are counterexamples. The easiest are the $k^2 + 3$ points $(1:0:0), (0:1:0), (0:0:1), (1:\epsilon_1:\epsilon_2)$, where $\epsilon_i^k = 1$, $k \geq 3$. Then $I = (x(y^k - z^k), y(x^k - z^k), z(x^k - y^k))$.

Question What about $k = 2$?

Question What about if one point is removed from the $k^2 + 3$ points?

Backelin and Oneto has shown that for the k^N points $(1:\epsilon_1:\dots:\epsilon_N)$ in P^N that $I^n = I^{(n)}$ for all n. What about the symbolic and ordinary powers if we adjoin the coordinate points, or just one coordinate point?

Question When is $I^2 = I^{(2)}$ for points in P^2? In P^N?

Question Is S/I Gorenstein if $I^2 = I^{(2)}$?

Question For $N + 2$ generic points in P^N, is $I^2 = I^{(2)}$?

Question If I is generated by s generic forms in S, what is the Hilbert series of S/I? If $N = 1$? If $N = 2$?

5. Powers vs symbolic powers for ideal of points in $\mathbb{P}^n$

Let $S = \mathbb{C}[x_0, \dots, x_n]$. If p is a point, then $P = I(p)$ is generated bu n linear forms. The ideal of s points $p_1, \dots, p_s$ is $I = P_1 \cap \dots \cap P_s$. Let $M_d = (a_{ij})$ be the $s \times \binom{n+d}{d}$-matrix with the columns indexed by the monomials $m_1, \dots, m_{\binom{n+d}{d}}$ monomials of degree d and the rows indexed by the points $p_1, \dots, p_s$ and let $a_{ij} = m_j(p_i)$ (the value of m_j in the point p_i). Then a solution to the system $(a_{ij}X = 0$ gives a form of degree d which is 0 in the points, so the Hilbert function in degree d is the rank of M_d. If the points are generic, then M_d has full rank, so the Hilbert series is $\sum \min(s, \binom{n+i}{i}) t^i$. The symbolic power $I^{(d)}$ of I is $I^d S_T \cap S$, $T = (P_1 \cup \dots \cup P_s)^c = P_1^d \cap \dots \cap P_s^d$. It is clear that $I^d \subseteq I^{(d)}$. I^d always have the P_i-primary components P_i^d. The primary decomposition of I^d is either $P_1^d \cap \dots \cap P_s^d$

or $P_1^d \cap \cdots \cap P_s^d \cap Q$, where Q is $(x_0, \ldots, x_n)$-primary. In the first case $I^d = I^{(d)}$ and is CM. In the second case I^d is not CM. If I is a complete intersection, then S/I^d is CM, so $I^d = I^{(d)}$. The big (impossible) question is: What is the Hilbert series of S/I^d and what is the primary decomposition of I^d?.

Geometrically $I^{(d)}$ consists of the forms for which all partial derivatives of order $\leq d-1$ are zero in the points. In fact it suffices to claim that all derivatives of exactly order $d-1$ are zero, because the other conditions follow from the Euler relation ($\deg F(\cdot F = \sum x_i \partial/\partial x_i(F)$. If the points are generic it is natural to assume that each point p should be replaced by $e(S/P)$ (the multiplicity) generic points. If $p = (1 : 0 \cdots : 0)$, then $P = (x_1, \ldots, x_n)$ and $e(S(P^d)) = e(S/(x_1, \ldots, x_n)^d) = e(\mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^d) = l(\mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^d) = \binom{n+d-1}{n}$. This is equivalent to that the derivatives of order $d-1$ give independent conditions. The "expected" Hilbert series of $S/I^{(d)}$ is $\sum \min(\binom{n+d-1}{n} s, \binom{n+i}{i}) t^i$. From now on I restrict to $n = d = 2$. For $d = n = 2$ the expected series is the actual except for $s = 2$ and $s = 5$ (Alexander-Hirschowitz). In fact it is easy to see that these are counterexamples. For $s = 2$ there is a line l through the points, so $l^2 \in I^{(2)}$. For $s = 5$ there is a quadric q through the points, so $q^2 \in I^{(2)}$. The expected series are $1 + 3t + 6t^2/(1-t)$ and $1 + 3t + 6t^2 + 10t^3 + 15t^4/(1-t)$, resp., so there should be no element of degree 2 resp. 4 in the ideal. I calculated the Hilbert series for S/I^2 for $s \leq 21$ points (and for $s = 28$). Here is the result (I list the difference between the Hilbert series of S/I^2 and $S/I^{/2)}$.

For $s = 1$, $s = 2$, $s = 4$ the difference is 0 since S/I is a complete intersection.
$s = 3$, $t3$
$s = 5$, t^5
$s = 6$, $3t^5$
$s = 7$, t^6
$s = 8$, t^6
$s = 9$, $3t^7$
$s = 10$, $6t^7$
$s = 11$, $2t^8 + 3t^7$
$s = 12$, $t^9 + 3t^8$
$s = 13$, $3t^9 + 3t^8$
$s = 14$, $t^9 + 2t^8$
$s = 15$, $10t^9$
$s = 16$, $3t^{10} + 7t^9$
$s = 17$, $t^{11} + 5t^{10} + 4t^9$
$s = 18$, $3t^{11} + 6t^{10} + t^9$
$s = 19$, $6t^{11} + 6t^{10}$
$s = 20$, $10t^{11} + 5t^{10}$
$s = 21$, $15t^{10} + 3t^{10}$
$s = 28$, $21t^{13} + 7t^{12}$

Alessandro suggested to use apolarity. For $s = 3$, for $I = (x, y) \cap (x, z) \cap (y, z)$, this amounts to that the difference between the dimensions of $(I^{(2)})_d^{-1} = (x_0^d, x_1^d, x_2^d)_{d+1} = \dim\langle x_i^{d+1}, x_i^d x_j\rangle$ and $\dim(I^2)_d^{-1} = \dim(x^2yz, xy^2z, xyz^2)_{d+1}^{-1}$ is 1 for $d = 3$ and 0 otherwise. In degree 3 the first contains all monomials of degree 3 except xyz, the second contains all monomials of degree 3. If $d > 3$ the first is $\langle x_i^{d+1}, x_i^d x_j\rangle$ of dimension 9, the second is the same. (If $d < 3$ both are empty.) One can do the same for $s = 4$, but that we know in any case. Next case is $s = 5$. Can this be done?

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