ON POSTNIKOV-SHAPIRO ALGEBRAS AND THEIR GENERALIZATIONS

GLEB NENASHEV

Abstract. In this paper we consider different generalizations of Postnikov-Shapiro algebra, see [8]. Firstly, for a given graph G and a positive integer t, we generalize the notion of Postnikov-Shapiro algebras counting forests in G to an algebra counting t-labelled forests. We also prove that for large t we can restore the Tutte polynomial of G from the Hilbert series of such algebra.

Secondly, we prove that the original Postnikov-Shapiro algebra counting forests depends only on the matroid of G. And conversely, we can reconstruct this matroid from the latter algebra. Similar facts hold for analogous algebras counting trees in connected graphs.

Thirdly, we present a generalization of such algebras for hypergraphs. Namely, we construct a certain family of algebras for a given hypergraph, such that for almost algebras from this family, their Hilbert series is the same. Finally, we present the definition of a hypergraphical matroid, whose Tutte polynomial allows us to calculate this generic Hilbert series.

1. Introduction

The famous matrix-tree theorem of Kirchhoff (see [6] and p. 138 in [10]) claims that the number of spanning trees of a given graph G equals to the determinant of the Laplacian matrix of G. It is also well known that the number of spanning forests of G (or equivalently trees for connected G) equals to $T_G(1,1)$ while the number of all subforests of G equals to $T_G(2,1)$, where T_G is the Tutte polynomial of G (see p. 237 in [10]).

There exist many generalizations of the classical matrix-tree theorem, e.g. for directed graphs, matrix-forest theorems, etc (see e.g. [3], see [2] and references therein). In particular, in [8] A. Postnikov and B. Shapiro constructed several algebras associated to G whose total dimensions are equal to the number of either spanning trees or forests of G. Below we extend constructions of [8] to a larger class of algebras.

We use the standard notation: $E(G)$ and $e(G)$ are the set and the number of edges of graph G; $V(G)$ and $v(G)$ are the set and the number of vertices of G; $T_G(x, y)$ is the Tutte polynomial of G.

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Notation 1. *Take an undirected graph G with n vertices and some field $\mathbb{K}$ of characteristic 0. Let $t > 0$ be a positive integer.*

(I) *Let $\Phi_G^{F_t}$ be the algebra over $\mathbb{K}$ generated by $\{\phi_e : e \in E(G)\}$ satisfying the relations $\phi_e^{t+1} = 0$, for any $e \in E(G)$.*

Fix any linear order of vertices of G. For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in G} c_{i,e}\phi_e,$$

where $c_{i,e} = \pm 1$ for vertices incident to e (for the smaller vertex v_i, $c_{i,e} = 1$; for the larger vertex v_j, $c_{j,e} = -1$) and 0 otherwise. Denote by $\mathcal{C}_G^{F_t}$ the subalgebra of $\Phi_G^{F_t}$ generated by $X_1, \ldots, X_n$.

(II) *Consider the ideal $J_G^{F_t}$ in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by*

$$p_I^F = \left(\sum_{i \in I} x_i\right)^{t \cdot D_I + 1},$$

where I ranges over all nonempty subsets of vertices, and D_I is the total number of edges between vertices in I and vertices outside I. Define the algebra $\mathcal{B}_G^{F_t}$ as the quotient $\mathbb{K}[x_1, \ldots, x_n]/J_G^{F_t}$.

Remark 1. *In the above definition of $\mathcal{C}_G^{F_t}$, we can separately choose the "smaller" vertex for each edge. In other words, if we change the signs of all $c_{i,e}$ for some edges, then we obtain an isomorphic algebra.*

For an orientation $\overline{G}$ of graph G, we define the subalgebra $\mathcal{C}_{\overline{G}}^{F_t}$ of $\{\phi_e : e \in E(G)\}$, where for each edge arrow goes to the "smaller" vertex.

The field $\mathbb{K}$ of characteristic 0 is fixed throughout this paper. By a graph we always mean an undirected graph without loops (multiple edges are allowed).

Notation 2. *Fix some linear order on the set $E(G)$ of edges of G. Let F be any subforest in G. By $act_G(F)$ denote the number of all externally active edges of F, i.e. the number of edges $e \in E(G) \setminus F$ such that* (i) *subgraph $F + e$ has a cycle;* (ii) *e is the minimal edge in this cycle in the above linear order.*

Denote by F^+ the set of edges of the forest F together with all externally active edges, and denote by $F^- = E(G) \setminus F^+$ the set of externally nonactive edges.

For $t = 1$, these algebras (denoted there by $\mathcal{B}_G^F$ and $\mathcal{C}_G^F$) were introduced by A. Postnikov and B. Shapiro in [8]. The following result was proved in this paper.

Theorem 1 (cf. [8])**.** *For any graph G, algebras $\mathcal{B}_G^F$ and $\mathcal{C}_G^F$ are isomorphic, their total dimension over $\mathbb{K}$ is equal to the number of subforests in G.*

Moreover, the dimension of the k-th graded component of these algebras equals the number of subforests F of G with external activity $e(G) - e(F) - k$.

In fact, the second part of Theorem 1 claims that the above Hilbert polynomial is a specialization of the Tutte polynomial of G.

There are a few generalizations of $\mathcal{B}_G^F$ and $\mathcal{B}_G^T$ in the literature (the latter algebra counts trees in G instead forests, see notation 7), see [1]. F. Ardila and A. Postnikov introduced similar algebras for zonotopes; they also introduced internal algebras (when the power equals $D_I - 1$ instead of $D_I + 1$). In [5] the definitions of $\mathcal{B}_G^\Delta$ and $\mathcal{C}_G^\Delta$ were given in terms of some simplex Δ on the set of vertices of G. With this definition algebras $\mathcal{B}_G^F$ and $\mathcal{B}_G^T$ become particular cases corresponding to different simplices.

In § 2 we generalize Theorem 1 for $t > 1$. In fact, these new algebras are isomorphic to the original algebras for the graph $\widehat{G}$, where each edge of G corresponds to t edges from $\widehat{G}$. We think this direction is important, because on one hand it is not a new object, however, on the another hand we can reconstruct Tutte polynomial from the Hilbert series of algebra corresponding to large t, see Theorem 4.

In § 3 we prove that the original Postnikov-Shapiro algebras depend only on the graphical matroid of G. And conversely, we can reconstruct the matroid from such an algebra, see Theorem 5. It means that these algebras store almost all information about the graphs. As a consequence, the same is true for t-algebras.

In § 4 we construct a family of algebras corresponding to a given hypergraph, and we also present a natural definition of a matroid of a hypergraph. For usual graphs, our definition gives graphical matroids. For a given hypergraph, we prove that almost all algebras in our family have the same Hilbert series, and this Hilbert series is a specialization of the Tutte polynomial corresponding to the hypergraphical matroid. This family of algebras is motivated by Postnikov-Shapiro-Shapiro algebras for vector configuration (see [9]).

In § 5 we discuss the algebras counting spanning trees of a graph. There is a similar result about such algebra and a graphical matroid, however, only with the additional condition, that G is connected.

In appendix § 6 we present an elementary bijection between G-parking functions and spanning trees and formulate a question about the existence of analog of G-parking functions for vector matroids. It is well known that the numbers of G-parking functions and spanning trees are same (for example, the first proof by bijective see in [4]). This fact was very useful in Postnikov-Shapiro's proof of Theorem 1.

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2. **Algebras associated with t-labelled forests**

Below we generalize Theorem 1 for $t > 1$, and show that the corresponding dimension coincides with the number of the so-called t-labelled forests. In Theorem 3 we prove that the Hilbert polynomial of $\mathcal{B}_G^{F_t}$ can be expressed in terms of the Tutte polynomial of G. And conversely, in Proposition 4 we show that, for any sufficiently large t, the Tutte polynomial of G can be restored from the Hilbert series of the algebra $\mathcal{B}_G^{F_t}$.

Consider a finite labelling set $\{1, 2, \dots, t\}$ containing t different labels; each label being to a number from 1 to t.

Definition 3. *A spanning forest of the graph G with a label from $\{1, 2, \dots, t\}$ on each edge is called a t-labelled forest. The weight of a t-labelled forest F, denoted by $\omega(F)$, is the sum of the labels of all its edges.*

Theorem 2. *For any graph G and a positive integer t, algebras $\mathcal{B}_G^{F_t}$ and $\mathcal{C}_G^{F_t}$ are isomorphic. Their total dimension over $\mathbb{K}$ is equal to the number of t-labelled forests in G.*

The dimension of the k-th graded component of the algebra $\mathcal{B}_G^{F_t}$ is equal to the number of t-labelled forests F of G with the weight $t \cdot (e(G) - act_G(F)) - k$.

Proof. Denote by $\widehat{G}$ the graph on n vertices and $t \cdot e(G)$ edges such that each edge of G corresponds to its t clones in the graph $\widehat{G}$. In other words, each edge of G is substituted by its t copies with labels $1, 2, \dots, t$. For each edge $e \in E(G)$, its clones $e_1, \dots, e_t \in E(\widehat{G})$ are ordered according to their numbers; clones of different edges have the same linear order as the original edges. Thus we obtain a linear order of the edges of $\widehat{G}$.

Consider the following bijection between t-labelled forests in G and usual forests in $\widehat{G}$: each t-labelled forest $F \subset G$ corresponds to the forest $F' \subset \widehat{G}$, such that for each edge $e \in E(F)$, the forest F' has the clone of the edge e whose number is identical with the label of edge e in the forest F.

Obviously,

$$act_{\widehat{G}}(F') = t \cdot act_G(F) + \omega(F) - e(F),$$

and $e(\widehat{G}) = t \cdot e(G)$. Since $\mathcal{B}_G^{F_t}$ and $\mathcal{B}_{\widehat{G}}^{F}$ are isomorphic, the Hilbert series of the algebra $\mathcal{B}_G^{F_t}$ coincides with the Hilbert series of the algebra $\mathcal{B}_{\widehat{G}}^{F}$, which settles the second part of Theorem 2.

To prove the first part of this Theorem, observe that $\mathcal{B}_G^{F_t}$ and $\mathcal{B}_{\widehat{G}}^{F}$ are isomorphic, and algebras $\mathcal{C}_{\widehat{G}}^{F_t}$ and $\mathcal{B}_{\widehat{G}}^{F}$ are isomorphic. Thus we must show that algebras $\mathcal{C}_{\widehat{G}}^{F}$ and $\mathcal{C}_G^{F_t}$ are isomorphic. This indeed true, because for every edge $e \in E(G)$, the elements $\phi_e, \dots, \phi_e^t$ are linearly independent in the algebra $\Phi_G^{F_t}$ with coefficients containing no ϕ_e. Also elements $(\phi_{e_1}+\cdots+\phi_{e_t}),\ldots,(\phi_{e_1}+\cdots+\phi_{e_t})^t$ are linearly independent in the algebra $\Phi_{\widehat{G}}^F$ with coefficients containing no $\phi_{e_1},\ldots,\phi_{e_t}$, and $(\phi_{e_1}+\cdots+\phi_{e_t})^{t+1}=0$. Moreover elements ϕ_{e_i} only occur in the sum $(\phi_{e_1}+\cdots+\phi_{e_t})$ in the algebra $\Phi_{\widehat{G}}^F$. □

Denote by $c(G)$ the number of connected components of the graph G.

Theorem 3. *Dimension of the k-th graded component of $\mathcal{B}_G^{F_t}$ is equal to the coefficient of the monomial $y^{t\cdot e(G)-c(G)+v(G)+1-k}$ in the polynomial*

$$\left(\frac{y^t-1}{y-1}\right)^{v(G)-c(G)}\cdot T_G\left(\frac{y^{t+1}-1}{y^{t+1}-y},y^t\right).$$

Proof. Consider the graph $\widehat{G}$ constructed in the proof of Theorem 2. Set $J_G(x,y):=T_{\widehat{G}}(x,y)$; we will use the following deletion-contraction recurrence for J_G, where $G-e$ denote the graph obtained by deleting e from G and $G\cdot e$ is the contraction of G by e.

Lemma 1. *Polynomial $J_G(x,y)$ satisfies the following:*

1. *If G is empty, then $J_G(x,y)=1$.*
2. *If e is a loop in G, then $J_G(x,y)=y^tJ_{G-e}(x,y)$.*
3. *If e is a bridge in G, then $J_G(x,y)=(y^{t-1}+\cdots+1)\cdot J_{G\cdot e}(x,y)+(x-1)\cdot J_{G-e}(x,y)$.*
4. *If e is not a loop or a bridge, then $J_G(x,y)=(y^{t-1}+\cdots+1)\cdot J_{G\cdot e}(x,y)+J_{G-e}(x,y)$.*

Proof. We prove these relations by using the deletion-contraction recurrence for the usual Tutte polynomial.

1. If G is empty, then $T_G(x,y)=1$.
2. If e is a loop in G, then $T_G(x,y)=y\cdot T_{G-e}(x,y)$.
3. If e is a bridge in G, then $T_G(x,y)=x\cdot T_{G-e}(x,y)$.
4. If e is not a loop or a bridge, then $T_G(x,y)=T_{G\cdot e}(x,y)+T_{G-e}(x,y)$.

Return to the proof of the Lemma.

(1). Graph $\widehat{G}$ is also empty, hence $J_G(x,y)=T_{\widehat{G}}(x,y)=1$.

(2). Clones $e_1,\ldots,e_t$ is also loops in graph $\widehat{G}$, therefore $T_{\widehat{G}}(x,y)=y^t\cdot T_{\widehat{G}-\{e_1,\ldots,e_t\}}(x,y)=y^t\cdot T_{\widehat{G-e}}(x,y)$, hence $J_G(x,y)=y^t\cdot T_{\widehat{G-e}}(x,y)=J_{G-e}(x,y)$.

(3). We calculate our polynomial using the deletion-contraction recurrence for the Tutte polynomial.

$$J_G(x,y)=T_{\widehat{G}}(x,y)=T_{\widehat{G}\cdot e_k}(x,y)+T_{\widehat{G}-e_t}(x,y)=$$

$$y^{t-1}\cdot T_{\widehat{G\cdot e}}(x,y)+T_{\widehat{G}-e_t}(x,y)=$$

$$y^{t-1}\cdot T_{\widehat{G\cdot e}}(x,y)+T_{(\widehat{G}-e_t)\cdot e_{t-1}}(x,y)+T_{\widehat{G}-e_t-e_{t-1}}(x,y)=$$

$$(y^{t-1}+y^{t-2})\cdot T_{\widehat{G\cdot e}}(x,y)+T_{\widehat{G}-e_t-e_{t-1}}(x,y)=$$
$$\dots$$
$$(y^{t-1}+y^{t-2}+\ldots+y)\cdot T_{\widehat{G\cdot e}}(x,y)+T_{\widehat{G}-e_t-\ldots-e_2}(x,y)=$$
$$(y^{t-1}+y^{t-2}+\ldots+y)\cdot T_{\widehat{G\cdot e}}(x,y)+x\cdot T_{\widehat{G}-e_t-\ldots-e_1}(x,y)=$$
$$(y^{t-1}+y^{t-2}+\ldots+y)\cdot T_{\widehat{G\cdot e}}(x,y)+x\cdot T_{\widehat{G-e}}(x,y)=$$
$$(y^{t-1}+y^{t-2}+\ldots+y+1)\cdot T_{\widehat{G\cdot e}}(x,y)+(x-1)\cdot T_{\widehat{G-e}}(x,y)=$$
$$(y^{t-1}+\cdots+1)\cdot J_{G\cdot e}(x,y)+(x-1)\cdot J_{G-e}(x,y).$$

(4). It is similar to (3), but now we have

$$T_{\widehat{G}-e_k-\ldots-e_2}(x,y)=T_{(\widehat{G}-e_k-\ldots-e_2)\cdot e_1}(x,y)+T_{\widehat{G}-e_k-\ldots-e_2-e_1}(x,y)=$$
$$T_{\widehat{G\cdot e}}(x,y)+T_{\widehat{G-e}}(x,y),$$

since in this case edge e_1 is not a bridge in $\widehat{G}-e_k-\ldots-e_2$.

□

Now let us rewrite $J_G(x,y)$ in terms of t-labelled forests using the deletion-contraction recurrence for $J_G(x,y)$ in the above linear order of edges of G. Obviously, the edges by which we contract the graph constitute a forest. Therefore, $J_G(x,y)=\sum_{F_u}a(F_u)$, where $a(F_u)$ depends only on G and the forest F_u. Now rewrite the latter equality in terms of t-labelled forests. When we contract edge e in G, the term y^{k-1} in the factor $(y^{t-1}+y^{t-2}+\ldots+y+1)$ corresponds to the choice of the k-th label for edge e, i.e. we have $J_G(x,y)=\sum_F y^{\omega(F)-e(F)}b(F)$. It remains to calculate $b(F)$. An edge for t-labelled forest F is a loop if and only if it is active, and the number of edges which are bridges in our recursion equals to $c(F)-c(G)=(v(G)-1-e(F))-c(G)=(v(G)-c(G))-1-e(F)$. Therefore, we have

$$J_G(x,y)=\sum_F y^{\omega(F)-e(F)}\cdot y^{t\cdot act_G(F)}\cdot(x-1)^{(v(G)-c(G))-1-e(F)},$$

$$J_G(x,y)=\sum_F y^{\omega(F)-e(F)+act_G(F)}\cdot(x-1)^{(c(G)-v(G))-1-e(F)},$$

$$J_G(1+\frac{1}{y},y)=\sum_F y^{\omega(F)-e(F)+t\cdot act_G(F)}\cdot(\frac{1}{y})^{(c(G)-v(G))-1-e(F)},$$

$$J_G(1+\frac{1}{y},y)=\sum_F y^{\omega(F)-e(F)+t\cdot act_G(F)-((c(G)-v(G))+1+e(F)},$$

$$J_G(1+\frac{1}{y},y)=\sum_F y^{\omega(F)+t\cdot act_G(F)-c(G)+v(G)+1}. \qquad (*)$$

By Theorem 2 the dimension of the k-th graded component of algebra $\mathcal{B}_G^{F_t}$ equals the number of t-labelled forests F of G with weight

$t\cdot(e(G)-act_G(F))-k$. Then the dimension of the k-th graded component is equal to the coefficient of the monomial $y^{t\cdot e(G)-k-c(G)+v(G)+1}$ in polynomial $J_G(1+\frac{1}{y},y)$.

Lemma 2.
$$J_G(1+\frac{1}{y},y)=\left(\frac{y^t-1}{y-1}\right)^{v-c(G)}\cdot T_G\left(\frac{y^{t+1}-1}{y^{t+1}-y},y^t\right).$$

Proof. Conditions 1, 2 and 4 of Lemma 1 hold for polynomial $\left(\frac{y^t-1}{y-1}\right)^{v-c(G)}\cdot T_G\left(\frac{y^{t+1}-1}{y^{t+1}-y},y^t\right)$. Now we can check the 3-rd condition. Set
$$z:=(y^{t-1}+\ldots+1)=\frac{y^t-1}{y-1}.$$
Then, $\frac{y^{t+1}-1}{y^{t+1}-y}=\frac{1}{zy}+1$. We have
$$z^{v(G)-c(G)}\cdot T_G\left(\frac{1}{zy}+1,y^t\right)=$$
$$z^{v(G)-c(G)}\cdot T_{G\cdot e}\left(\frac{1}{zy}+1,y^t\right)+\frac{1}{zy}\cdot z^{v(G)-c(G)}\cdot T_{G-e}\left(\frac{1}{zy}+1,y^t\right)=$$
$$(y^{t-1}+\ldots+1)\cdot z^{v(G\cdot e)-c(G\cdot e)}\cdot T_{G\cdot e}\left(\frac{1}{zy}+1,y^t\right)+$$
$$\frac{1}{y}\cdot z^{v(G-e)-c(G-e)}\cdot T_{G-e}\left(\frac{1}{zy}+1,y^t\right).$$

Hence, the 3-rd condition holds as well. Therefore, if we calculate these polynomials using the recursion method we get the same results. Hence, these polynomials coincide. □

Lemma 2 implies Theorem 3. □

Theorem 4. *For any positive integer $t\geqslant n$, it is possible to restore the Tutte polynomial of any connected graph G on n vertices knowing only the dimensions of each graded component of the algebra $\mathcal{B}_G^{F_t}$.*

Proof. Choose a integer $t\geqslant n$. By Theorem 2 we know that the degree of the maximal nonempty graded component of $\mathcal{B}_G^{F_{tn}}$ equals to the maximum of $t\cdot(e(G)-act_G(F))-\omega(F)$ taken over F. It attains its maximal value for the empty forest (i.e. $F=\emptyset$). Then we know the value of $t\cdot e(G)$, hence, we know the number of edges of the graph G.

By Theorem 3 we also know the polynomial
$$\left(\frac{y^t-1}{y-1}\right)^{v(G)-c(G)}\cdot T_G\left(\frac{y^{t+1}-1}{y^{t+1}-y},y^t\right),$$
because G is connected (i.e. $c(G)=1$). This polynomial equals to
$$\sum_F y^{\omega(F)+t\cdot act_G(F)-c(G)+v(G)+1}=\sum_F y^{\omega(F)+t\cdot act_G(F)+v(G)},$$

where the summation is taken over all t-labelled forests (see eq. (*) and Lemma 2). Rewriting it in terms of the usual subforests, we can calculate

$$\sum_{F_u}(y+\ldots+y^t)^{e(F_u)}\cdot y^{t\cdot act_G(F_u)+v(G)}.$$

Hence, we also know the sum

$$\sum_{F_u}(1+\ldots+y^{(t-1)})^{e(F_u)}\cdot y^{e(F_u)+t\cdot act_G(F_u)}. \qquad (**)$$

Since $e(F_u)<t$, we can compute the number of usual subforests with a fixed pair of parameters $e(F_u)$ and $act_G(F_u)$. Indeed, consider the monomial of the minimal degree in polynomial ($\oplus$), and present it in the form $s\cdot y^m$. Observe that s is the number of subforests F_u s.t. $F_u\equiv m\ (mod\ t)$ and $act_G(F_u)=\left[\frac{m}{t}\right]$. Remove from the polynomial $(**)$ all summands for these subforests, and repeat this operation until we get 0.

It is well known that $T_G(x,y)=\sum_{a,b}\#\{F_u : e(F_u)=a,\ act(F_u)=b\}\cdot(x-1)^{n-1-a}\cdot y^b$. Therefore we know the whole Tutte polynomial of G, since we know the number of usual subforests with any fixed number of edges and any fixed external activity. $\square$

3. **Algebras and matroids**

Obviously, the original Postnikov-Shapiro algebra as well as our t-algebras corresponding to a disconnected graph G are the Cartesian products of the algebras corresponding to the connected components of G. In particular, it is also true for 2-connected components (maximal connected subgraphs such that they remain connected, after removal of any vertex). The same fact is also true for matroids. In this section we prove the follow result.

Theorem 5. *Algebras $\mathcal{B}^F_{G_1}$ and $\mathcal{B}^F_{G_2}$ for graphs G_1 and G_2 are isomorphic if and only if the graphical matroids of these graphs coincide.*

Results of the previous section together with this fact imply the following:

Corollary 1. (I) *Algebras $\mathcal{B}^{F_t}_{G_1}$ and $\mathcal{B}^{F_t}_{G_2}$ for graphs G_1 and G_2 are isomorphic if and only if the graphical matroids of these graphs are the same.*

(II) *For any graphs G_1 and G_2 and positive integers a and b, algebras $\mathcal{B}^{F_a}_{G_1}$ and $\mathcal{B}^{F_a}_{G_2}$ are isomorphic if and only if $\mathcal{B}^{F_b}_{G_1}$ and $\mathcal{B}^{F_b}_{G_2}$ are isomorphic.*

(III) *For any positive integer t, there is the unique Tutte polynomial corresponding to $\mathcal{B}^{F_t}_G$.*

In the proof of the above theorem we use the following theorem of H. Whitney.

Theorem 6 (Whitney's 2-isomorphism theorem, see [11] or [7])**.** *Let G_1 and G_2 be two graphs. Then their graphical matroids are isomorphic if and only if G_1 can be transformed to a graph, which is isomorphic to G_2 by a sequence of operations of vertex identification, cleaving and twisting.*

These three operation are defined below:
1a) *Identification*: Let v and v' be vertices from different connected components of the graph. We modify the graph by identifying v and v' as a new vertex v''.
1b) *Cleaving* (the inverse of identification): A graph can only be cleft at a cut-vertex or at a vertex incident with a loop.
2) *Twisting*: Suppose that the graph G is obtained from two disjoint graphs G_1 and G_2 by identifying vertices u_1 of G_1 and u_2 of G_2 as the vertex u of G and additionally identifying vertices v_1 of G_1 and v_2 of G_2 as the vertex v of G. In a twisting of G about $\{u, v\}$, we identify u_1 with v_2 and u_2 with v_1 to get a new graph G'.

We split our proof of Theorem 5 in two parts presented in § 3.1 and in § 3.2.

3.1. **Algebras are isomorphic if their matroids are isomorphic.** Because algebras $\mathcal{B}_G^F$ and $\mathcal{C}_G^F$ are isomorphic for any graph by Theorem 1, we can proof Theorem 5 for algebras $\mathcal{C}_{G_1}^F$ and $\mathcal{C}_{G_1}^F$.

Lemma 3. *If graphs G and G' differ by a sequence of Whitney's deformations, then the algebras $\mathcal{C}_G^F$ and $\mathcal{C}_{G'}^F$ are isomorphic.*

Proof. It is sufficient to check the claim for each deformation separately.

1° *Identification and Cleaving.* We need to prove our fact only for cleaving, because identification is the inverse of cleaving. In this case algebras doesn't change, because the linear subspace defined by X_i for vertices doesn't change. This holds, because if we split a vertex k into k' and k'', then in the new graph, $X_{k'}$ equals to the minus sum of X_i corresponding to the vertices from its component except k' (sum of $X_.$ from one connected component is zero), i.e. $X_{k'}$ belongs to the linear space $<X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n>$. Similarly $X_{k''}$ belongs to the linear space $<X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n>$. Hence, $<X_1, \dots, X_{k'}, X_{k''}, \dots, X_n>$ is a subspace of the linear space $<X_1, \dots, X_n>$. The equation, $X_n = X_{n'} + X_{n''}$ implies that these linear spaces coincide.

2° *Whitney's deformation of the second kind.* Define the digraph $\overline{G}$ as the orientation of the graph G, where each arrow goes to the "smaller" vertex (see Remark 1).

Let us make a twist of the vertices u and v. Let $\overline{G_1}$ and $\overline{G_2}$ be the orientations of G_1 and G_2 corresponding to $\overline{G}$, and $\overline{G'}$ be the orientation of G' corresponding to the gluing $\overline{G_1}$ and $\overline{G_2}$ with reversing each arrow from $\overline{G_2}$. Vertex u' in G' obtained by gluing of u_1 and v_2; v' is obtained by gluing of v_1 and u_2.

Let X_k, $X_{1,k}$, $X_{2,k}$ and X'_k be the sums of edges with signs of incident to vertex k in graphs $\overline{G}$, $\overline{G_1}$, $\overline{G_2}$ and $\overline{G'}$.
For a vertex k of G_1 except u_1 and v_1:
$X_k = X_{1,k} = X'_k$.
For a vertex k of G_2 except u_2 and v_2:
$X_k = X_{2,k} = -X'_k$, because we reverse the orientation in the second part of twisting.
For other vertices we have:
$X_u = X_{1,u_1} + X_{2,u_2}$;
$X_v = X_{1,v_1} + X_{2,v_2}$;
$X'_{u'} = X_{1,u_1} - X_{2,v_2} = X_u - (X_{2,u_2} + X_{2,v_2})$;
$X'_{v'} = X_{1,v_1} - X_{2,u_2} = X_v - (X_{2,u_2} + X_{2,v_2})$.
We know that the sum of variables corresponding to the vertices of any graph is zero, because each edge goes with plus to one vertex and with minus to another. We have

$$\sum_{k \in G_2} X_{2,k} = 0,$$

$$\sum_{k \in G_2 \setminus u_2, v_2} X_{2,k} + X_{2,u_2} + X_{2,v_2} = 0,$$

$$X_{2,u_2} + X_{2,v_2} = - \sum_{k \in G_2 \setminus \{u_2, v_2\}} X_{2,k} = - \sum_{k \in G_2 \setminus \{u_2, v_2\}} X_k.$$

Hence, $X'_{u'}$ and $X'_{v'}$ belong to the linear space generated by X_k, where $k \in G$. In other words, the linear space for $\overline{G'}$ is a linear subspace of the space for $\overline{G}$. Similarly we can prove the converse. Then, the linear spaces coincide, and since we have the same relations ($\phi_e^2 = 0$ for any edge), i.e. the algebras are same in these bases. □

We have proved the first part of Theorem 5, because if the corresponding graphical matroids are ismorphic, then there exist such a sequence of Whitney's operations.

Corollary 2. *The algebra corresponding to a graph G is the Cartesian product of the algebras corresponding to the 2-connected components of G.*

3.2. **Reconstructing matroid from the algebra.**

Lemma 4. *It is possible to reconstruct the matroid of a graph G from the algebra $\mathcal{C}_G^F$.*

Remark 2. *We know $\mathcal{C}_G^F$, and only it. In particular, we do not know the basis corresponding to the vertices of G, and we have no information about the graded components of $\mathcal{C}_G^F$.*

Proof. For an element $Y \in \mathcal{C}_G^F$, we define its length $\ell(Y)$ as the minimal number such that $Y^{\ell+1}$ is zero (the length can be infinite).

We call an element $Y \in \mathcal{C}_G^F$ irreducible if there is no representation $Y = \sum_i Z_{2i-1}Z_{2i}$ such that $\ell(Z_j)$ is finite for any j.

Consider a basis $\{Y_1, ..., Y_m\}$ of the algebra $\mathcal{C}_G^F$ with the following conditions:

- Each Y_j is irreducible;
- For any $k \leqslant m$, different nonzero numbers $r_1, \dots, r_k \in \mathbb{K}$, any $r'_1, \dots, r'_k \in \mathbb{K}$ and different $i_1, \dots, i_k \in [1, m]$, we have
$$\ell(r_1 Y_{i_1} + \dots + r_k Y_{i_k}) \geqslant \ell(r'_1 Y_{i_1} + \dots + r'_k Y_{i_k});$$
- For any linear combination Y of $\{Y_1, ..., Y_m\}$ and on reducible Z,
$$\ell(Y) \leqslant \ell(Y + Z);$$
- $\sum_i \ell(Y_i)$ is minimal.

Such a basis of $\mathcal{C}_G^F$ always exists. For example, the basis $X_1, \dots, X_n$ (corresponding to the vertices) satisfies to the first three conditions. However, the sum of lengths of X_i is not always minimal.

For an element $Y \in \mathcal{C}_G^F$, we define its linear part as $\overline{Y}$ and its remainder as $\widehat{Y}$, where $Y = \overline{Y} + \widehat{Y}$, where $\overline{Y}$ belongs to the 1-graded component of $\mathcal{C}_G^F$ and $\widehat{Y}$ belongs to the linear span of other graded components. Observe that we do not know this representation, because we do not know the 1-graded component. We say that an edge e belongs to $\overline{Y}$ if $\overline{Y}$ includes the variable ϕ_e corresponding to edge e with a

Proposition 7. *A basis $\{Y_1, ..., Y_m\}$ as above satisfies additionally the following conditions:*

1. *The set $\{\overline{Y_1}, ..., \overline{Y_m}\}$ is a basis;*
2. *For any linear combination Y of $\{Y_1, ..., Y_m\}$,*
$$\ell(Y) = \ell(\overline{Y});$$
3. *Each edge belong to one or two $\overline{Y_i}$ and $\overline{Y_j}$. If in two, then with opposite coefficients;*
4. *Each $\overline{Y_j}$ has edges only from one 2-connected component.*

Proof. **(1).** We know that $\{Y_1, ..., Y_m\}$ is a basis, hence, the 1-graded component of $\mathcal{C}_G^F$ is a linear space of $<\overline{Y_1}, ..., \overline{Y_m}>$. Then, $<\overline{Y_1}, ..., \overline{Y_m}>$ is also a basis.

(2). For any $i \in [1, m]$, the part $\widehat{Y_i}$ is reducible, otherwise $\ell(Y_i)$ is infinite, and the sum in last condition is infinite. However for the basis corresponding to the vertices this sum is finite. Then for any linear combination Y of $\{Y_1, ..., Y_m\}$, we have $\ell(Y) \leqslant \ell(\overline{Y}) < \infty$. However, it is clear that $\ell(Y) \geqslant \ell(\overline{Y})$. Hence, $\ell(Y) = \ell(\overline{Y})$ for any Y from the linear space $<Y_1, ..., Y_m>$.

(3). Obviously, each edge belongs to at least one $\overline{Y_j}$, because any edge belongs to some X_i and this X_i is a linear combination of $\overline{Y_1}, \ldots, \overline{Y_m}$.

Assume that there is an edge e, which is in $\overline{Y_{i_1}}$, $\overline{Y_{i_2}}$ and $\overline{Y_{i_3}}$. Then there are different nonzero numbers r_1, r_2 and r_3 such that, for $Y = r_1Y_1 + r_2Y_2 + r_3Y_3$, $\overline{Y}$ without e. Then $\ell(\overline{Y})$ is at most the number of different edges in $\overline{Y_{i_1}}$,$\overline{Y_{i_2}}$ and $\overline{Y_{i_3}}$ minus one.

Consider general r_1', r_2' and r_3', then, for $Y' = r_1'Y_1 + r_2'Y_2 + r_3'Y_3$, $\ell(\overline{Y'})$ is the number of different edges in $\overline{Y_{i_1}}$, $\overline{Y_{i_2}}$ and $\overline{Y_{i_3}}$. We have

$$\ell(\overline{r_1Y_1 + r_2Y_2 + r_3Y_3}) < \ell(\overline{r_1'Y_1 + r_2'Y_2 + r_3'Y_3}).$$

Using (2) we also have

$$\ell(r_1Y_1 + r_2Y_2 + r_3Y_3) < \ell(r_1'Y_1 + r_2'Y_2 + r_3'Y_3),$$

which contradicts with our choice of a basis.

We proved the first part of condition; the proof of the second part is the same, but only for two different r_1 and r_2.

(4). Assume the opposite, i.e. that there is $\overline{Y_i}$ which has edges belonging to two different 2-connected components.

We know that the algebra is the Cartesian product of the subalgebras corresponding to 2-connected components. Then there are Z_1 and Z_2 from the 1-graded component, such that $\overline{Y_i} = Z_1 + Z_2$ and $\ell(Z_1), \ell(Z_2) < \ell(\overline{Y_i})$.(For example, Z_1 is a part corresponding to some 2-connected component and Z_2 is another part).

Because Z_1 and Z_2 belong to the linear space $<\overline{Y_1}, ..., \overline{Y_m}>$, we can change $\overline{Y_i}$ to Z_1 or Z_2 in the basis $\{\overline{Y_1}, ..., \overline{Y_m}\}$. (Indeed if we can not do it, then Z_1 and Z_2 belong to $<\overline{Y_1}, \ldots \overline{Y_{i-1}}, \overline{Y_{i-1}}, \ldots \overline{Y_m}>$. Therefore$\overline{Y_i} = Z_1 + Z_2$ also belongs to the latter space). Let us change to Z_1.

We have a new basis $\{\overline{Y_1}, \ldots \overline{Y_{i-1}}, Z_1, \overline{Y_{i-1}}, \ldots \overline{Y_m}\}$, whose sum of lengths is less than the sum of lengths of $\{\overline{Y_1}, \ldots, \overline{Y_m}\}$, which equals to the sum of lengths of $\{Y_1, \ldots, Y_m\}$. And for basis $\{\overline{Y_1}, \ldots \overline{Y_{i-1}}, Z_1, \overline{Y_{i-1}}, \ldots \overline{Y_m}\}$, the first three conditions of choice basis are holds, and sum of lengths is less, then we should to choose this basis instead $\{Y_1, \ldots, Y_m\}$. Contradiction. □

Let us now construct the cut space of G. This will finish the proof, because we can define the graphical matroid in terms of the cut space of a graph. By a cut we mean a set C of edges such that the subgraph $G \setminus C$ has more connected components than G; by an elementary cut we mean a minimal cut, i.e. a cut, whose arbitrary subset is not a cut.

The sum $\overline{\sum 2^i Y_i} = \sum 2^i \overline{Y_i}$ has each edge with a nonzero coefficient by (2) of Proposition 7. Hence,

$$e(G) = \ell\left(\sum_{i=1}^{m} 2^i \overline{Y_i}\right) = \ell\left(\sum_{i=1}^{m} 2^i Y_i\right).$$

Therefore we know the number of edges in the graph.

Consider the set $\tau = \{\psi_1, \ldots, \psi_{e(G)}\}$ consisting of $e(G)$ elements and a family of subsets $K_1, \ldots, K_m$ constructed by the following rules.

- For each pair i and j, we choose $\frac{\ell(Y_i)+\ell(Y_j)-\ell(Y_i+Y_j)}{2}$ own elements from τ and add they to both Z_i and Z_j;
- For every i, we choose $\ell(Y_i) - \sum_{j\neq i}(\frac{\ell(Y_i)+\ell(Y_j)-\ell(Y_i+Y_j)}{2})$ own elements from τ and add they to Z_i.

In fact, for any edge e from G, we choose the corresponding element $\psi_{k(e)}$ and add it to Z_i if and only if e belongs to $\overline{Y_i}$.

Consider the space Γ of subsets in τ with the operation Δ (symmetric difference) generated by $Z_1, \ldots, Z_m$. We want to prove that Γ is isomorphic to the cut space of G.

Let C be an elementary cut of G. There is X_C in 1-graded components which equals to the sum of X_i corresponding to the vertices, which belong to some new component of $G \setminus C$. Then X_C has an edge with nonzero coefficient if and only if the edge belong to C. Consider the minimal t such that there is a linear combination $X_C = a_1\overline{Y_{i_1}} + \ldots + a_t\overline{Y_{i_t}}$; consider the sum $X'_C = \overline{Y_{i_1}} + \ldots + \overline{Y_{i_t}}$. Obviously, X'_C is nonzero and has edges only from the cut C, because an edge belongs to sum X'_C if and only if it is exactly in one of $\overline{Y_{i_1}}, \ldots, \overline{Y_{i_t}}$.

Assume that X'_C has not all edges from X_C. Let C' be a subset of the edges of C belonging to X'_C. The set of edges C' is not a cut of G, because C is an elementary cut. Hence, for any edge $e \in C'$, there is a path $e_1, \ldots, e_k$ in $G\setminus C'$, such that $e, e_1, \ldots, e_k$ is a simple cycle. Let the edge e connect vertices b_k and b_0, and the edge e_i connect b_{i-1} and b_i for any $i \in [1, k]$. Because X'_C belongs to the 1-graded component, then $X'_C = \sum_{i=1}^n a_i X_i$, where $a_i \in \mathbb{K}$ and X_i are elements corresponding to vertices of G. For $i \in [1, k]$, the edge e_i does not belong to X'_C, however belongs only to variables $X_{b_{i-1}}$ and X_{b_i} from $\{X_1, \ldots, X_n\}$. Furthermore to one with coefficient 1 and to another with -1, hence, $a_{b_{i-1}} = a_{b_i}$. Then, we also have $a_{b_0} = a_{b_k}$, hence the variable ϕ_e is in X'_C with a zero coefficient. Contradiction.

We concluded that a subset of τ corresponding to an elementary cut belongs to the space Γ. To finish the proof, we need to show, that if a subset of τ belongs to Γ, that it either corresponds to a cut or to the empty set.

Assume the contrary, i.e. assume that there is $Z_{i_1}\Delta Z_{i_2}\Delta \cdots Z_{i_s}$ not belonging to a cut. Let C be a set of edges from $\overline{Y_{i_1}} + \overline{Y_{i_2}} + \cdots \overline{Y_{i_s}}$, then C is not a cut in G. By Proposition 7 we can split the summation into the summations in individual connected components (and even in 2-connected component). Then, for any connected component it is not a cut.

Let $\overline{Y_{i_{j_1}}}+\overline{Y_{i_{j_2}}}+\cdots\overline{Y_{i_{j_r}}}$ correspond to a connected component G', and has the set of edges C'. Then we have

$$\overline{Y_{i_{j_1}}}+\overline{Y_{i_{j_2}}}+\cdots\overline{Y_{i_{j_r}}}=\sum_{v\in V(G')} a_vX_v.$$

We know that $G'\setminus C'$ is connected. Therefore there is a spanning tree T in $G'\setminus C'$. For any edge v_iv_j from T, we have $a_{v_i}=a_{v_j}$, otherwise the edge v_iv_j belongs to $\sum_{v\in V(G')} a_vX_v$ with a nonzero coefficient. Since T is a spanning tree of G' then all coefficients a_v are the same. Thus $\sum_{v\in V(G')} a_vX_v = a(\sum_{v\in V(G')} X_v)=0$, the last is zero, because sum of variables corresponding to vertices from connected component is zero. Hence before splitting by components, we also have $\overline{Y_{i_1}}+\overline{Y_{i_2}}+\cdots\overline{Y_{i_s}}=0$, hence, $Z_{i_1}\Delta Z_{i_2}\Delta\cdots Z_{i_s}$ is the empty set.

Therefore the space Γ is isomorphic to the cut space of G, i.e. there is a unique possible graphical matroid corresponding to $\mathcal{C}_G^F$. □

4. Vector configurations and hypergraphs

4.1. **Algebras corresponding to vector configuration.** The following algebra was introduced by A. Postnikov, B. Shapiro and M. Shapiro in [9].

Notation 4. *Given a finite set A of vectors $a_1,\dots,a_m$ in $\mathbb{K}^n$, let Φ_m^F be the algebra over $\mathbb{K}$ generated by $\{\phi_i:\ i\in[1,m]\}$ with relations $\phi_i^2=0$, for each $i\in[1,m]$.*

For $i=1,\dots,n$, set $X_i=\sum_{k\in[1,m]} a_{k,i}\phi_k$. Denote by $\mathcal{C}_A^F$ the subalgebra of Φ_m^F generated by $X_1,\dots,X_n$.

The Hilbert series of $\mathcal{C}_A^F$ also corresponds to a specialization of the Tutte polynomial of the corresponding vector matroid, see [9].

The number of different vector configuration is uncountable. Furthermore there are uncountably many non-isomorphic algebras corresponding to vector configuration. However the number of matroids is countable. Furthermore it is finite for a fixed number of vectors. It means that there are at least two different vector configurations with the same corresponding matroid, i.e. it is impossible to reconstruct such a vector configuration and algebra from the corresponding matroid.

4.2. **Family of algebras for a hypergraph.**

Notation 5. *Given a hypergraph H on n vertices, let us associate commuting variables $\phi_e, e\in H$ to all edges of H.*

Set Φ_G^H to be the algebra generated by $\{\phi_e:\ e\in H\}$ with relations $\phi_e^2=0$, for any $e\in H$.

Let $C=\{c_{i,e}\in\mathbb{K}: i\in[1,n],\ e\in E(H)\}$ be the set of parameters, such that $c_{i,e}=0$ for vertices non-incident to e, and $\sum_{i=1}^n c_{i,e}=0$ for any edge $e\in E(H)$.

For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in H} c_{i,e} \phi_e,$$

Denote by $\widehat{\mathcal{C}}_H(C)$ the subalgebra of Φ_G^F generated by $X_1, \ldots, X_n$, and denote by $\widehat{\mathcal{C}}_H$ the set of such subalgebras.

Proposition 8. (I) *For a hypergraph H, the dimension of the space of parameters is $\sum_{e \in E}(|e| - 1)$.*

(II) *Given set of parameters C and non-zero numbers a_e $e \in E$. Let C' be the set of parameters such that $c'_{i,e} = a_e c_{i,e}$ for any $i \in [1, n]$ and $e \in E$. Then the subalgebras for C and for C' are isomorphic.*

Proof. (I) For any edge $e \in E$, we have $|e|$ own parameters and only one relation.

(II) Change the variables $\phi'_e = a_e \phi_e$; this set of variables also generates the algebra Φ_H^F, because $a_e \neq 0$ for any edge. Now we write the subalgebra generated by the set C' in the new variables, i.e. $X_i = \sum_{e \in H} c'_{i,e} \phi_e = \sum_{e \in H} a_e c_{i,e} \phi_e = \sum_{e \in H} c_{i,e} \phi'_e,$. Relations in the variables ϕ'_e in Φ_G^F are $(\phi'_e)^2 = 0$. The set of relations is the same that for the set C of parameters in variables ϕ_e, hence, the subalgebras are isomorphic. □

Corollary 3. *For a usual graph G, almost all algebras from $\widehat{\mathcal{C}}_G$ are isomorphic to $\mathcal{C}_G$.*

We define the hypergraphical matroid using the definition of an independent set of edges of a hypergraph.

Definition 6. *Let H be a hypergraph on n vertices. A set K of edges is called independent if there is a set of vectors in $\mathbb{K}^n$ corresponding to these edges such that*

- *If a vertex is not in an edge $e \in K$, then the corresponding coordinate of the vector (cor. to e) is zero;*
- *The sum of coordinates in each vector is zero;*
- *Vectors corresponding to edges of K are linearly independent.*

In this definition, set of the vectors corresponding to all edges (for each edge exactly one vector) it is the following set of parameters C, where $(c_{1,e}, \ldots, c_{n,e})$ vector corresponding to an edge e.

Theorem 9. *For any hypergraph H, the hypergraphical matroid (defined above) is a matroid.*

Proof. Let B be the set of all maximal independent subsets of edges. We need to check that the properties of a matroid are satisfied.

B is not empty. This is trivial, because if our hypergraph is not empty, then it has edges. Any edge is independent, hence, there is maximal independent set which contains a given edge e.

Basis exchange property.

The space of coordinates of vectors is isomorphic to $\mathbb{K}^{\sum_{e\in E}(|e|-1)}$, see Proposition 8.

The set of parameters (or the same set of vectors corresponding to edges) corresponds to some point of $\mathbb{K}^{\sum_{e\in E}(|e|-1)}$. Let E' be some independent subset of edges. If there is a point in $\mathbb{K}^{\sum_{e\in E}(|e|-1)}$, such that the vectors corresponding to edges of E' are linearly independent, then these vectors are independent for points from some Zariski open subset of $\mathbb{K}^{\sum_{e\in E}(|e|-1)}$.

Introduce Z as the intersection of these Zariski open subsets corresponding to all independent sets of edges. Then Z is also a Zariski open subset of $\mathbb{K}^{\sum_{e\in E}(|e|-1)}$; vector matroids corresponding to points from Z are the same, and this generic matroid is our hypergraphical matroid. Hence, our definition is correct. □

Question 1. *What information about a hypergraph contains this matroid?*

Remark 3. *Condition that the sum of any edge is zero is very important. Otherwise: firstly, for usual graph, this definition isn't same that for a graphical matroid; and secondly it will give a smaller class, because we can add to a hypergraph a vertex* 0 *and add it to each edge, and our hypergraphical matroid for the latter hypergraph is same that for a former hypergraph without extra conditions.*

Theorem 10. *Almost all algebras from $\widehat{\mathcal{C}}_H$ have the same Hilbert series, and this Hilbert series is possible to restore from the Tutte polynomial of the corresponding hypergraphical matroid.*

Proof. For any set of parameters C, we can calculate the Hilbert series of $\widehat{\mathcal{C}}_H(C)$ from the Tutte polynomial of the vector matroid corresponding to C. For almost all set of parameters, vector matroids are the same, and this matroid is our hypergraphical matroid. Hence, we can calculate the generic Hilbert series of almost all set of parameters from the Tutte polynomial of the hypergraphical matroid. □

Question 2. (I) *Let two hypergraphs H_1 and H_2 have the same hypergraphical matroid. Is it true that for a generic set of parameters C_1 for H_1, there is a set of parameters C_2, such that subalgebras $\widehat{\mathcal{C}}_{H_1}(C_1)$ and $\widehat{\mathcal{C}}_{H_2}(C_2)$ are isomorphic?*

(II) *And conversely, assume that for a generic set of parameters C_1 for H_1, there is a set of parameters C_2, such that subalgebras $\widehat{\mathcal{C}}_{H_1}(C_1)$ and $\widehat{\mathcal{C}}_{H_2}(C_2)$ are isomorphic. Is it true that H_1 and H_2 have the same hypergraphical matroid?*

5. **Algebras corresponding to spanning trees**

In this section we discuss analogous algebras counting spanning trees. Recall the definition for algebras $\mathcal{B}_G^T$ and $\mathcal{C}_G^T$ borrowed from [8].

Notation 7. *Take an undirected graph G with n vertices.*

(I) *Let Φ_G^T be the algebra over $\mathbb{K}$ generated by $\{\phi_e : e \in G\}$ with relations $\phi_e^2 = 0$, for any $e \in G$, and $\prod_{e\in H} \phi_e = 0$, for any $H \subset E(G)$ such that $G \setminus H$ is disconnected.*

Fix a linear order of vertices of G. For $i = 1, \dots, n$, set

$$X_i = \sum_{e\in E(G)} c_{i,e}\phi_e,$$

where $c_{i,e} = \pm 1$ for vertices incident to e (for the smaller vertex v_i, $c_{i,e} = 1$, for the larger vertex v_j, $c_{j,e} = -1$) and 0 otherwise. Denote by $\mathcal{C}_G^T$ the subalgebra of Φ_G^T generated by $X_1, \dots, X_n$.

(II) *Consider the ideal J_G^T in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by*

$$p_I^T = \left(\sum_{i\in I} x_i\right)^{D_I},$$

where I ranges over all nonempty subsets of vertices, and D_I is the total number of edges between vertices in I and vertices outside I. Define the algebra $\mathcal{B}_G^T$ as the quotient $\mathbb{K}[x_1, \dots, x_n]/J_G^T$.

The case, when the graph G is disconnected, is not interesting, because both algebras are trivial. In paper [8] the following result was proved:

Theorem 11 (cf. [8])**.** *For any graph G, algebras $\mathcal{B}_G^T$ and $\mathcal{C}_G^T$ are isomorphic, their total dimension over $\mathbb{K}$ is equal to the number of spanning trees in G.*

Moreover, the dimension of the k-th graded component of these algebras equals the number of spanning trees of G with external activity $e(G) - v(G) + 1 - k$.

5.1. *t*-**labelled trees.**

It is possible to introduce similar algebras which enumerate t-labelled trees, but it is not very interesting. Let $\mathcal{B}_G^{T_t}$ be an algebra in which we change generators of the ideal to $(\sum_{i\in I} x_i)^{tD_I}$. The definition of $\mathcal{C}_G^{T_t}$ will change in a more complicated way.

However, there is no result about a reconstruction of the Tutte polynomial from the Hilbert series, because all trees have same number of edges and then $HP(\mathcal{B}_G^{T_t}, x) = (1 + x)^{n-1} HP(\mathcal{B}_G^T, x^t)$, where HP is Hilbert polynomials and n is number of vertices. Then, Hilbert series for $\mathcal{B}_G^{T_t}$ and for $\mathcal{B}_G^T$ have the same information about the graph.

5.2. **Algebras and matroids.**

Theorem 12. *Algebras $\mathcal{B}^T_{G_1}$ and $\mathcal{B}^T_{G_2}$ for connected graphs G_1 and G_2 are isomorphic if and only if the graphical matroids of these graphs coincide.*

Proof. (I) *If the graphical matroids are the same, then the algebras are isomorphic.* In Theorem 5 we defined orientations on edges so that 1-graded components were the same, and as a consequence these algebras were the same. For the first Whitney's deformations, we did not to change orientations on edges; for second, we needed to change all arrows on edges in the second part of twisting.

Let $\overline{G_1}$ and $\overline{G_2}$ be these orientations. We know that $\Phi^F_{G_1}$ and $\Phi^F_{G_2}$ are the same, and with these orientation the algebras $\mathcal{C}^F_{G_1}$ and $\mathcal{C}^F_{G_2}$ are the same as subalgebras of $\Phi^F_{G_i}$.

Let I be the ideal generated by products of edges from cuts of G_1 in $\Phi^F_{G_1}$. Because the variables on edges in G_1 and G_2 are the same and C is a cut in G_1 if and only if C is a cut in G_2, then I is also the ideal generated by cuts of G_2.

Thus $\Phi^T_{G_1} = \Phi^F_{G_1}/I$, hence,

$$\mathcal{C}^T_{G_1} = \mathcal{C}^F_{G_1}/I,$$

similarly

$$\mathcal{C}^T_{G_2} = \mathcal{C}^F_{G_2}/I.$$

It means that the algebras $\mathcal{C}^T_{G_1}$ and $\mathcal{C}^T_{G_2}$ are also the same in orientations $\overline{G_1}$ and $\overline{G_2}$.

(II) *For an algebra $\mathcal{C}^T_G$, there is a unique graphical matroid.* The proof is the same, because, in fact, we worked with the 1-graded component. However, we need to change our definition of lengths. For an element $Y \in \mathcal{C}^T_G$, we define the length $\ell(Y)$ as the minimal number such that Y^ℓ is zero (the length can be infinite). The rest of the proof goes without change. □

Remark 4. *Connectivity of graphs is important, because for disconnected graphs, algebras counting trees are trivial.*

5.3. **Algebras for hypergraphs.** There is a problem with a definition of algebra counting trees in this case.

For a usual graph its matroid stores cuts; in particular we can define the algebra $\mathcal{B}^T_G$ from its graphical matroid if we know that a graph is connected.

In some sense our hypergraphical matroid also store "cuts". These cuts are in matroid terms, but not in usual sense. Add condition that the product of edges of any any such cut is zero. But we still need a condition that hypergraph to be connected (otherwise, for usual graph, it is not the same); in combinatorial sense a hypergraph with one edge, which contains all vertices, is connected, however, there is no independent set with $n - 1$ vertices. In particular for a definition of this algebra we need to construct something similar to G-parking functions, see appendix 6 below.

6. Appendix: G-parking functions

Definition 8. *For a graph (directed graph) G on $n+1$ vertices, its G-parking function is a tuple $(b_0, b_1, \dots, b_n)$, where $b_0 = -1$ and $b_i \in \mathbb{N}_0$ such that, for any subset I of vertices, there is a vertex $j \in I$ whose outdegree to $[0, n] \setminus I$ is less than b_j.*

The following algorithm give a bijection between G-parking functions and rooted trees with a root at the vertex 0:

Let $(b_0, b_1, \dots, b_n)$ be a G-parking function. Construct the tree by using the following algorithm: zero level $T_0 = 0$ is vertex 0; we define next levels by the following rules:

- Consider the set $A_k = [0, n] \setminus (T_0 \cup .. \cup T_{k-1})$, subset T_k consists of all vertices of this subset s.t. the number of outcoming edges from a vertex of A_k to the complement of A_k is less than the number in the G-parking function corresponding to this vertex.
- Our rooted tree has the following edges between levels T_{k-1} and T_k: an edge from a vertex v of T_k to m-th (in ascending) vertex of $T_{k-1} \cap N_v$, where $m = b_v - |\,(T_0 \cup .. \cup T_{k-1}) \cap N_v|$.

Why is this algorithm working? The set T_k isn't empty, because $(b_0, b_1, \dots, b_n)$ is a G-parking function. And m is correct, because $b_v \geq |\,(T_0 \cup .. \cup T_{k-1}) \cap N_v|$ and $b_v < |\,(T_0 \cup .. \cup T_k) \cap N_v|$.

Why is it an injection? Obviously our algorithm gives different result for different G-parking functions.

Why is it a surjection? We can reverse the algorithm. Consider a tree T, let $(b_0, b_1, \dots, b_n)$ correspond to the tree T. We must prove that this sequence is a G-parking function. We assume the contrary, i.e. that there is subset I of $[1, n]$, s.t. the condition of G-parking functions is violated. Consider the first moment when we choose a vertex from I, hence there is $I' \supset I$ and $z \in I$ s.t. the number of edges from z to $\bar{I}'$ is larger than b_z, then the number of edges from z to $\bar{I}$ is also larger than b_z, which is a contradiction.

Question 3. *Is there an analog of G-parking functions for vector matroids or hypergraphs? Is it possible to define them in terms of subsets of cuts, i.e. such that the first type conditions are conditions of a G-parking function, and other conditions are some coordination between cuts?*

Usual G-parking functions were defined in terms of sets of vertices, in particular, in terms of cuts. However, for some hypergraph there are cuts, which say that there is no spanning trees without edges from a given cut, but in combinatorial sense the hypergraph is still connected.

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Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden,

E-mail address: nenashev@math.su.se