LEFSCHETZ PROPERTIES OF GRADED ALGEBRAS

The background for the name is the hard Lefschetz theorem: Let X be an n-dimensional non-singular complex projective variety in P^N. Then in the cohomology ring $H^*(X)$ of X, the k-fold product $[L]^k\cdot$ with the cohomology class $[L]$ (which has degree 2), of a hyperplane gives an isomorphism between H^{n-k} and H^{n+k}.

1. Definitions

In the sequel $S = \mathbb{C}[x_1, \ldots, x_n]$, I a homogeneous ideal, and $A = S/I$. Let f be homogeneous of degree d. The multiplication $A \xrightarrow{f\cdot} A$ maps A_k to A_{k+d}. Suppose this is injective for all k. Then

$$0 \longrightarrow A \xrightarrow{f\cdot} A \longrightarrow A/f \longrightarrow 0$$

is exact. Thus $\dim A_k - \dim A_{k+d} + \dim(A/f)_{d+k} = 0$. Multiply with t^{d+k} and sum for all $k \geq -d$. This gives $(A/f)(t) = (1 - t^d)A(t)$ (where $R(t)$ is the Hilbert series of R). That the multiplication is injective means that f is a NZD. If $f_1, \ldots, f_2$ is an $S = \mathbb{C}[x_1, \ldots, x_n]$-sequnce with $\deg(f_i) = d_i$, then the complete intersection $A = S/(f_1, \ldots, f_s)$ has $A(t) = \prod_{i=1}^{s}(1 - t^{d_i})/(1 - t)^n$. If A is Artinian, there are no NZD's. The best one can hope for is that the multiplication is either injective or surjective (has maximal rank) for all k. In the sequel A is Artinian.

Definitions A has the weak Lefschetz property (WLP) if the multiplication with a generic linear form is injective or surjective. A has the strong Lefschetz property (SLP) if the multiplication with the k'th power of a generic linear form is injective or surjective for all k. A has the general maximal rank property (MRP) if the multiplication with a generic form of degree k is injective or surjective for all k. A has the semistrong Lefschetz property (SSLP) if the multiplication with a product with k generic linear form is injective or surjective for all k.

Remark The last is not standard, it is my suggestion. We have SLP $\Longrightarrow$ SSLP $\Longrightarrow$ GLP $\Longrightarrow$ WLP.

2. Results

First some helpful results.

Definitions A is level if the socle lies in only one degree. A sequence $(h_i)_{i=1}^{N}$ is unimodal if $h_0 \leq h_1 \leq \cdots \leq h_k \geq h_{k+1} \geq \cdots \geq h_N$ for some k.

Proposition Let A be an artinian standard graded algebra, and let l be a general linear form. Consider the homomorphisms $\phi_d \colon A_d \to A_{d+1}$ defined by multiplication by l, for $d \geq 0$.

(a) If ϕ_{d_0} is surjective for some d_0, then ϕ_d is surjective for all $d \geq d_0$.

(b) If A is level and ϕ_{d_0} is injective for some $d_0 \geq 0$, then ϕ_d is injective for all $d \leq d_0$.

(c) In particular, if A is level and $\dim A_{d_0} = \dim A_{d_0+1}$ for some d_0 then A has the WLP if and only if ϕ_{d_0} is injective (and hence is an isomorphism).

(d) If A has the WLP property, then the Hilbert function is unimodal.

Proof For (b) consider the canonical module.

Proposition Every Artinian ideal in $\mathbb{C}[x, y]$ has the Strong Lefschetz property.

Proof This uses gin.

Theorem Every monomial complete intersection satisfies SLP.

Every complete intersection in 3 variables satifies WLP.

Example An example of an ideal with the Weak Lefschetz property but not the Strong Lefschetz property:

Let I be the lex-segment ideal with generators $x_1^2, x_1x_2, x_1x_3^2, x_2^3, x_2^2x_3^2, x_2x_3^3, x_3^5$. This has Hilbert function (1,3,4,3,1), and one can check that for multiplication by a general linear form L we have maximal rank between consecutive components, while L^2 has the element x_1 in the kernel of the multiplication from degree 1 to degree 3.

Question Do all artinian complete intersections have the WLP or the SLP?

A Gorenstein ideals with the Weak Lefschetz property but not the Strong Lefschetz property. One uses the theory of inverse systems.

Example Let A be the ring $\mathbb{C}[u, v, x, y, z]$ and let $f = u(xu^2 + yuv + zv^2)$. The dual of f gives a Gorenstein algebra with Hilbert function (1,5,6,5,1) (this can be checked, for instance, with the computer program Macaulay2 using the script <l_from_dual). This algebra has the Weak Lefschetz property but not the Strong Lefschetz property.

Conjecture Any artinian Gorenstein algebra presented by quadrics, has the WLP.

Definition An O-sequence is a sequence of positive integers that occurs as the Hilbert function of some graded algebra.

Proposition Let $h = (1, h_1, h_2, ..., h_s)$ be a finite sequence of positive integers. Then h is the Hilbert function of a graded artinian algebra with the WLP if and only if the positive part of the first difference is an O-sequence and after that the first difference is non-positive until h reaches 0. Furthermore, this is also a necessary and sufficient condition for h to be the Hilbert function of a graded artinian algebra with the SLP.

Definition 2 Let n and d be positive integers. The d-binomial expansion of n is $n = n_{(d)} = \binom{n_d}{d} + \binom{n_{d-1}}{d-1} + \cdots + \binom{n_j}{j}$, where $n_d > n_{d1} > \cdots > n_j \geq j \geq 1$. This is unique. Also, define, for a and b integers, $n = (n_{(d)})^a_b = \binom{n_d+a}{d+b} + \binom{n_{d-1}+a}{d-1+b} + \cdots + \binom{n_j+a}{j+b}$.

Theorem Let $H = 1, h_1 = r, h_2, \ldots h_e, h_{e+1} = 0$ be an O-sequence, and let t be the smallest integer such that $h_t \leq t$. Then all the artinian algebras having Hilbert function H enjoy SLP, MRP, WLP if and only if:

i) $r = 2$; or

ii) $r > 2$, $h_t \leq 2$, and, for all indices $i = 1, 2, \dots, t-1$, $h_{i-1} = ((h_i)_i)^{-1}_{-1}$.

Question Does a general artinian reduction of a reduced, arithmetically Gorenstein set of points in P_n have the WLP?

Theorem Let $A = k[x, y, z]/I$, $I = (L_1^{a_1}, \dots, L_m^{a_m})$ be any ideal generated by powers of linear forms. Then A has the WLP.

Example If $I = (L_1^3, L_2^3, L_3^3, L_4^3)$ (where L_i is general in $\mathbb{C}[x, y, z]$), then $(\cdot l^3)$ fails to have maximal rank.

Example Consider the ideal $I = (x_1^N, x_2^N, x_3^N, x_4^N, L^N)$ for a general linear form L in $\mathbb{C}[x_1, x_2, x_3, x_4]$. By computation using CoCoA, S/I fails the WLP, for $N = 3, ..., 12$.

Problems 1. Prove the failure of the WLP in the Example for all $N \geq 3$.
2. What happens for mixed powers?
3. What happens for almost complete intersections, that is, for $r+1$ powers of general linear forms in r variables when $r > 4$?
4. What about more than $r+1$ powers of general linear forms?

Theorem Let $I = (L_1^t, \dots, L_n^t)$ in $\mathbb{C}[x_1, x_2, x_3, x_4]$, with L_i linear generic. If $n \in \{5, 6, 7, 8\}$, then the WLP fails, respectively, for $t \geq \{3, 27, 140, 704\}$.

Theorem Let $I = (L_1^t, \dots, L_{2k+1}^t)$ in $S = \mathbb{C}[x_1, \dots, x_{2k}]$ with L_i generic linear forms and $k \geq 2$. Then S/I fails the WLP if and only if $t > 1$.

Theorem Let $I = (L_1^t, \dots, L_8^t)$ in $\mathbb{C}[x_1, \dots, x_7]$, where $L_1, \dots, L_8$ are general linear forms.
If $t = 2$, then S/I has the WLP.
If $t \geq 4$, then S/I fails the WLP.

Theorem Let L be a general linear form and let $I = \mathbb{C}[x_1.x_2, x_3, x_4, x_5]/(x_1^d, x_2^d, x_3^d, x_4^d, x_5^d, L^d)$. Then the ring S/I fails the WLP if and only if $d > 3$.

Conjecture For $I = (L_1^t, \dots, L_n^t)$ in $\mathbb{C}[x_1, ..., x_r]$ with L_i linear generic and $n \geq r+1 \geq 5$, the WLP fails for all $t >> 0$.

Conjecture Let $S = k[x_1, \dots, x_{2n+1}]$. Let $L_1, \dots, L_{2n+2}$ be general linear forms and $I = (L_1^d, \dots, L_{2n+1}^d, L_{2n+2}^d)$.
If $n = 3$ and $d = 3$, then S/I fails the WLP.
If $n \geq 4$, then S/I fails the WLP if and only if $d > 1$.

Conjecture G Let $g_1, \dots, g_s$ be generic forms of degrees $d_1, \dots, d_s$ in $S = \mathbb{C}[x_1, \dots, x_n]$. Then the Hilbert series of $S/(g_1, \dots, g_s)$ is $\lceil \prod_{i=1}^s (1 - t^{d_i})/(1-t)^n \rceil$, where $\lceil , \rceil$ means that one truncates the series as soon as a coefficient is ≤ 0. (This is equivalent to the fact that all $S/(g_1, \dots, g_s)$ satisfies MRP.)

Proposition (a) If Conjecture G is true for all ideals generated by general forms in r variables, then all ideals generated by general forms in $r+1$ variables have the WLP.

(b) Let $S = \mathbb{C}[x_1, \ldots, x_{r+1}]$, let l be a general linear form and let $A = S/(l) \equiv \mathbb{C}[x_1, \ldots, x_r]$. Fix positive integers $s, d_1, ..., d_s, d_{s+1}$. Let $L_1, L_2, \ldots, L_s, L_{s+1}$ be linear forms.

(i) The ideal $I = (L_1^{d_1}, \ldots, L_s^{d_s})$ has the WLP.

ii) The multiplication $\cdot L^{d_{s+1}}$ has maximal rank

Then $S/(L^{d_1}, \ldots, L^{d_{s+1}})$ has the WLP.

Corollary Let $S = \mathbb{C}[x_1, \ldots, x_{r+1}]$, let l be a general linear form and let $A = S/(l) \equiv \mathbb{C}[x_1, \ldots, x_r]$. For integers $d_1, \ldots, d_{r+2}$, if an ideal of the form $(L_1^{d_1}, \ldots, L_{r+2}^{d_{r+2}})$, L_i general linear, fails to have WRP, then the ideal $(\bar{L}_1^{d_1}, \ldots, \bar{L}_{r+2}^{d_{r+2}})$ fails to satisfy Conjecture G.

Every generic complete intersection satisfies SLP. (This follows from the monomial case.)

Theorem Every ideal generated by generic forms in 4 variables satisfies WLP.

Example $\mathbb{C}[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3, (x_1x_2 + x_3)^3)$ satisfies MRP but not SLP.

Proposition Let $F_1, \ldots, F_r$ be general forms of degree d in $S = \mathbb{C}[x_1, \ldots, x_n]$. Set $A = S/(F1, \ldots, F_r)$. Assume $\binom{d+t_0+2}{d+t_0} - r\binom{t_0+2}{t_0} \geq 0$. Then, $\dim_{\mathbb{C}}\{RF_j\}_{j=1,\ldots,r}$ spans a vector space of maximum dimension for all $t \leq t_0$.

Questions Is SSLP equivalent to SLP (very unlikely)? Is SSLP equivalent to MRP (less unlikely)?

Something about Betti numbers.

Two interesting articles (more geometric) are:

Ordinary Curves, Webs and the Ubiquity of the Weak Lefschetz Property by Rosa M. Miro -Roig

Laplace Equations and the Weak Lefschetz Property by Emilia Mezzetti, Rosa M. Miro -Roig, and Giorgio Ottaviani

I have mainly used: JOURNAL OF COMMUTATIVE ALGEBRA Volume 5, Number 3, Fall 2013 SURVEY ARTICLE: A TOUR OF THE WEAK AND STRONG LEFSCHETZ PROPERTIES BY JUAN MIGLIORE AND UWE NAGEL and the references in there.