GORENSTEIN ALGEBRAS

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1. Definition of Gorenstein

To begin with, we consider finite dimensional (thus Artinian) k-algebras A, k a field of characteristic 0, so $A = k \oplus A_1 \oplus \cdots \oplus A_s = k[x_1, \dots, x_n]/I$, I homogeneous.

Definition $\mathrm{Soc}(A) = \{f; fx_i = 0 \ \forall i\}$.

Definition A is Gorenstein if $\mathrm{Soc}(A)$ is 1-dimensional, so if $\mathrm{Soc}(A) = A_s$ and A_s is 1-dimensional.

V, W k-spaces, $B\colon V \times W \to k$ bilinear (so $V \otimes W \to k$ linear). B induces linear maps
$L\colon V \to W^*, v \mapsto (w \mapsto B(v,w))$
$R\colon W \to V^*, w \mapsto (v \mapsto B(v,w))$

Definition B is called a perfect (non-singular, non-degenerate) pairing if L and R are isomorphisms.

Proposition If V and W are finite dimensional with bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$, then B is a perfect pairing if and only if $\forall w, B(v,w) = 0 \ \forall v \implies w = 0$ and if and only if the matrix $(B(e_i, f_j))$ is non-singular.

Theorem A is Gorenstein if and only if $A_t \times A_{s-t}$ (multiplication) is a perfect pairing for $0 \le t \le s$.

Proof $\Longrightarrow$: Let $\{m_i\}$ be all monomials of degree $s-t$ and let $\deg(a) = t < s$, and suppose $am_i = 0 \ \forall i$. Now let m' be a monomial of degree $s-t-1$. Then $am'x_i = 0 \ \forall i$, so $am' \in \mathrm{Soc}(A)$. But $\deg am' = s-1$, so $am' = 0$. Continue! Finally $ax_i = 0 \ \forall i$, so $a \in \mathrm{Soc}(A)$, $\deg(a) = t$, so $a = 0$.
$\Longleftarrow$: $A_s = A_0^*$, so $\dim_k A_s = 1$. Suppose $a \in \mathrm{Soc}(A)$, $\deg(A) = t < s$. Then $aA_{s-t} = 0$, so not perfect pairing.

Corollary The Hilbert series of a Gorenstein ring is symmetric, if $H_A(t) = \sum_{i=0}^{s} a_i t^i$, then $a_i = a_{s-i}$.

Remark There are other ways to define Gorenstein rings, e.g. (0) is an irreducible ideal, or A is selfinjective. To define Gorenstein for rings of dimension $> d$, one can say: A is Gorenstein if and only if A/x is Gorenstein for some (any) nonzerodivisor x. Complete intersecions are Gorenstein, and Gorenstein are Cohen-Macaulay.

2. Perp ideals

Let $A = k[x_1, \dots, x_n]$ and $B = k[\partial_1, \dots, \partial_n]$. A is a B-module with $\partial_i x_j = \delta_{i,j}$ and extended. (A is the injective envelope of k, the Macaulay's inverse system.)

Theorem The pairing $B_j \times A_j \to k$ is perfect.

Example $j = 3$, $A = k[x_1, x_2]$. Take bases $x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$ and $\partial_1^3, \partial_1^2\partial_2, \partial_1\partial_2^2, \partial_2^3$. Then $\begin{pmatrix} 3! & 0 & 0 & 0 \\ 0 & 2! & 0 & 0 \\ 0 & 0 & 2! & 0 \\ 0 & 0 & 0 & 3! \end{pmatrix}$ is nonsingular.

Definition For $F \in A_s$, $F^\perp = \{\partial \in B; \partial F = 0\} = \mathrm{Ann}_B(F)$ (perp ideal).

Lemma If $V_1 \subseteq V$ and $V \times W \to k$ is perfect, $\dim V_1 = m, \dim V = \dim W = n$, then $\dim V_1^\perp = n - m$.

Theorem $F^\perp$ is an ideal and $B/F^\perp$ is Gorenstein (artinian).

Proof That $F^\perp$ is an ideal is clear. If $\deg(\partial) > s$, then $\partial \in F^\perp$, so $B/F^\perp$ is artinian. $\langle F \rangle \subseteq A_s$ and $\dim\langle F \rangle = 1$, so $\dim\langle F \rangle^\perp = \dim A_s - 1 = \dim B_s - 1$, so $\dim(B/F^\perp)_s = 1$. (With $\langle F \rangle$, I mean the vector space generated by F.) Suppose $\deg(\partial) = t < s, \partial \notin F^\perp$, and that $\partial\partial_i \in F^\perp$ for all i. ∂F has degree $s - t > 0$, so at least one ∂_i has $\partial\partial_i F = \partial_i(\partial F) \neq 0$.

Example If $F = x_1^{i_1} \cdots x_n^{i_n}$ is a monomial, then $F^\perp = (\partial_1^{i_1+1}, \dots, \partial_n^{i_n+1})$.

Problem 1 For which F is $F^\perp$ a complete intersection?

Problem 2 For which F is $F^\perp$ generated in degree 2?

Problem 3 Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of d, and $F = \mathbf{l}^\mu$, where $\mathbf{l}^\mu = l_1^{\mu_1} \cdots l_k^{\mu_k}$, l_i generic linear forms. What is $F^\perp$?

a.s.o.

3. Inverse system of ideals

Definition Let I be a homogeneous ideal in B. Then $I^{-1} = \{f \in A; I \circ f = 0\} = \{f \in A; gf = 0 \ \forall g \in I\}$.

Remarks I^{-1} is a graded B-submodule of A, not finitely generated in general, and not an ideal in general. If $I = (g_1, \dots, g_k)$, then $I^{-1} = \{f \in A; g_i f = 0 \ \forall i\}$.

Notation $\langle a_1, \dots, a_n \rangle$ denotes the vector space generated by $a_1, \dots, a_n$.

Example If $I = (\partial_1^2)$, then $(I^{-1})_i = \langle x_1, x_2 \rangle, (I^{-1})_2 = \{f = ax_1^2 + bx_1x_2 + cx_2^2; \partial_1^2 f = 0\} = \langle x_1x_2, x_2^2 \rangle, (I^{-1})_3 = \{f = ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3; \partial_1^2 f = 0\} = \langle x_1x_2^2, x_2^3 \rangle$ a.s.o.

Theorem $(I^{-1})_j = I_j^\perp$

Proof $I_j \times I_j^\perp \to 0$, so $(I^{-1})_j \subseteq I_j^\perp$. Let $G \in I_j^\perp$, then $hG = 0 \ \forall h \in I_j$. Claim $FG = 0 \ \forall F \in I$. Clear if $\deg(F) > j$. Suppose then $\deg(F) < j$, and let f be an element,

$\deg(f) = j - \deg(F)$. Then $\deg(fF) = j$ and $fF \in I_j$, so $fFG = 0$, so FG is annihilated by all f of degree $j - \deg(F)$. Apolarity gives $FG = 0$.

Corollary $\dim_k(I^{-1})_j = \dim_k(B_j/I_j)$.

Proof $(I^{-1})_j = I_j^{\perp}$ and $\dim I_j^{\perp} = \dim A_j - \dim I_j = \dim B_j - \dim I_j$.

Corollary I^{-1} is finitely generated if and only if B/I is Artinian.

Proof I^{-1} is finitely generated if and only if $(I^{-1})_j = 0, j >> 0$ if and only if $(R/J)_j = 0, j >> 0$.

Corollary If I is generated by monomials, then I^{-1} consists of those monomials which do not belong to "I". (The reason for the quotation marks is that I is a monomial ideal in B, not in A.)

Lemma If $V_1, V_2 \subseteq V$ and $V \times W \to k$ is perfect, then $(V_1 + V_2)^{\perp} = V_1^{\perp} \cap V_2^{\perp}$ and $(V_1 \cap V_2)^{\perp} = (V_1 + V_2)^{\perp} = V_1^{\perp} + V_2^{\perp}$.

4. Fat points

A point $P \in \mathbb{P}^n$ has a prime ideal p generated by n linear forms. A point is called fat of order t if, for all $F \in p$ all the partial derivatives of F of order $\leq t$ vanish at P, thus if and only if $F \in p^{t+1}$, if and only if P is a singular point of $V(F)$ having multiplicity $\geq t+1$. A set of s fat points $P_1, \ldots, P_s$ of order t has ideal $I = p_1^{t+1} \cap \cdots \cap p_s^{t+1}$, the *symbolic power* of $p_1 \cap \cdots \cap p_s$. We have $(p_1 \cap \cdots \cap p_s)^{t+1} \subseteq p_1^{t+1} \cap \cdots \cap p_s^{t+1}$. Let $P_1, \ldots, P_s$ be points of $\mathbb{P}^n$ and suppose that $P_i = [p_{i0} : p_{i1} : \cdots : p_{in}]$. Let $L_{Pi} = p_{i0}x_0 + p_{i1}x_1 + \cdots + p_{in}x_n \in k[x_0, \ldots, x_n] = A$. Then, if $I = p^{t+1} \cap \cdots \cap p_s^{t+1} \subseteq k[\partial_0, \ldots, \partial_n]$ we have $(I^{-1})_j$ is the jth graded piece of the ideal $(L_{P_1}^{j-t}, \ldots, L_{P_s}^{j-t})$ for $j \geq t$ and $(I^{-1})_j = A_j$ if $j < t$.

Corollary $\dim_k(A/I)_j = \dim_k(L_{P_1}^{j-t}, \ldots, L_{P_s}^{j-t})_j$.

Problem Are there any other classes of ideals I (besides monomial ideals and ideals of fat points) where one can describe I^{-1} concretely?

Hard problem For which ideals is the symbolic powers equal to the real power?

5. Macaulay's theorem

Theorem[Macaulay] B/I is Gorenstein with socle in degree s if and only if there exists an $F \in A_s$ such that $I = \{g \in B; gF = 0\} = \mathrm{Ann}(F)$ if and only if I^{-1} is generated by F as B-module.

Proof We have proved one half of the theorem. Suppose B/I is Gorenstein with socle degree s. If $g \in (B/I)_t$, $t < s$, we must have $g(B/I)_{s-t} \subseteq (B/I)_t$. Since $(B/I)_t$ is 1-dimensional, we have that I_t has dimension $\dim B_t - 1 = \dim A_t - 1$, so $I_t = \langle F \rangle^{\perp}$ for some $F \in A_t$. Let $(\langle F \rangle^{\perp} \colon B_{s-t}) = \{g \in B_t; gB_{s-t} \subseteq \langle F \rangle^{\perp}\}$. Consider $J = \oplus_{i=t}^{1}(\langle F \rangle^{\perp} \colon B_i) \oplus (\langle F \rangle^{\perp})$. It is not hard to see that J is an ideal (in fact the largest ideal which is equal to I in degrees $\geq t$). We claim that $J = \mathrm{Ann}_B(F)$. This is true in degrees $\geq s$. Let $G \in J_t$, $t < s$, so $GB_{s-t} \subseteq \langle f \rangle^{\perp}$. Now $\deg(GF) = s - t$. Let $f \in B_{s-t}$ so $\deg(fG) = s$, so $fG \in \langle F \rangle^{\perp}$, so

$(fG)F = f(GF) = 0$. Since $B_{s-t} \times A_{s-t} \to k$ is perfect, we have $GF = 0$, so $J \subseteq \mathrm{Ann}_B(F)$. Now suppose $G \in (\mathrm{Ann}_B(F))_t$ and $f \in B_{s-t}$. Then $Gf \in \langle F\rangle^\perp$, so $GB_{s-t} \subseteq \langle F\rangle^\perp$, so $G \in J$. What remains to show is that $I = J$. This is true in degrees $\geq s$. Let $f \in I_t$, $t < s$. We have $fB_{s-t} \subseteq I_s$, so $f \in J$. The pairing $B_t/I_t \times B_{s-t}/I_{s-t} \to B_s/I_s = k$ is perfect. Let $f \in J_t$, so $fB_{s-t} \subseteq I_s$. Then $\overline{f}\overline{h} = 0$ for all $h \in B_{s-t}$, so $\bar{f} = 0$, so $f \in I_t$.

6. Macaulay2

It is possible to make calculations in Macaulay2. Here is an example:

$R = QQ[x, y, z]$
$g = \mathrm{matrix}\{\{x * y * z\}\}$
$f = \mathrm{fromDual}\ g$
$h = \mathrm{toDual}\ (3, f)$

f is a matrix of generators for the perp ideal of g, and $\mathrm{Ann}(h) = f$. The number d (here 3) is needed. It should be so high so $x^{d+1} = 0$ for all variables in the Gorenstein ring. (So in the example we could have $d = 1$.)