INEQUALITIES FOR HILBERT FUNCTIONS

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Let $S_n = \mathbb{C}[x_1, \ldots, x_n]$, $T_n = S_n/(x_1^2, x_2^2, \ldots, x_n^2)$, $E_n = \mathbb{C}\langle x_1, \ldots, x_n\rangle/(x_ix_j + x_ix_j, x_i^2)$.

1. Homogeneous ideals in S_n

Let $R = S_n/I$, I homogeneous, let $h_i = \dim_{\mathbb{C}} R_i$. Given h_d, we want an upper bound for h_{d+1}.

Let $m, d > 0$. Write $m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ with $k_d > k_{d-1} > \cdots > k_1 \geq 0$. There is a unique way to do this.

Example $d = 3, m = 28$. $28 = \binom{6}{3} + \binom{4}{2} + \binom{2}{1}$.

For $m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$, let $m^{\langle d\rangle} = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d-1+1} + \cdots + \binom{k_1+1}{1+1}$, so $28^{\langle 3\rangle} = \binom{7}{4} + \binom{5}{3} + \binom{3}{2} = 48$.

Theorem 1 (Macaulay)**.** $h_{d+1} \leq h_d^{\langle d\rangle}$.

Let M be a set of monomials of degree d. Order the monomials lexicographically with $x_1 < x_2 < \cdots < x_n$. M is called a lexsegment if $v \in M, u > v \implies u \in M$. It is easy to see that an ideal generated by a lexsegment in degree d generates a lexsegment in degree $d + 1$.

It is natural that if we want an ideal generated by monomials of degree d which generates as little as possible, we should choose a lexsegment ideal. For a lexsegment ideal of degree d we have equality in the theorem.

If M is a lexsegment of degree 3 in S_5 such that $h_3(S_5/(M)) = 28$, then $\#M = \binom{7}{3} - 28 = 7$, so $M = \{x_5^3, x_5^2x_4, x_5^2x_3, x_5^2x_2, x_5^2x_1, x_5x_4^2, x_5x_4x_3\}$. The nonzero monomials of degree 3 in $\mathbb{C}[x_1, x_2, x_3, x_4]$ are $\binom{6}{3}$. The nonzero monomials in $x_5\mathbb{C}[x_1, x_2, x_3]$ are $\binom{4}{2}$. The nonzero ideals in $x_5x_4\mathbb{C}[x_1, x_2]$ are $\binom{2}{1}$. M generates the lexsegment $\{x_5^4, \ldots, x_5x_4x_3x_1\}$ in degree 4. The number of nonzero monomials of degree 4 in $\mathbb{C}[x_1, \ldots, x_4]$ are $\binom{7}{4}$, thoose in $x_5\mathbb{C}[x_1, x_2, x_3]$ are $\binom{5}{3}$, thoose in $x_5x_4\mathbb{C}[x_1, x_2]$ are $\binom{3}{2}$.

In this way one can prove that there is equality for lexsegment ideals. The theorem is proved in general in [1]. They use a theorem by Green. For $m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ let $m_{\langle d\rangle} = \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_1-1}{1}$.

Theorem 2 (Green)**.** *Let R be a homogeneous $\mathbb{C}$-algebra, and let $d \geq 1$ be an integer. Then $h_d(R/hR) \leq h_d(R)_{\langle d\rangle}$ for a general linear form h.*

Macaulay's theorem is extended in Gotzmann's persistence theorem.

Theorem 3. *If for $R = S_n/I$ we have equality in Macaulay's theorem between degrees d and $d+1$, and if I has no generator of degree $> d+1$, then we have equality for all higher degrees.*

2. Homogeneous ideals in E_n

Let Δ be a simplicial complex, and let f_i be the number of i-dimensional faces in Δ.

If Δ is the 2-dimensional simplicial complex with maximal faces $\{x_1, x_2, x_3\}$ and $\{x_3, x_4\}$, then $f_{-1} = 1, f_0 = 4, f_1 = 4, f_2 = 1$. Consider the ideal I_Δ in E_n generated by all non-faces of Δ. In the example $I_\Delta = (x_1x_4, x_2x_4)$. It is clear that the Hilbert series of E_n/I_Δ (called the indicator algebra) is $\sum f_i t^{i+1}$. (This resembles the Stanley-Reisner algebra of Δ, which is defined as $\mathbb{C}[\Delta] = S_n/I_\Delta$. If the Hilbert series of $\mathbb{C}[\Delta]$ is $g(t)$ and the Hilbert series of the indicator algebra of Δ is $h(t)$, then $k(t) = h(t/(1-t))$.

For $m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ let ${}_{\langle d\rangle}m = \binom{k_d}{d-1} + \binom{k_{d-1}}{d-1-1} + \cdots + \binom{k_1}{1-1}$.
For $m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ let ${}^{\langle d\rangle}m = \binom{k_d+1}{d} + \binom{k_{d-1}+1}{d-1} + \cdots + \binom{k_1+1}{1}$.
The following is proved in [2].

Theorem 4. *(1) If $(f_{-1}, f_0, \ldots, f_{d-1})$ is the f-vector of a pure (all maximal faces have the same dimension) simplicial complex, then ${}_{\langle i\rangle}f_i \le f_{i-1}$ for $1 \le i \le d-1$.*

(2) If ${}_{\langle i\rangle}f_i = f_{i-1}$ for some i in a pure simplicial complex, then ${}_{\langle j\rangle}f_j = f_{j-1}$ for all $1 \le j \le i$.

(3) If $1 + \sum_0^n h_i t^i$ is the Hilbert series of a graded $\mathbb{C}$-algebra E_n/I, then $h_{i+1} \le {}^{\langle i\rangle}h_i$ for $0 < i \le n-1$.

(4) If $h_{i+1} = {}^{\langle i\rangle}h_i$ for some i, and I has no generator of degree $> i+1$, then $h_{j+1} = {}^{\langle j\rangle}h_j$ if $j \ge i$.

(1) was proved by Kruskal and Katona independently.

3. Generic forms in T_n and E_n

T_n and E_n are isomorphic as graded vector spaces, but not as rings. (In E_n every odd element has square 0.)

3.1. One generic form in T_n.

Theorem 5. *The Hilbert series of $T_n/(f)$, f generic of degree d is $[(1-t^d)(1+t)^n]$. (Take only terms as long as they are positive.)*

Proof It suffices to get one example since we have an equality and the generic case is the worst. Take the sum of all squarefree monomials of degree d. Let the squarefree monomials $\{m_i\}$ of degree $i-d$ denote the rows, and the squarefree monomials $\{n_i\}$ of degree i denote the columns in a matrix. The multiplication matrix then is an incidence matrix, in place (j,k) there is a 1 if $m_j | n_k$ and 0 otherwise. That this matrix has full rank is well-known.

3.2. **One generic form of even degree in E_n.** It is proved in [3] that the same formula as for T_n is true.

3.3. **One generic form of odd degree in E_n.** We have that

$$0 \longrightarrow \mathrm{Ann}(f)(-d) \longrightarrow E_n(-d) \xrightarrow{f\cdot} E_n \longrightarrow E_n/(f) \longrightarrow 0$$

is exact. It is clear that $(f) \subseteq \mathrm{Ann}(f)$. Sometimes the inclusion is strict, i.e. $(f) \neq \mathrm{Ann}(f)$. E.g. when $d = 3$ and $n = 16$ the differens in Hilbert series is $16t^9 + 120t^8 + 559t^7$. If $d > 3$ (and odd) the differens is much smaller. If $n - d = 0, 1, 3, 4, 5$, the differens seems to be 0. If $n - d = 2$, the differens seems to be t. If $n - d = 6$, the differens is t^3 if $d = 5, 9, 13$. If $n - d = 11$, $d = 5$ the differens is t^6.

Conjectures If $n - d = 0, 1, 3, 4, 5$, the differens is 0. If $n - d = 2$, the differens is t. If $n - d = 6$, the differens is t^3 if $d = 1 \bmod 4$. For fixed d, $\mathrm{Ann}(f) = (f)$ if $n >> 0$.

3.4. **Several generic forms.**

Theorem 6. *Consider the exterior algebra E_5 over a vector space with basis $\{e_1, ..., e_5\}$ (over any field). Let $f_1 = \sum c_{ij} e_i \wedge e_j$ and $f_2 = \sum d_{ij} e_i \wedge e_j$ be two forms in E_5. Then $\{e_i \wedge f_j\}$ is linearly dependent.*

Proof It suffices to prove the theorem for generic forms (so we can suppose that the c_{ij}'s and d_{ij}'s are algebraically independent over the prime field of k). I calculated a relation.

In [3] they calculate the Hilbert series for two quadratic forms in $n \leq 13$ variables. It differs from the "expected" series $(1 - t^2)^2(1 + t)^n$ with t^3 for $n = 5$, with t^4 for $n = 7, 8$, with t^5 for $n = 9$, with $10t^5$ for $n = 10$, with $t^5 + t^6$ for $n = 11$, with $64t^6$ for $n = 12$, and with $13t^6 + t^7$ for $n = 13$.

Conjectures There is no difference for two forms of degree 2 from the "expected" series $[(1 - t^2)^2(1 + t)^n]$ in degrees $< [(n + 1)/2]$. For two forms of even degree $d > 2$, the series is the "expected" $[(1 - t^d)^2(1 + t)^n]$.

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*.

[2] A. Aramova, J. Herzog, T. Hibi, *Gotzmann Theorems for Exterior Algebras and Combinatorics*, J. OF ALGEBRA **191**, 174–211, 1997.

[3] G. Moreno and J. Snellman, *SOME CONJECTURES ABOUT THE HILBERT SERIES OF GENERIC IDEALS IN THE EXTERIOR ALGEBRA*, Homology, Homotopy and Applications, vol.**4(2)**, 2002, pp.409–426.