ON SECANT VARIETIES OF VARIETIES OF POWERS

RALF FRÖBERG, ALESSANDRO ONETO, GIORGIO OTTAVIANI, AND BORIS SHAPIRO

Abstract. In [FOS12], the authors began the study of kth Waring rank of general forms, namely what is the minimal number of degree d forms to write the general form of degree kd as sum of their k th powers. Motivated by this problem, we study the secant varieties to the variety of kth powers. We conjecture the dimension of each secant variety and we show how this would follow from the Fröberg conjecture on the Hilbert series of general ideals.

1. Introduction

Let $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{i\geq 0} S_i$ be the polynomial ring in $n+1$ variables and complex coefficients with standard gradation, namely S_i is the $\mathbb{C}$-vector space of homogeneous polynomials, or forms, of degree i.

In [FOS12], the authors introduced a Waring problem for polynomials of degree kd written as sum of kth powers. Precisely, given a form $F \in S_{kd}$, we define the kth **Waring rank** of F to be the minimal number of degree d forms such that F can be written as their sum; i.e.

$$\mathrm{rk}_k(F) := \min\{s \mid \exists G_1, \ldots, G_s \in S_d \text{ s.t. } F = G_1^k + \ldots + G_s^k\}.$$

Problem 1. Let k, d be positive integers.

(i) what is the kth Waring rank of the *general* form of degree kd?
It will be denoted by $\mathrm{rk}_{gen}(n, k, d)$.

(ii) given a form $F \in S_{kd}$, what is the kth Waring rank of F?

The $d = 1$ case is the *classic* Waring problem for polynomials where we look at forms of degree k written as sum of kth powers of linear forms. This problem has a long story and the complete answer to the question (i) regarding the general form is been given by J. Alexander and A. Hirschowitz in 1995, after a series of celebrated papers, [AH92a, AH92b, AH95]. On the other hand, question (ii) is in general still open. We have an exhaustive answer only in the two variables case, due to J.J. Sylvester [Sy51] and in the case of monomials [CCG12].

In 2012, R. Fröberg, G. Ottaviani and B. Shapiro began the study of Problem 1 in the case $d \geq 2$. Their main result is the following.

Theorem 1.1. [FOS12, Theorem 4] *The general form of degree kd can be written as sum of at most k^n kth powers of degree d forms; namely,*

$$\mathrm{rk}_{gen}(n, k, d) \leq k^n.$$

A surprising fact of this result is that it does not depend on the parameter d and actually, for d large enough, it gives a sharp bound. The aim of this paper is to understand the behavior of $\mathrm{rk}_{gen}(n, k, d)$ before reaching the bound of k^n.

As regard the question of the kth Waring rank of a given form not much is been done and we only have some partial results in the case of monomials, see [CO15].

The main object of studies in this paper are *secant varieties to varieties of kth powers*.

Definition 1. Given any projective variety $X \subset \mathbb{P}^n$, we define the sth **secant variety** of X to be the variety of the secant $\mathbb{P}^{s-1}$'s to X; namely,

$$\sigma_s(X) := \overline{\bigcup_{P_1,\ldots,P_s \in X} \langle P_1,\ldots,P_s\rangle}^{\text{Zariski}} \subset \mathbb{P}^n.$$

Consider the **variety of k th powers of degree d forms**, namely

$$\mathrm{VP}_{n,k,d} := \{[G^k] \mid G \in S_d\} \subset \mathbb{P}(S_{kd}).$$

In the $d = 1$ case, $\mathrm{VP}_{n,k,1}$ is simply the classical Veronese variety given by the degree k Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$, $N = \binom{n+d}{n} - 1$.

Remark 1. The relation between secant varieties to $\mathrm{VP}_{n,k,d}$ and the Waring problem is given by the following,

$$\mathrm{rk}_{gen}(n,k,d) = \min\{s \mid \sigma_s(\mathrm{VP}_{n,k,d}) = \mathbb{P}(S_{kd})\}.$$

Indeed, if $\sigma_s(\mathrm{VP}_{n,k,d})$ fills the ambient space, we have that the general form $[F] \in \mathbb{P}(S_{kd})$ lies on a $\mathbb{P}^{s-1}$ spanned by s independent points of the variety $\mathrm{VP}_{n,k,d}$, namely F can be written as a linear combination of kth powers of some degree d forms $G_1^k,\ldots,G_s^k$. Then, an exhaustive answer to our Problem 1(i) would follow from an answer to the following geometrical problem.

Problem 2. What is the dimension of the sth secant variety of $\mathrm{VP}_{n,k,d}$?

In general, there is a very naïve expectation for the dimension of the sth secant variety to a priojective variety $X \subset \mathbb{P}^n$. Indeed, simply by counting parameters, we have that

$$exp. \dim \sigma_s(X) = \min\{s \dim(X) + s - 1, n\}.$$

However, this is not always the case and, if the actual dimension of $\sigma_s(X)$ is smaller than the expected one, we say that X is *s-defective* and the difference is called *defect*.

In this way, we can have an expectation for the general kth Waring rank, namely

$$exp.\mathrm{rk}_{gen}(n,k,d) = \left\lceil \frac{\binom{n+kd}{n}}{\binom{n+d}{d}} \right\rceil.$$

Problem 2 is strictly related to the Fröberg conjecture on the Hilbert series of general ideals, as a consequence we conjecture the dimension of the secant varieties to $\mathrm{VP}_{n,k,d}$ as follows.

Conjecture 1. *Consider n,k,d positive integers with $d \geq 2$; then,*

$$\dim \sigma_s(\mathrm{VP}_{n,k,d}) = \begin{cases} \min\left\{s\binom{n+d}{n} - \binom{s}{2}, \binom{n+2d}{n}\right\} - 1 & \textit{for } k = 2; \\ \min\left\{s\binom{n+d}{n}, \binom{n+kd}{n}\right\} - 1 & \textit{otherwise}; \end{cases}$$

In particular, $\mathrm{VP}_{n,k,d}$ *is defective if and only if* $k = 2$.

In Section 2.1, we give a geometrical description of variety of powers in relation to Veronese varieties. This approach doesn't lead us in any conclusion, but we include it to help the reader in relating these varieties and their secants to largely study classical objects. In Section 2.2, we use Terracini's lemma to link our computations on the dimension of secants to varieties of powers with the Fröberg conjecture on Hilbert series of general ideals. In Section 2.3, we resume the consequences of our analysis on the dimension of $\sigma_s(\mathrm{VP}_{n,k,d})$ from the prospective of the related Waring problem described above.

2. Dimension of secant varieties of varieties of powers

2.1. **A first look to varieties of powers and their secants.** Given the positive integers n, k, d, we consider the variety of kth powers of degree d forms in $n+1$ variables $\mathrm{VP}_{n,k,d} \subset \mathbb{P}^N$, where $N = \dim S_{kd} - 1 = \binom{n+kd}{n} - 1$. It is a projective variety of dimension $M = \dim S_d = \binom{n+d}{n} - 1$ which corresponds to the image of the regular map

$$\nu_{n,k,d} : \mathbb{P}(S_d) \longrightarrow \mathbb{P}(S_{kd}),\ [G] \longmapsto [G^k].$$

As mentioned before, in the $d = 1$ case, we have that $\nu_{n,k,1}$ is the degree k Veronese embedding of $\mathbb{P}^n$. However, in the $d \geq 2$ case, we can see our variety $\mathrm{VP}_{n,k,d}$ as a linear projection of a Veronese variety. Consider the standard monomial basis of S_d, namely

$$X_{i_0,\ldots,i_n} := x_0^{i_0} \ldots x_n^{i_n},\ \text{with } i_j \geq 0,\ i_0 + \ldots + i_n = d,$$

then, any form F of degree d in the x's can be seen as a linear form in X's. We will write $lin(F)$ when we refer to F in the latter way. Hence, we can look at the Veronese embedding

$$\nu_{M,k} : \mathbb{P}(S_d) \longrightarrow \mathbb{P}(\mathcal{S}ym^k(S_d)),\ [lin(F)] \longmapsto [lin(F)^k],$$

where $\mathcal{S}ym^k(S_d)$ is the degree k part of the symmetric algebra associated to the vector space S_d. We denote the coordinates of $\mathbb{P}(\mathcal{S}ym^k(S_d))$ as $Z_{i_{1,0}\ldots,i_{1,n};\ldots;i_{k,0},\ldots,i_{k,n}}$ where $i_{j,0} + \ldots + i_{j,n} = d$ for any $j = 1, \ldots, k$.

We have a natural embedding of S_{kd} in $\mathcal{S}ym^k(S_d)$ and we can look at the projection onto $\mathbb{P}(S_{kd})$. This is given by the multiplication map from $\mathcal{S}ym^k(S_d)$ to S_{kd}, but geometrically corresponds to the linear projection on $\mathbb{P}(S_{kd})$ from the linear space E defined by the equations

$$Z_{i_{1,0}\ldots,i_{1,n};\ldots;i_{k,0},\ldots,i_{k,n}} - Z_{j_{1,0}\ldots,j_{1,n};\ldots;j_{k,0},\ldots,j_{k,n}} = 0,$$

where $i_{1,h} + \ldots + i_{k,h} = j_{1,h} + \ldots + j_{k,h}$ for any $h = 0, \ldots, n$; in other words, we just *remind* oursself that, under these numerical assumptions, as monomials in S_{kd}, the following equation holds,

$$X_{i_{1,0},\ldots,i_{1,n}} \ldots X_{i_{k,0},\ldots,i_{k,n}} = X_{j_{1,0},\ldots,j_{1,n}} \ldots X_{j_{k,0},\ldots,j_{k,n}}.$$

By definition, we have that $\nu_{n,k,d}([F]) = \pi_{n,k,d}(\nu_{M,k}([lin(F)]))$; moreover, it is easy to observe that actually our variety $\mathrm{VP}_{n,k,d}$ coincides with the linear projection of the Veronese variety $\nu_{M,k}(\mathbb{P}^M)$ via $\pi_{n,k,d}$.

Example 1. Consider $n = 1$, $k = 2$ and $d = 2$. We consider the monomial basis of S_2 given by

$$X_{20} = x_0^2,\ X_{11} = x_0x_1,\ X_{02} = x_1^2$$

and we consider the basis of $\mathcal{S}ym^2(S_2)$ as $Z_{20;20} = X_{20}^2, Z_{20;11} = X_{20}X_{11}, \ldots, Z_{02;02} = X_{02}^2$. In the space $\mathcal{S}ym^2(S_2)$ we have that $X_{20}X_{02} \neq X_{11}^2$, even though they both coincide to $x_0^2x_1^2$ once we project in S_4. Hence, in this case, we have that the linear space $E \subset \mathbb{P}^5 = \mathbb{P}(Sym^2(S_2)$ from where we are projecting is given by the equations

$$\begin{cases} Z_{20;20} = Z_{20;11} = Z_{11;02} = Z_{02;02} = 0; \\ Z_{20;02} - Z_{11;11} = 0; \end{cases}$$

namely, we have that $\pi_{n,k,d} : \mathbb{P}(\mathcal{S}ym^2(S_2)) \to \mathbb{P}(S_4)$ is the linear projection of $\mathbb{P}^5$ on $\mathbb{P}^4$ from the point $[0 : 0 : 1 : -1 : 0 : 0]$.

As described in the introduction, we are interested in the secant varieties to $\mathrm{VP}_{n,k,d}$ and in particular in their dimensions. As a consequence of the result by J. Alexander and A. Hirschowitz related to the classic Waring problem, we know all the dimensions of the secant varieties to the Veronese varieties and then we might study the relation between the projection of the secant varieties to the Veronese $\nu_{M,k}(\mathbb{P}^M)$ via $\pi_{n,k,d}$ and the secant varieties to $\mathrm{VP}_{n,k,d}$.

It is a trivial observation that $\pi_{n,k,d}(\sigma_s(\nu_{M,k}(\mathbb{P}^M))) \subset \sigma_s(\mathrm{VP}_{n,k,d})$, but this is not enough to compute the dimension of $\sigma_s(\mathrm{VP}_{n,k,d})$. We need a much deeper understanding of this projection map, in particular we need to understand how the secant varieties of the Veronese $\nu_{M,k}(\mathbb{P}^M)$ intersect the linear space where we are projecting from in order to know the general fiber of such projection. Unfortunately, we are not able to continue this geometrical approach; however, we have a more algebraic way to attack our problem.

2.2. **Terracini's Lemma and Fröberg conjecture.** In order to compute the dimension of the secant variety $\sigma_s(\mathrm{VP}_{n,k,d})$, we just need to compute the dimension of the general tangent space and then we can start by rephrasing in our setup Terracini's lemma [Te11].

Lemma 2.1 (Terracini's Lemma). *Let X be a projective variety in $\mathbb{P}^n$. Consider a general point P on the span of general points $P_1, \ldots, P_s \in X$; then, the tangent space to $\sigma_s X$ at the point P is given by the projective span of the tangent spaces to X at the P_i's; namely,*

$$T_P\sigma_s(X) = \langle T_{P_1}X, \ldots, T_{P_s}X\rangle.$$

Thus, if we want to look at the tangent space to $\sigma_s(\mathrm{VP}_{n,k,d})$ at a general point, we need to look at the projective span of s general hyperplanes to our variety of powers $\mathrm{VP}_{n,k,d}$. It is easy to check that

$$T_{[G^k]}\mathrm{VP}_{n,k,d} = \{[G^{k-1}H] \mid H \in S_d\} \subset \mathbb{P}(S_{kd}).$$

Then, we have that the general tangent space to our secant variety $\sigma_s(\mathrm{VP}_{n,k,d})$ is given by the degree kd homogeneous part of the ideal generated by $(k-1)$th powers of generic forms of degree d. In this way, we turn our Problem 2 on the dimension of secant varieties of $\mathrm{VP}_{n,k,d}$ into an algebraic question.

Definition 2. Given a homogeneous ideal $I \subset S$, we define the **Hilbert function** of S/I as

$$\mathrm{HF}_{S/I}(i) := \dim_{\mathbb{C}}[S/I]_i;$$

and the **Hilbert series** as

$$\mathrm{HS}_{S/I}(t) := \sum_{i\geq 0} \mathrm{HF}_{S/I}(i)t^i.$$

Thus, we rephrase Problem 2 as follows.

Problem 3. Consider n, k, d positive integers. Given $I = (G_1^{k-1}, \dots, G_s^{k-1})$ generated by $(k-1)$th powers of degree d general forms, what is $\mathrm{HF}_{S/I}(kd)$?

2.2.1. *The $k = 2$ case.* In this case, we are looking at the variety of squares $\mathrm{VP}_{n,2,d}$. As explained before, in order to compute the dimension of $\sigma_s\mathrm{VP}_{n,2,d}$, we need to calculate the Hilbert function in degree $2d$ of the idea $I = (G_1, \dots, G_s)$ generated by general forms of degree d. In 1985, R. Fröberg conjectured a formula for the Hilbert series of these ideals.

Conjecture 2 (Fröberg conjecture, [Fr85])**.** *Let $I = (G_1, \dots, G_s)$ be an ideal generated by general forms of degree $\deg G_i = d_i$, $i = 1, \dots, s$, in $n + 1$ variables; then*

$$\mathrm{HS}_{S/I}(t) = \left\lceil \frac{\prod_{i=1}^{s}(1 - t^{d_i})}{(1 - t)^{n+1}} \right\rceil,$$

where $\lceil \sum_i a_i t^i \rceil := \sum_i b_i t^i$ where $b_i := a_i$, if $a_j > 0$ for all $j \leq i$, and $b_i := 0$, otherwise.

Even if largely studies, this conjecture is known to be true only in a few cases. When $s \leq n + 1$ it is an easy exercise since I turns to be a complete intersection; $s = n + 2$ is due to R. Stanley, see [Fr85]; $n = 1$ is due to R. Fröberg [Fr85]; $n = 2$ is due to D. Anick [An86]. Other numerical cases are known to be true, e.g. see [FH94, Ni15], but the latters are the most exhaustive results on this conjecture.

Theorem 2.2. *Fröberg conjecture implies Conjecture 1 for $k = 2$.*

Proof. Let $I = (G_1, \dots, G_s)$ be an ideal generated by general forms of degree d. Then,

$$\begin{aligned}\operatorname{codim} \sigma_s(\mathrm{VP}_{n,2,d}) = \mathrm{HF}_{S/I}(2d) = \mathrm{coeff}_{2d} \left\lceil \frac{(1 - t^d)^s}{(1 - t)^{n+1}} \right\rceil = \\ = \max\left\{ \binom{n+2d}{n} - s\binom{n+d}{n} + \binom{s}{2}, 0 \right\}.\end{aligned}$$

Then, since $\dim \mathbb{P}(S_{2d}) = \binom{n+2d}{n} - 1$,

$$\dim \sigma_s(\mathrm{VP}_{n,2,d}) = \min\left\{ s\binom{n+d}{n} - \binom{s}{2}, \binom{n+2d}{n} \right\} - 1.$$

□

Remark 2. The fact that $\mathrm{VP}_{n,2,d}$ is defective and that the defect of each secant variety $\sigma_s(\mathrm{VP}_{n,2,d})$ is equal to $\binom{s}{2}$ was expected as consequence of an easy geometrical observation. Indeed, the dimension of a secant variety $\sigma_s(X)$ is smaller than the expected one when s general tangent spaces to X have pairwise non-empty intersection. In our case, we have that

$$T_{[G^2]}\mathrm{VP}_{n,2,d} \cap T_{[H^2]}\mathrm{VP}_{n,2,d} = [GH];$$

thus when we compute the dimension of the secant variety $\sigma_s(\mathrm{VP}_{n,2,d})$ by using Terracini's Lemma, we have *at least* a defect of $\binom{s}{2}$ due to the non-empty intersection of every pair of the s tangent spaces to $\mathrm{VP}_{n,2,d}$ spanning the tangent space to $\sigma_s(\mathrm{VP}_{n,2,d})$.

2.2.2. *The $k > 2$ case.* In this case, we are looking at ideals generated by powers of general forms. As observed in [FH94], powers of general linear forms are not necessarely generic and they can fail to have the Hilbert series conjectured by Fröberg. However, several computer experiments support the following conjecture, see [Ni15].

Conjecture 3. *An ideal $I = (G_1^m, \ldots, G_s^m)$ generated by mth powers of general forms of degree d, with $d \geq 2$, has the Hilbert series conjectured by Fröberg.*

Theorem 2.3. *Conjecture 3 implies Conjecture 1 for $k \geq 3$.*

Proof. Let $I = (G_1^{k-1}, \ldots, G_s^{k-1})$ be an ideal generated by $(k-1)$th powers of general forms of degree d. Then,

$$\begin{aligned}\operatorname{codim} \sigma_s(\mathrm{VP}_{n,k,d}) = \mathrm{HF}_{S/I}(kd) = \mathrm{coeff}_{kd}\left\lceil \frac{(1-t^{(k-1)d})^s}{(1-t)^{n+1}} \right\rceil = \\ = \max\left\{ \binom{n+kd}{n} - s\binom{n+d}{n}, 0 \right\}.\end{aligned}$$

Then, since $\dim \mathbb{P}(S_{kd}) = \binom{n+kd}{n} - 1$,

$$\dim \sigma_s(\mathrm{VP}_{n,k,d}) = \min\left\{ s\binom{n+d}{n}, \binom{n+kd}{n} \right\} - 1.$$

□

2.3. **Conclusions.**

(1) The Fröberg conjecture is known to be true in the case of two variables; moreover, it is easy to check that also Conjecture 3 is true for $n = 1$. Indeed, we can specialize the ideal $I = (G_1^m, \ldots, G_s^m)$ to a power ideal $(L_1^{dm}, \ldots, L_s^{dm})$ which, for two variables, have the Hilbert series conjectured by Fröberg, as easily observed in [GS98]. By semicontinuity, we conclude that Conjecture 3 is true for binary forms.

Corollary 1. *The general* k*th Waring rank in the binary case is never defective; namely,*

$$\mathrm{rk}_{gen}(1,k,d) = \left\lceil \frac{kd+1}{n+1} \right\rceil.$$

This fact is been already proven independently with different methods by Reznick [Re13].

(2) The Fröberg conjecture is known to be true in the case of three variables; hence, we can conclude the following

Corollary 2. *For sum of squares in three variables ($k = 2,\ n = 2$),*

$$\mathrm{rk}_{gen}(2,2,d) = \begin{cases} \left\lceil \frac{\binom{2d+2}{2}}{\binom{n+2}{2}} \right\rceil + 1 & \text{if } d = 3,4 \\ \left\lceil \frac{\binom{2d+2}{2}}{\binom{n+2}{2}} \right\rceil & \text{if } d = 2,\ d \geq 5 \end{cases}$$

(3) Since we conjecture that $\mathrm{VP}_{n,k,d}$ is non-defective for $k \geq 3$, we expect that
$$\mathrm{rk}_{gen}(n,k,d) = \left\lceil \frac{\binom{n+kd}{n}}{\binom{n+d}{d}} \right\rceil, \text{ for } k \geq 3;$$

(4) in the $k=2$, the defectiveness of the variety of powers $\mathrm{VP}_{n,2,d}$ allows us to have cases in which the general kth Waring rank is bigger than the expected one, as observed in (2) in the three variable case. With the support of a computer, we have checked up to $n = 14$. TThe behavior looks unfortunately quite random and, since there are no reason to expect the opposite, we believe that this list of exceptions is actually infinite.

n	d
2	3,4
3	2
4	2,3,5
5	2,5,20
6	2,3,12,16
7	2,3,4,7
8	2,3,4,5,6
9	2,3,4,6
10	2,3,4,5,6,7
11	2,3,4,5,9
12	2,3,4,5,6,7
13	2,3,4,5,6,7,10
14	2,3,4,5,7,8,9,13

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(R. Fröberg, A. Oneto. B. Shapiro) Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: ralff@math.su.se, oneto@math.su.se, shapiro@math.su.se

(G. Ottaviani) Dipartimento di Matematica e Informatica U. Dini, Università degli studi di Firenze, Firenze, Italy
E-mail address: ottaviani@math.unifi.it