Remarks on the symmetric rank of symmetric tensors

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Abstract

We give sufficient conditions on a symmetric tensor $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n$ to satisfy the equality: the symmetric rank of $\mathcal{S}$, denoted as $\mathrm{srank}\,\mathcal{S}$, is equal to the rank of $\mathcal{S}$, denoted as $\mathrm{rank}\,\mathcal{S}$. This is done by considering the rank of the unfolded $\mathcal{S}$ viewed as a matrix $A(\mathcal{S})$. The condition is: $\mathrm{rank}\,\mathcal{S} \in \{\mathrm{rank}\, A(\mathcal{S}), \mathrm{rank}\, A(\mathcal{S})+1\}$. In particular, $\mathrm{srank}\,\mathcal{S} = \mathrm{rank}\,\mathcal{S}$ for $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$ for the cases $(d,n) \in \{(3,2),(4,2),(3,3)\}$.

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1 Introduction

For a field $\mathbb{F}$ let $\otimes^d\mathbb{F}^n \supset \mathrm{S}^d\mathbb{F}^n$ denote d-mode tensors and the subspace of symmetric tensors on $\mathbb{F}^n$. Let $\mathcal{T} \in \otimes^d\mathbb{F}^n$. Denote by $\mathrm{rank}\,\mathcal{T}$ the rank of the tensor $\mathcal{T}$. That is, for $\mathcal{T} \neq 0$ $\mathrm{rank}\,\mathcal{T}$ is the minimal number k such that $\mathcal{T}$ is a sum of k rank one tensors. ($\mathrm{rank}\, 0 = 0$.) Assume that $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n \setminus \{0\}$. Suppose that $|\mathbb{F}| \geq d$, i.e. $\mathbb{F}$ has at least d elements. Then it is known that $\mathcal{S}$ is a sum of k symmetric rank one tensors [7, Proposition 7.2]. See [1] for the case $|\mathbb{F}| = \infty$, i.e. $\mathbb{F}$ has an infiite number of elements. The minimal k is the symmetric rank of $\mathcal{S}$, denoted as $\mathrm{srank}\,\mathcal{S}$. Clearly, $\mathrm{rank}\,\mathcal{S} \leq \mathrm{srank}\,\mathcal{S}$. In what follows we assume that $d \geq 3$. For $\mathbb{F} = \mathbb{C}$ it is conjectured that $\mathrm{rank}\,\mathcal{S} = \mathrm{srank}\,\mathcal{S}$. (This conjecture is attributed to P. Comon.) In [4] it is shown that this conjecture holds in the first nontrivial case: $\mathrm{rank}\,\mathcal{S} = 2$.

For a finite field the situation is more complicated: Observe first that for $\mathbb{F} = \mathbb{Z}_2$ and the symmetric matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have the the inequality $\mathrm{rank}\, A = 2 < \mathrm{srank}\, A = 3$. ($A$ is a sum of all three distinct symmetric rank one matrices in $\mathrm{S}^2\mathbb{Z}_2^2$.) Second, it is shown in [7, Proposition 7.1] that over a finite field there exist symmetric tensors that are not a sum symmetric rank one tensors.

To state our result we need the following notions: For $n \in \mathbb{N}$ denote $[n] = \{1,\ldots,n\}$. Let $\mathcal{S} = [s_{i_1,\ldots,i_d}]_{i_1,\ldots,i_d \in [n]} \in \mathrm{S}^d\mathbb{F}^n$. Denote by $A(\mathcal{S})$ an $n \times n^{d-1}$ matrix with entries $b_{\alpha\boldsymbol{\beta}}$ where $\alpha \in [n]$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_{d-1}) \in [n]^{d-1}$. Then $b_{\alpha\boldsymbol{\beta}} := s_{\alpha,\beta_1,\ldots,\beta_{d-1}}$. ($A(\mathcal{S}) \in \mathbb{F}^{n\times n^{d-1}}$ is the unfolding of $\mathcal{S}$ in the direction $k \in [d]$.) Hence $\mathrm{rank}\, A(\mathcal{S}) \leq n$. If $m := \mathrm{rank}\, A(\mathcal{S}) < n$ it means that we can choose another basis

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so that $\mathcal{S}$ is represented as $\mathcal{S}' \in \mathrm{S}^d\mathbb{F}^m$. Recall that $\operatorname{rank} \mathcal{S} \ge \operatorname{rank} A(\mathcal{S})$. (See for example the arguments in [6] for $d = 3$.). Thus, to study Comon's conjecture we can assume without loss of generality that $\operatorname{rank} \mathcal{S} = n$.

Denote by $\Sigma(n, d, \mathbb{F})$ and $\Sigma_s(d, n, \mathbb{F})$ the Segre variety of rank one tensors plus the zero tensor and the subvariety of symmetric tensors of at most rank one in $(\mathbb{F}^n)^{\otimes d}$.

Let $F_{d,n,k} : \Sigma(n, d, \mathbb{F})^k \to (\mathbb{F}^n)^{\otimes d}$ be the polynomial map:

$$F_{d,n,k}((\mathcal{T}_1, \dots, \mathcal{T}_k)) := \sum_{j=1}^{k} \mathcal{T}_j. \tag{1.1}$$

Let $\mathcal{T} = F_{d,n,k}((\mathcal{T}_1, \dots, \mathcal{T}_k))$. In what follows we say that the decomposition $\mathcal{T} = \sum_{j=1}^{k} \mathcal{T}_j$ is unique if $\operatorname{rank} \mathcal{T} = k$ and any decomposition of $\mathcal{T}$ to a sum of r rank one tensors is obtained by permuting the order of the summands in $\mathcal{T} = \sum_{j=1}^{k} \mathcal{T}_j$.

Denote by $G_{d,n,k}$ the restriction of the map $F_{d,n,k}$ to : $\Sigma_s(n, d, \mathbb{F})^k$. Thus $F_{d,n,k}(\Sigma(n, d, \mathbb{F})^k)$ and $G_{d,n,k}(\Sigma_s(n, d, \mathbb{F})^k)$ are the sets of of d-mode tensors on $\mathbb{F}^n$ tensors of at most rank k and of symmetric tensors of at most symmetric rank k.

Chevalley's theorem yields that $F_{d,n,k}(\Sigma(n, d, \mathbb{C})^k)$ and $G_{d,n,k}(\Sigma_s(n, d, \mathbb{C})^k)$ are constructible sets. Hence the dimension of $G_{d,n,k}(\Sigma_s(n, d, \mathbb{C})^k)$ is the maximal rank of the Jacobian of the map $G_{d,n,k}$.

$\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$ is said to have a generic symmetric rank k if the following conditions hold: First, the dimension of the constructible set $G_{d,n,k}(\Sigma_s(n, d, \mathbb{C})^k)$ is greater than the dimension of $G_{d,n,k-1}(\Sigma_s(n, d, \mathbb{C})^{k-1})$. Second, there exists a strict subvariety $O \subset \Sigma(n, d, \mathbb{C})^k$, such that $\mathcal{S} \in G_{d,n,k}(\Sigma_s(n, d, \mathbb{C})^k \setminus O)$. Let

$$k_{n,d} := \frac{\binom{n+d-1}{d}}{n}. \tag{1.2}$$

Chiantini, Ottaviani and Vannieuwenhoven showed recenlty [3] that if $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$ has a generic symmetric rank $k < k_{n,d}$ then $k = \operatorname{rank} S$. It is much easier to establish this kind of result for smaller values of k using Kruskal's theorem. See [7, Theorem 7.6].

The aim of this paper is to establish a much weaker result on Comon's conjecture, which does not use the term *generic*. In particular we show that Comon's conjecture holds for symmetric tensors of at most rank 3 and for 3-symmetric tensors of at most rank 5 over $\mathbb{C}$.

Our main result is

Theorem 1.1 *Let $d \ge 3$, $|\mathbb{F}| \ge 3$ and $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n$. Suppose that* $\operatorname{rank} \mathcal{S} \le \operatorname{rank} A(\mathcal{S}) + 1$. *Then* $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$.

We now summarize briefly the content of this paper. In §2 we recall Kruskal's theorem on the rank of 3-tensor. In §3 we prove Theorem 1.1 for the case $\operatorname{rank} \mathcal{S} = \operatorname{rank} A(\mathcal{S})$. In §4 we show that each $\mathcal{S} \in \mathrm{S}^3\mathbb{F}^2$, where $|\mathbb{F}| \ge 3$, satisfies $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. In §5 we prove Theorem 1.1 in the case $d = 3$ and $\operatorname{rank} \mathcal{S} = \operatorname{rank} A(\mathcal{S}) + 1$. In §6 we prove Theorem 1.1 for $d \ge 4$. In §7 we summarize our results for $\mathbb{F} = \mathbb{C}$.

2 Kruskal's theorem

We recall Kruskal's theorem for 3-tensors and any field $\mathbb{F}$. For p vectors $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \mathbb{F}^n$ denote by $[\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_p]$ the $n \times p$ matrix whose columns are $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Kruskal's rank of $[\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_p]$, denoted as $\mathrm{Krank}\,(\mathbf{x}_1, \ldots, \mathbf{x}_p)$ is the maximal k such that any k vectors in the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ are linearly independent. (If $\mathbf{x}_i = 0$ for some $i \in [p]$ then $\mathrm{Krank}\,(\mathbf{x}_1, \ldots, \mathbf{x}_p) = -\infty$.)

Theorem 2.1 *(Kruskal) Let $\mathbb{F}$ be a field, $r \in \mathbb{N}$ and $\mathbf{x}_i \in \mathbb{F}^m, \mathbf{y}_i \in \mathbb{F}^n, \mathbf{z}_i \in \mathbb{F}^p$ for $i \in [r]$. Assume that*

$$\mathcal{T} = \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i. \tag{2.1}$$

Suppose that

$$2r + 2 \leq \mathrm{Krank}\,(\mathbf{x}_1, \ldots, \mathbf{x}_r) + \mathrm{Krank}\,(\mathbf{y}_1, \ldots, \mathbf{y}_r) + \mathrm{Krank}\,(\mathbf{z}_1, \ldots, \mathbf{z}_r). \tag{2.2}$$

Then rank $\mathcal{T} = r$. *Furthermore, the decomposition* (2.1) *is unique.*

In what follows we need a following simple corollary of Kruskal's theorem:

Lemma 2.2 *Let $3 \leq d \in \mathbb{N}$. Assume that $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,r} \in \mathbb{F}^{n_j}$ are linearly independent for each $j \in [d]$. Let*

$$\mathcal{T} = \sum_{i=1}^{r} \otimes_{j=1}^{d} \mathbf{x}_{j,i}. \tag{2.3}$$

Then rank $\mathcal{T} = r$. *Furthermore, the decomposition* (2.3) *is unique.*

Proof. Observe first that $\otimes_{j=1}^{p} \mathbf{x}_{j,1}, \ldots, \otimes_{j=1}^{p} \mathbf{x}_{j,r}$ linearly independent for $p = 1, \ldots, d$. Clearly, this is true for $p = 1$ and $p = 2$. Use the induction to prove this statement for $p \geq 3$ by observing that $\otimes_{j=1}^{p} \mathbf{x}_{j,i} = (\otimes_{j=1}^{p-1} \mathbf{x}_{j,i}) \otimes \mathbf{x}_{p,i}$ for $p = 3, \ldots, d$.

Consider $\mathcal{T}$ given by (2.3). Suppose first that $r = 1$. Then $\mathcal{T}$ is a rank one tensor and its decomposition is unique. Assume that $r \geq 2$. Consider $\mathcal{T}$ as a 3-tensor on the 3-tensor product $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes (\otimes_{j=3}^{d} \mathbb{F}^{n_j})$. Clearly

$$Krank(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{r,1}) = \mathrm{Krank}\,(\mathbf{x}_{1,2}, \ldots, \mathbf{x}_{r,2}) = \mathrm{Krank}\,(\otimes_{j=3}^{d} \mathbf{x}_{j,1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j,r}) = r.$$

As $3r - 2 \geq 2r$, Kruskal's theorem yields that the rank of $\mathcal{T}$ as 3-tensor is r. Hence rank $\mathcal{T}$ as d tensor is r too. Furthermore the decomposition (2.3) of $\mathcal{T}$ as a 3-tensor is unique. Hence the decomposition(2.3) is unique. $\square$

In what follows we need the following lemma.

Lemma 2.3 *Let $d \geq 3$ and $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n$. Assume that*

$$\mathcal{S} = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{j,i}. \tag{2.4}$$

Then $\mathcal{S} = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j),i}$ for any permutation σ of $[d]$. Suppose that the following inequality holds:

$$\begin{aligned} &2k + 2 \leq K(\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,k}) + K(\mathbf{x}_{2,1}, \ldots, \mathbf{x}_{2,k}) + \\ &K(\otimes_{j=3}^{d} \mathbf{x}_{j,1}, \ldots, \otimes_{j=3}^{d} \mathbf{x}_{j,k}). \end{aligned} \tag{2.5}$$

Then $\operatorname{rank} \mathcal{S} = \operatorname{srank} \mathcal{S} = k$*, i.e.* $\operatorname{span}(\mathbf{x}_{1,i}) = \ldots = \operatorname{span}(\mathbf{x}_{d,i})$ *for each* $i \in [k]$*. Furthermore, the decomposition* (2.4) *is unique.*

Proof. Assume that (2.4) holds. Since $\mathcal{S}$ symmetric we deduce that $\mathcal{S} = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j),i}$ for any permutation σ of $[d]$. Suppose that (2.5) holds. Kruskal's theorem yields that the decomposition of $\mathcal{S}$ as a 3-tensor on $\mathbb{F}^n \otimes \mathbb{F}^n \otimes (\otimes^{d-2}\mathbb{F}^n)$ is unique. In particular, the decomposition (2.4) is unique. Hence $\operatorname{rank} \mathcal{S} = k$. Let σ be the transposition on $[d]$ satisfying $\sigma(1) = 2, \sigma(2) = 1$. Then $\mathcal{S} = \sum_{i=1}^{k} \otimes_{j=1}^{d} \mathbf{x}_{\sigma(j),i}$. The uniqueness of the decomposition (2.4) yields that $\mathbf{x}_{1,i}$ and $\mathbf{x}_{2,i}$ are colinear for each $i \in [n]$. Let σ be a transposition on $[d]$ satisfying $\sigma(2) = j, \sigma(j) = 2$ for some $j \geq 3$. The uniqueness of the decomposition (2.4) yields that $\mathbf{x}_{2,i}$ and $\mathbf{x}_{j,i}$ are colinear for each $i \in [n]$. Hence $\operatorname{span}(\mathbf{x}_{1,i}) = \ldots = \operatorname{span}(\mathbf{x}_{d,i})$ for each $i \in [d]$. Therefore the decomposition (2.4) is a decomposition to a sum of symmetric rank one tensors. Hence $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. □

3 The case $\operatorname{rank} \mathcal{S} = \operatorname{rank} A(\mathcal{S})$

Theorem 3.1 *Let* $d \geq 3$ *and* $\mathcal{S} \in \mathrm{S}(d, \mathbb{F}^n)$*. Suppose that* $\operatorname{rank} \mathcal{S} = \operatorname{rank} A(\mathcal{S})$*. Then* $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$*. Furthermore,* $\mathcal{S}$ *has has a unique rank one decomposition.*

Proof. We can assume without loss of generality that $\operatorname{rank} A(\mathcal{S}) = n$. So (2.4) holds for $k = n$. Clearly, $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n}$ are linearly independent for each $j \in [d]$. Combine the proof of Lemma 2.2 with Lemma 2.3 to deduce the theorem. □

The following corollary generalizes [4, Proposition 5.5] to any field $\mathbb{F}$:

Corollary 3.2 *Let* $\mathbb{F}$ *be a field,* $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n \setminus \{0\}, d \geq 3$*. Assume that* $\operatorname{rank} \mathcal{S} \leq 2$*. Then* $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$*.*

Proof. Clearly, $\operatorname{rank} A(S) \in \{1, 2\}$. If $\operatorname{rank} A(S) = 1$ then $\mathcal{S} = s \otimes^d \mathbf{u}$. Hence $\operatorname{rank} \mathcal{S} = \operatorname{srank} \mathcal{S} = 1$. If $\operatorname{rank} A(\mathcal{S}) = 2$ then $\operatorname{rank} \mathcal{S} = 2$ and we conclude the result from Theorem 3.1. □

4 The case $\mathrm{S}^3\mathbb{F}^2$

Theorem 4.1 *Let* $\mathcal{S} \in \mathrm{S}^3\mathbb{F}^2$*. Assume that* $|\mathbb{F}| \geq 3$*. Then* $\operatorname{rank} \mathcal{S} = \operatorname{srank} \mathcal{S} \leq 3$*.*

Proof. In view of Corollary 3.2 it is enough to consider the case where $\operatorname{rank} \mathcal{S} \geq 3$. Let $\mathcal{S} = [s_{i,j,k}]_{i,j,k \in [2]}$.

1. Assume that $s_{1,1,2}s_{1,2,2} \neq 0$. Let

$$\mathcal{S} = \sum_{i=1}^{3} t_i \otimes^3 \mathbf{u}_i, \ \mathbf{u}_1 = (1, b)^\top, \quad \mathbf{u}_2 = (1, 0)^\top, \quad \mathbf{u}_3 = (0, 1)^\top. \tag{4.1}$$

Then

$$s_{1,1,2} = t_1 b, \quad s_{1,2,2} = t_1 b^2 \Rightarrow t_1 = \tfrac{s_{1,1,2}^2}{s_{1,2,2}}, \quad b = \tfrac{s_{1,2,2}}{s_{1,1,2}},$$
$$t_2 = s_{1,1,1} - t_1, \quad t_3 = s_{2,2,2} - t_1 b^3.$$

Hence $\operatorname{rank}\mathcal{S} \le 3$. Our assumption yields that $\operatorname{rank}\mathcal{S} = 3$ and (4.1) is a minimal decomposition of $\mathcal{S}$ to rank one tensors. This decomposition shows that $\operatorname{rank}\mathcal{S} = \operatorname{srank}\mathcal{S}$.

2. Assume that $s_{1,1,2} = s_{1,2,2} = 0$ then $\mathcal{S} = s_{1,1,1} \otimes^3 (1,0)^\top + s_{2,2,2} \otimes^3 (0,1)^\top$. This contradicts our assumption that $\operatorname{rank}\mathcal{S} \ge 3$.

3. It is left to discuss the case where $\operatorname{rank}\mathcal{S} \ge 3$ and $s_{1,1,2} = 0$ and $s_{1,2,2} \neq 0$. The homogeneous polynomial of degree 3 corresponding to $\mathcal{S}$ is

$$f(x_1, x_2) = s_{1,1,1}x_1^3 + s_{1,2,2}x_1x_2^2 + s_{2,2,2}x_2^3.$$

   (a) Assume that the characteristic of $\mathbb{F}$ is 3. Make the following change of variables: $x_1 = y_1, x_2 = y_1 + y_2$. The new tensor $\mathcal{S}'$ satisfies *1*.

   (b) Assume that the characteristic of $\mathbb{F}$ is not 3.

      i. Assume that $s_{1,1,1} \neq 0$. Make the following change of variables: $x_1 = y_1 + ay_2, x_2 = y_2$. Then

$$f(y_1, y_2) = \alpha y_1^3 + \beta y_1^2 y_2 + \gamma y_1 y_2^2 + \delta,\ \beta = 3as_{1,1,1}, \gamma = s_{1,2,2} + 3a^2 s_{1,1,1}.$$

      Then choose a nonzero a such that $s_{1,2,2} + 3a^2 s_{1,1,1} \neq 0$. (This is always possible if $|\mathbb{F}| \ge 4$ as we assumed that $\mathbb{F} \neq \mathbb{Z}_3$ and $|\mathbb{F}| \ge 3$.) The new tensor $\mathcal{S}'$ satisfies *1*.

      ii. Assume that $s_{1,1,1} = s_{2,2,2} = 0$. Make the following change of variables: $x_1 = y_1, x_2 = (y_1 + y_2)$. Then we are either in the case *1* if the characteristic of $\mathbb{F}$ is not 2 or in the case *3*(b)i if the characteristic of $\mathbb{F}$ is 2.

      iii. Assume that $s_{1,1,1} = 0$ and $s_{2,2,2} \neq 0$. Make the following change of variables: $y_2 = s_{1,2,2}x_1 + s_{2,2,2}x_2, y_2 = x_2$. Then we are in the case *3*(b)ii. □

Note that if $|\mathbb{F}| \gg 1$ then using the change of coordinates and then the above procedure we obtain that if $\operatorname{rank}\mathcal{S} = 3, \operatorname{rank} A(\mathcal{S}) = 2$ we have many presentation of $\mathcal{S}$ as sum of three rank one symmetric tensors.

Observe next that for $\mathbb{F} = \mathbb{Z}_2$ not every symmetric tensor $\mathcal{S} \in \mathrm{S}^3\mathbb{Z}_2^2$ is a sum of rank one symmetric tensors. The number of all symmetric tensors in $\mathrm{S}^3\mathbb{Z}_2^2$ is 2^4. The number of all nonzero symmetric tensors which are sum of rank one symmetric tensors is $2^3 - 1$. Hence Theorem 4.1 does not hold for $\mathbb{F} = \mathbb{Z}_2$.

Corollary 4.2 *Let $\mathcal{S} \in \mathrm{S}(3, \mathbb{F}^n)$. Assume that $|\mathbb{F}| \ge 3$ and* $\operatorname{rank}\mathcal{S} = 3$. *Then* $\operatorname{srank}\mathcal{S} = \operatorname{rank}\mathcal{S}$.

Proof. Clearly, $\operatorname{rank} A(\mathcal{S}) \in \{2, 3\}$. If $\operatorname{rank} A(\mathcal{S}) = 3$ we deduce the corollary from Theorem 3.1. If $\operatorname{rank} A(\mathcal{S}) = 2$ we deduce the corollary from Theorem 4.1. □

5 The case $d = 3$ and $\mathrm{rank}\ \mathcal{S} = \mathrm{rank}\ A(\mathcal{S}) + 1$

In this section we prove Theorem 1.1 for $d = 3$. In view of Theorem 3.1 it is enough to consider the case $\mathrm{rank}\ \mathcal{S} = \mathrm{rank}\ A(\mathcal{S}) + 1$. Furthermore, in view of Theorem 4.1 it is enough to consider the case $\mathrm{rank}\ A(\mathcal{S}) \geq 3$. We first give the following obvious lemma:

Lemma 5.1 *Let* $|\mathbb{F}| \geq 3$ *and* $\mathcal{S} \in \mathrm{S}(3, \mathbb{F}^n)$. *Suppose that* $\mathrm{rank}\ \mathcal{S} = \mathrm{rank}\ A(\mathcal{S}) + 1$. *Assume furthermore that there exists a decomposition of* $\mathcal{S}$ *to* $\mathrm{rank}\ A(\mathcal{S}) + 1$ *rank one tensors such that at least one of them is symmetric, i.e.* $s \otimes^3 \mathbf{u}$. *Let* $\mathcal{S}' = \mathcal{S} - s\otimes^3\mathbf{u}$. *Then* $\mathrm{rank}\ \mathcal{S}' = \mathrm{rank}\ \mathcal{S} - 1$ *and* $\mathrm{rank}\ A(\mathcal{S}') \in \{\mathrm{rank}\ A(\mathcal{S}) - 1, \mathrm{rank}\ A(\mathcal{S})\}$. *Furthermore:*

1. *If* $\mathrm{rank}\ A(\mathcal{S}') = \mathrm{rank}\ A(\mathcal{S})$ *then* $\mathrm{rank}\ \mathcal{S}' = \mathrm{rank}\ A(\mathcal{S}')$ *and* $\mathrm{rank}\ \mathcal{S}' = \mathrm{srank}\ \mathcal{S}'$. *Hence* $\mathrm{rank}\ \mathcal{S} = \mathrm{srank}\ \mathcal{S}$.

2. *If* $\mathrm{rank}\ A(\mathcal{S}') = \mathrm{rank}\ A(\mathcal{S}) - 1$ *then* $\mathrm{rank}\ \mathcal{S}' = \mathrm{rank}\ A(\mathcal{S}') + 1$.

5.1 The case $\mathrm{rank}\ A(\mathcal{S}) = 3$

We now discuss Theorem 1.1 where $\mathcal{S} \in \mathrm{S}(3, \mathbb{F}^n)$, where $\mathrm{rank}\ \mathcal{S} = 4, \mathrm{rank}\ A(\mathcal{S}) = 3$. Without loss of generality we can assume that $n = 3$. Then

$$\mathcal{S} = \sum_{i=1}^{4} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i. \tag{5.1}$$

Suppose first that there is a decomposition (5.1) such that $\mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ is symmetric for some $i \in [4]$. Then we can use Lemma 5.1. Apply Theorems 3.1 and 4.1 to deduce that $\mathrm{srank}\ \mathcal{S} = \mathrm{rank}\ \mathcal{S} = 4$.

Assume the *Assumption*: there no is a decomposition (5.1) such that $\mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ is symmetric for some $i \in [4]$. The first par of Lemma 2.3 yields:

$$0 = \sum_{i=1}^{4} \mathbf{x}_i \otimes (\mathbf{y}_i \otimes \mathbf{z}_i - \mathbf{z}_i \otimes \mathbf{y}_i).$$

As $\mathrm{rank}\ A(\mathcal{S}) = 3$ we can assume that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. Then $\mathbf{x}_4 = \sum_{j=1}^{3} a_j \mathbf{x}_j$. Then the above equality yields:

$$0 = \sum_{j=1}^{3} \mathbf{x}_j \otimes (\mathbf{y}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{y}_j + a_j(\mathbf{y}_4 \otimes \mathbf{z}_4 - \mathbf{z}_4 \otimes \mathbf{y}_4)). \tag{5.2}$$

As $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent it follows that

$$\mathbf{y}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{y}_j = -a_j(\mathbf{y}_4 \otimes \mathbf{z}_4 - \mathbf{z}_4 \otimes \mathbf{y}_4) \text{ for } j \in [3]. \tag{5.3}$$

Assume first that $\mathbf{y}_4$ and $\mathbf{z}_4$ are colinear. Then $\mathbf{y}_j$ and $\mathbf{z}_j$ are colinear for $j \in [3]$. Hence we w.l.o.g we can assume that $\mathbf{y}_i = \mathbf{z}_i = \mathbf{u}_i$ for $i \in [4]$. So $\mathcal{S} = \sum_{i=1}^{4} \mathbf{x}_i \otimes \mathbf{u}_i \otimes \mathbf{u}_i$. Since $\mathcal{S}$ is symmetric we can we obtain that $\mathcal{S} = \sum_{i=1}^{4} \mathbf{u}_i \otimes \mathbf{u}_i \otimes \mathbf{x}_i$. Renaming the vectors we can assume that in the original decomposition (5.1) we have that $\mathbf{x}_i$ and $\mathbf{y}_i$ are colinear for $i \in [4]$. Since we assumed that no rank one tensor $\mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$

is not symmetric, we deduce that each pair $\mathbf{y}_i, \mathbf{z}_i$ in the original decomposition (5.1) is not colinear. In particular, it is enough to study the case where $\mathbf{y}_4$ and $\mathbf{z}_4$ are not colinear, i.e. $\mathbf{y}_4 \otimes \mathbf{z}_4 - \mathbf{z}_4 \otimes \mathbf{y}_4 \neq 0$.

Suppose that $a_j \neq 0$ for some $j \in [3]$. Then (5.3) yields that $\mathbf{y}_j$ and $\mathbf{z}_j$ are not colinear and $\mathbf{y}_j, \mathbf{z}_j \in \operatorname{span}(\mathbf{y}_4, \mathbf{z}_4)$. Assume that $a_j = 0$. Then $\mathbf{y}_j$ and $\mathbf{z}_j$ are colinear.

Assume first that $a_1 a_2 a_3 \neq 0$. Then the above arguments yields that $\operatorname{span}(\mathbf{y}_1, \ldots, \mathbf{y}_4) \subset \operatorname{span}(\mathbf{y}_4, \mathbf{z}_4)$, which contradicts the assumption that $\operatorname{span}(\mathbf{y}_1, \ldots, \mathbf{y}_4) = \mathbb{F}^3$.

So we need to assume that at least one of $a_i = 0$. Assume first that exactly one $a_i = 0$. Without loss of generality we can assume that in (5.2) $a_1 = 0$ and $a_2 a_3 \neq 0$. This yields that $\mathbf{y}_1$ and $\mathbf{z}_1$ are colinear. Our *Assumption* yields that $\mathbf{x}_1$ and $\mathbf{y}_1$ are not colinear. Furthermore $\operatorname{span}(\mathbf{y}_2, \mathbf{z}_2) = \operatorname{span}(\mathbf{y}_3, \mathbf{z}_3) = \operatorname{span}(\mathbf{y}_4, \mathbf{z}_4)$. As span $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ is the whole space, we deduce that $\mathbf{y}_1 \notin \operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4)$. Similarly $\mathbf{y}_1 \notin \operatorname{span}(\mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4)$. Furthermore, $\operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4)$ is at least 2 dimensional. Hence $\operatorname{span}(\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \operatorname{span}(\mathbf{y}_4, \mathbf{z}_4)$. We now recall that $\mathcal{S} = \sum_{i=1}^4 \mathbf{y}_i \otimes \mathbf{x}_i \otimes \mathbf{z}_i$. Again, by renaming the indices $2, 3, 4$ we can assume that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, are linearly independent. Moreover, $\mathbf{y}_4 = a_2 \mathbf{y}_2 + a_3 \mathbf{y}_3$. Let $a_1 = 0$. So we get a similar equality to (5.2):

$$0 = \sum_{j=1}^{3} \mathbf{y}_j \otimes (\mathbf{x}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{x}_j + a_j(\mathbf{x}_4 \otimes \mathbf{z}_4 - \mathbf{x}_4 \otimes \mathbf{z}_4)).$$

Since $a_1 = 0$ it follows that $\mathbf{x}_1$ and $\mathbf{z}_1$ are colinear. But $\mathbf{y}_1$ and $\mathbf{z}_1$ are also colinear. So we have a contradiction to our *Assumption*.

Finally let us assume that $a_i = a_j = 0$ for some two distinct indices $i, j \in [3]$. W.l.o.g. we can assume that $\mathbf{x}_4 = \mathbf{x}_3$, i.e. $a_1 = a_2 = 0, a_3 = 1$. This implies that $\mathbf{y}_i$ and $\mathbf{z}_i$ are colinear for $i = 1, 2$. Furthermore

$$C := \mathbf{y}_3 \otimes \mathbf{z}_3 + \mathbf{y}_4 \otimes \mathbf{z}_4 = \mathbf{z}_3 \otimes \mathbf{y}_3 + \mathbf{z}_4 \otimes \mathbf{y}_4.$$

So C is a symmetric matrix. Note that C is a rank two matrix. Otherwise $\mathbf{y}_3 \otimes \mathbf{z}_3$ and $\mathbf{y}_4 \otimes \mathbf{z}_4$ are colinear. Then $\mathbf{x}_3 \otimes \mathbf{y}_3 \otimes \mathbf{z}_3 + \mathbf{x}_3 \otimes \mathbf{y}_4 \otimes \mathbf{z}_4$ is a rank one tensor. So rank $\mathcal{S} \leq 3$, contrary to our assumptions. Thus we can assume that $C = \mathbf{y}_3 \otimes \mathbf{y}_4 + \mathbf{y}_4 \otimes \mathbf{y}_3$ and $\mathbf{y}_3, \mathbf{y}_4$ are linearly independent. Hence we can assume that

$$\begin{aligned}\mathcal{S} = \mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{y}_1 + \mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{y}_2 + \mathbf{x}_3 \otimes (\mathbf{y}_3 \otimes \mathbf{y}_4 + \mathbf{y}_4 \otimes \mathbf{y}_3) = \\ \mathbf{y}_1 \otimes \mathbf{x}_1 \otimes \mathbf{y}_1 + \mathbf{y}_2 \otimes \mathbf{x}_2 \otimes \mathbf{y}_2 + \mathbf{y}_3 \otimes \mathbf{x}_3 \otimes \mathbf{y}_4 + \mathbf{y}_4 \otimes \mathbf{x}_3 \otimes \mathbf{y}_3.\end{aligned}$$

Our *Assumption* yields that the pairs $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$ are linearly independent. Hence $Q := \mathbf{x}_1 \otimes \mathbf{y}_1 - \mathbf{y}_1 \otimes \mathbf{x}_1 \neq 0$. The above arguments yield:

$$\begin{aligned}&\mathbf{y}_1 \otimes (\mathbf{x}_1 \otimes \mathbf{y}_1 - \mathbf{y}_1 \otimes \mathbf{x}_1) + \mathbf{y}_2 \otimes (\mathbf{x}_2 \otimes \mathbf{y}_2 - \mathbf{y}_2 \otimes \mathbf{x}_2) + \\ &\mathbf{y}_3 \otimes (\mathbf{x}_3 \otimes \mathbf{y}_4 - \mathbf{y}_4 \otimes \mathbf{x}_3) + \mathbf{y}_4 \otimes (\mathbf{x}_3 \otimes \mathbf{y}_3 - \mathbf{y}_3 \otimes \mathbf{x}_3) = 0\end{aligned}$$

Without loss in generality we may assume that $\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ are linearly independent. So $\mathbf{y}_1 = a_2 \mathbf{y}_2 + a_3 \mathbf{y}_3 + a_4 \mathbf{y}_4$. Substitute in the above equality this expression for $\mathbf{y}_1$ only for the $\mathbf{y}_1$ appearing in the left-hand side to obtain

$$\mathbf{y}_2 \otimes (\mathbf{x}_2 \otimes \mathbf{y}_2 - \mathbf{y}_2 \otimes \mathbf{x}_2 + a_2 Q_2) + \mathbf{y}_3 (\mathbf{x}_3 \otimes \mathbf{y}_4 - \mathbf{y}_4 \otimes \mathbf{x}_3 + a_3 Q) + \mathbf{y}_4 (\mathbf{x}_3 \otimes \mathbf{y}_3 - \mathbf{y}_3 \otimes \mathbf{x}_3 + a_4 Q) = 0.$$

Hence

$$\mathbf{x}_2 \otimes \mathbf{y}_2 - \mathbf{y}_2 \otimes \mathbf{x}_2 + a_2 Q_2 = \mathbf{x}_3 \otimes \mathbf{y}_4 - \mathbf{y}_4 \otimes \mathbf{x}_3 + a_3 Q = \mathbf{x}_3 \otimes \mathbf{y}_3 - \mathbf{y}_3 \otimes \mathbf{x}_3 + a_4 Q = 0.$$

Note that our *Assumption* yields that $a_2 \neq 0$. Hence $\text{span}(\mathbf{x}_2, \mathbf{y}_2) = \text{span}(\mathbf{x}_1, \mathbf{y}_1)$. Suppose first that $a_3 \neq 0$. Then $\text{span}(\mathbf{x}_3, \mathbf{y}_4) \subset \text{span}(\mathbf{x}_1, \mathbf{y}_1)$. This contradicts the assumption that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. Similarly, we get a contradiction if $a_4 \neq 0$. Hence $a_3 = a_4 = 0$. So $\mathbf{y}_3, \mathbf{y}_4 \in \text{span}(\mathbf{x}_3)$. This contradicts the assumption that $\mathbf{y}_3$ and $\mathbf{y}_4$ are linearly independent.

In conclusion we showed that our *Assumption* never holds. The proof of this case of Theorem 1.1 is concluded. □

5.2 Case rank $A(\mathcal{S}) \geq 4$

Proof. By induction on $r = \text{rank } A(\mathcal{S}) \geq 3$. For $r = 3$ the proof follows from the results above. Assume that Theorem holds for rank $\mathcal{S} = r+1$. Assume now that rank $A(\mathcal{S}) = r + 1$ and rank $\mathcal{S} = r + 2$. Without loss of generality we can assume that $n = r + 1$. Suppose first that the assumptions of Lemma 5.1 hold. If we are in the case *1* then srank $\mathcal{S}$ = rank $\mathcal{S}$. If we are in the case *2.* then we deduce from the induction hypothesis that srank $\mathcal{S}$ = rank $\mathcal{S}$.

As in the proof of the case rank $\mathcal{S} = 3$ we assume the *Assumption*: There does not exist a decomposition of $\mathcal{S}$ to rank $A(\mathcal{S}) + 1$ rank one tensors such that at least one of them is symmetric. We will show that we will obtain a contradiction.

Suppose $\mathcal{S} = \sum_{i=1}^{n+1} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$. The we have the equality

$$0 = \sum_{i=1}^{n+1} \mathbf{x}_i \otimes (\mathbf{y}_i \otimes \mathbf{z}_i - \mathbf{z}_i \otimes \mathbf{y}_i)$$

and the fact that span of all x's, y's and z's is $\mathbb{F}^n$.

Without loss of generality we may assume that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent. So $\mathbf{x}_{n+1} = \sum_{i=1}^{n} a_i \mathbf{x}_i$. Hence

$$\sum_{i+1}^{n} \mathbf{x}_i \otimes (\mathbf{y}_i \otimes \mathbf{z}_i - \mathbf{z}_i \otimes \mathbf{y}_i + a_i(\mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1} - \mathbf{z}_{n+1} \otimes \mathbf{y}_{n+1})) = 0. \tag{5.4}$$

As in the case $n = 3$ we can assume that $\mathbf{y}_{n+1}$ and $\mathbf{z}_{n+1}$ are not colinear. Thus if $a_i = 0$ we deduce that $\mathbf{y}_i$ and $\mathbf{z}_i$ are colinear. If $a_i \neq 0$ we deduce that $\text{span}(\mathbf{y}_i, \mathbf{z}_i) = \text{span}(\mathbf{y}_{n+1}, \mathbf{z}_{n+1})$. Since $\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ span $\mathbb{F}^n$ we can have at most two nonzero a_i. Since $\mathbf{x}_{n+1} \neq 0$ we must have at least one nonzero a_i. Assume first that $n - 1$ out of $\{a_1, \ldots, a_n\}$ are zero. We may assume without loss of generality that $a_1 = \ldots = a_{n-1} = 0$ and $a_n = 1$. So $\mathbf{x}_{n+1} = \mathbf{x}_n$. Without loss of generality we may assume that

$$\mathcal{S} = \mathbf{x}_n \otimes (\mathbf{y}_n \otimes \mathbf{z}_n + \mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}) + \sum_{i=1}^{n-1} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{y}_i.$$

Since $\mathcal{S}$ is symmetric as in case $n = 3$ we deduce that $\mathbf{y}_n \otimes \mathbf{z}_n + \mathbf{y}_{n+1} \otimes \mathbf{z}_{n+1}$ is symmetric and has rank two. So we can assume that $\mathbf{z}_n = \mathbf{y}_{n+1}, \mathbf{z}_{n+1} = \mathbf{y}_n$ and

$\dim \operatorname{span}(\mathbf{y}_n, \mathbf{y}_{n+1}) = 2$. We now repeat the arguments in the proof of this case for $n = 3$ to deduce the contradiction.

Suppose finally that exaclty $n-2$ out of $\{a_1, \ldots, a_n\}$ are zero. We may assume without loss of generality that $a_1 = \ldots = a_{n-2} = 0$ and $a_{n-1}, a_n \neq 0$. So $\operatorname{span}(\mathbf{y}_{n-1}, \mathbf{z}_{n-1}) = \operatorname{span}(\mathbf{y}_n, \mathbf{z}_n) = \operatorname{span}(\mathbf{y}_{n+1}, \mathbf{z}_{n+1})$. Hence $\mathbf{y}_{n-1}, \mathbf{y}_n, \mathbf{y}_{n+1}$ are linearly dependent. Since $\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ span the whole space we must have that $\dim \operatorname{span}(\mathbf{y}_{n-1}, \mathbf{y}_n, \mathbf{y}_{n+1}) = 2$. Without loss of generality we may assume the following: First, $\mathbf{y}_n, \mathbf{y}_{n+1}$ are linearly independent and $\mathbf{y}_{n-1} = a\mathbf{y}_n + b\mathbf{y}_{n+1}$. Second $\mathbf{z}_k = \mathbf{y}_k$ for $k = 1, \ldots, n-2$. So we can assume that

$$\mathcal{S} = \sum_{j=1}^{n-2} \mathbf{x}_j \otimes \mathbf{y}_j \otimes \mathbf{y}_j + \sum_{j=n-1}^{n+1} \mathbf{x}_j \otimes \mathbf{y}_j \otimes \mathbf{z}_j = \sum_{j=1}^{n-2} \mathbf{y}_j \otimes \mathbf{x}_j \otimes \mathbf{y}_j + \sum_{j=n-1}^{n+1} \mathbf{y}_j \otimes \mathbf{x}_j \otimes \mathbf{z}_j.$$

Permuting the las two factors in the last part of the above identity we obtain:

$$0 = \sum_{j=1}^{n-2} \mathbf{y}_j \otimes (\mathbf{x}_j \otimes \mathbf{y}_j - \mathbf{y}_j \otimes \mathbf{x}_j) + \sum_{j=n-1}^{n+1} \mathbf{y}_j(\mathbf{x}_j \otimes \mathbf{z}_j - \mathbf{z}_j \otimes \mathbf{x}_j).$$

Substitute $\mathbf{y}_{n-1} = a\mathbf{y}_n + b\mathbf{y}_{n+1}$ and recall that $\mathbf{y}_1, \ldots, \mathbf{y}_{n-2}, \mathbf{y}_n, \mathbf{y}_{n+1}$ are linearly independent. Hence $\mathbf{x}_i$ and $\mathbf{y}_i$ are colinear for $i = 1, \ldots, n-2 \geq 2$. This contradicts our *Assumption*. $\square$

6 Theorem 1.1 for $d \geq 4$

In this section we show Theorem for 1.1 for $d \geq 4$. Theorem 3.1 yields that it is enough to consider the case where $\operatorname{rank} \mathcal{S} = \operatorname{rank} A(\mathcal{S}) + 1$. We need the following lemma:

Lemma 6.1 *Let $2 \leq d \in \mathbb{N}$. Assume that $\mathbf{x}_{j,1}, \ldots \mathbf{x}_{j,n+1} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ and $\operatorname{span}(\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n+1}) = \mathbb{F}^n$ for $j \in [d]$. Consider the $n+1$ rank one d-tensors $\otimes_{j=1}^d \mathbf{x}_{j,i}, i \in [n+1]$. Then either all of them are linearly independent or n of these tensors are linearly independent and the other one is a multiple of one of the n linearly independent tensors.*

Proof. It is enough to consider the case where the $n+1$ rank one d-tensors $\otimes_{j=1}^d \mathbf{x}_{j,i}, i \in [n+1]$ are linearly dependent. Without loss of generality we may assume that $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,n}$ are linearly independent. Hence the n tensors $\otimes_{j=1}^d \mathbf{x}_{j,i}, i \in [n]$ are linearly independent as rank one matrices $\mathbf{x}_{1,i} \otimes (\otimes_{j=2}^d \mathbf{x}_{j,i})$ for $i \in [n]$. Assume that $\mathbf{x}_{1,n+1} = \sum_{j=1}^n a_j \mathbf{x}_{1,j}$ where not a_j are zero. Since we assumed that $\otimes_{j=1}^d \mathbf{x}_{j,i}, i \in [n+1]$ are linearly dependent it follows that $\otimes_{j=1}^d \mathbf{x}_{j,n+1} = \sum_{i=1}^n b_i \otimes_{j=1}^d \mathbf{x}_{j,i}$. So we obtain the indentity $\sum_{i=1}^n \mathbf{x}_{1,i} \otimes \mathcal{T}_i = 0$. Here $\mathcal{T}_i \in \otimes^{d-1}\mathbb{F}^n$ is a tensor of at most rank 2. Since $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,n}$ are linearly independent if follows that each $\mathcal{T}_i$ is zero. Hence if $a_i \neq 0$ it follows that b_i is not zero and $\otimes_{j=2}\mathbf{x}_{j,n+1}$ and $\otimes_{j=2}\mathbf{x}_{j,i}$ are colinear. Therefore $\mathbf{x}_{j,i}$ and $\mathbf{x}_{j,n+1}$ are colinear for $j = 2, \ldots, d$. Since $\dim \operatorname{span}(\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n+1}) = n$, we can't have another $a_k \neq 0$. So $\otimes_{j=1}^d \mathbf{x}_{j,n+1}$ is colinear with $\otimes_{j=1}\mathbf{x}_{j,i}$ as we claimed. $\square$

Proof of Theorem 1.1 for $d \geq 4$ **and** $\operatorname{rank} \mathcal{S} = \operatorname{rank} A(\mathcal{S}) + 1$. Without loss of generality we may assume that $n = \operatorname{rank} A(\mathcal{S}) \geq 2$. Assume that $\mathcal{S} = \sum_{i=1}^{n+1} \otimes_{j=1}^{d} \mathbf{x}_{j,i}$. Clearly, the assumptions of Lemma 6.1 holds. Consider the $d-2$ rank one tensors $\otimes_{j\in[d]\setminus\{p,q\}}\mathbf{x}_{j,i}$ for fixed $p \neq q \in [d]$ and $i \in [n+1]$. Suppose that these $n+1$ rank one tensors are linearly independent. We claim that $\mathbf{x}_{p,i}$ and $\mathbf{x}_{p,i}$ colinear for each $i \in [n+1]$. Without loss of generality we may assume that $p = 1, q = 2$. By interchanging the first two factors in the representation of $\mathcal{S}$ as a rank $n+1$ tensor we deduce:

$$\sum_{i=1}^{n+1} (\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} - \mathbf{x}_{2,i} \otimes \mathbf{x}_{1,i}) \otimes (\otimes_{j=3}^{d} \mathbf{x}_{j,i}) = 0.$$

As $\otimes_{j=3}^{d}\mathbf{x}_{j,i}, i \in [n+1]$ are linearly independent we deduce that $\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i} - \mathbf{x}_{2,i} \otimes \mathbf{x}_{1,i} = 0$ for each $i \in [n+1]$. I.e., $\mathbf{x}_{1,i}$ and $\mathbf{x}_{2,i}$ are colinear for each $i \in [n+1]$.

Suppose first that for each pair of integers $1 \leq p < q \leq d \otimes_{j\in[d]\setminus\{p,q\}}\mathbf{x}_{j,i}, i \in [n+1]$ are linearly independent. Hence $\mathbf{x}_{j,i} \in \operatorname{span}(\mathbf{x}_{1,i})$ for $j \in [d]$ and $i \in [n+1]$. Therefore $\otimes_{j=1}^{d}\mathbf{x}_{j,i}$ is a rank one symmetric tensor for each $i \in [n+1]$. Thus $\operatorname{srank} S = \operatorname{rank} S$.

Assume now, without loss of generality, that $\otimes_{j=3}^{d}\mathbf{x}_{j,i}, i \in [n+1]$ are linearly dependent. By applying Lemma 6.1 we can assume without loss of generality that $\otimes_{j=3}^{d}\mathbf{x}_{j,i}, i \in [n]$ are linearly independent and $\otimes_{j=3}^{d}\mathbf{x}_{j,n+1} = \otimes_{j=3}^{d}\mathbf{x}_{j,n}$. Without loss of generality we may assume that $\mathbf{x}_{j,n+1} = \mathbf{x}_{j,n}$ for $j \geq 3$. (We may need to rescale the vectors $\mathbf{x}_{2,n+1}, \ldots, \mathbf{x}_{d,n+1}$.) Hence $\mathbf{x}_{j,1}, \ldots, \mathbf{x}_{j,n}$ are linearly independent for each $j \geq 3$. Therefore we have the following decomposition of $\mathcal{S}$ as a 3-tensor in $(\otimes^2\mathbb{F}^n) \otimes \mathbb{F}^n \otimes (\otimes^{d-3}\mathbb{F}^n)$:

$$\begin{aligned}\mathcal{S} = \quad & (\mathbf{x}_{1,n} \otimes \mathbf{x}_{2,n} + \mathbf{x}_{1,n+1} \otimes \mathbf{x}_{2,n+1}) \otimes \mathbf{x}_{3,n} \otimes (\otimes_{j=4}^{d}\mathbf{x}_{j,n}) + \\ & \sum_{i=1}^{n-1} (\mathbf{x}_{1,i} \otimes \mathbf{x}_{2,i}) \otimes \mathbf{x}_{3,i} \otimes (\otimes_{j=4}^{d}\mathbf{x}_{j,i}) = \sum_{i=1}^{n} \mathcal{T}_i \otimes (\otimes_{j=4}^{d}\mathbf{x}_{j,i}).\end{aligned} \tag{6.1}$$

Clearly, $\otimes_{j=4}^{d}\mathbf{x}_{j,1}, \ldots, \otimes_{j=4}^{d}\mathbf{x}_{j,n}$ are linearly independent. Since $\mathcal{S}$ is symmetric by interchanging every two distinct factors $p, q \in [3]$ in $\otimes^d\mathbb{F}^n$ we deduce that $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are symmetric 3-tensors. Consider the symmetric tensor $\mathcal{T}_n = (\mathbf{x}_{1,n} \otimes \mathbf{x}_{2,n} + \mathbf{x}_{1,n+1} \otimes \mathbf{x}_{2,n+1}) \otimes \mathbf{x}_{3,n}$. As the rank of $A(\mathcal{T}_n)$ in the the third coordinate is 1 it follows that $\operatorname{rank} A(S) = 1$. Hence $\operatorname{rank} \mathcal{T}_n = 1$. Therefore $\operatorname{rank} \mathcal{S} \leq n$ contrary to our assumptions. □

Corollary 6.2 *Let* $|\mathbb{F}| \geq 3, d \geq 3, n \geq 2$. *Suppose that* $\mathcal{S} \in \mathrm{S}^d\mathbb{F}^n$ *and* $\operatorname{rank} \mathcal{S} \leq 3$. *Then* $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$.

Proof. It is enough to consider the case where $\operatorname{rank} A(\mathcal{S}) \geq 2$. Then $\operatorname{rank} \mathcal{S} \in \{2, 3\}$. Theorem 1.1 yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. □

7 Symmetric tensors over $\mathbb{C}$

Recall the known maximal value of the symmetric rank in $\mathrm{S}^d\mathbb{C}^n$, denoted as $\mu(d, n)$:

1. $\mu(d,2) = d$ [2, §3.1];

2. $\mu(3,3) = 5$ [13, §96], [5] and [10];

3. $\mu(3,4) = 7$ [13, §97];

4. $\mu(4,3) = 7$ [8, 11].

Theorem 7.1 *Let $\mathbb{F} = \mathbb{C}$ and $\mathcal{S}$ be a symmetric tensor in $\mathrm{S}^d\mathbb{C}^n$. Then* $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$ *in the following cases:*

1. *$d \geq 3, n \geq 2$ and $\operatorname{rank} \mathcal{S} \in \{\operatorname{rank} A(\mathcal{S}), \operatorname{rank} A(\mathcal{S}) + 1\}$.*

2. *For $n = 2$ and $d = 3$.*

3. *For $n = 2$ and $d = 4$*

4. *$n = d = 3$.*

5. *$\mathcal{S} \in \mathrm{S}^3\mathbb{C}^n$ and $\operatorname{rank} \mathcal{S} \leq 5$.*

Proof. Assume that $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$. Clearly, it is enough to prove the theorem for the case $\operatorname{rank} A(\mathcal{S}) \geq 2$. Then

$$2 \leq \operatorname{rank} \mathcal{S} \leq \operatorname{srank} \mathcal{S} \leq \mu(d,n). \tag{7.1}$$

1. follows from Theorem 1.1.
2. Assume $\mathcal{S} \in \mathrm{S}^3\mathbb{C}^2$. As $\mu(3,2) = 3$ we deduce the theorem from *1.*
3. Assume that $\mathcal{S} \in \mathrm{S}^4\mathbb{C}^2$. Suppose that $\operatorname{rank} \mathcal{S} \in \{2,3\}$. Then *1.* yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$.

Suppose that $\operatorname{rank} \mathcal{S} \geq 4$. As $\mu(4,2) = 4$ in view of (7.1) it follows that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S} = 4$.
4. Assume now that $\mathcal{S} \in \mathrm{S}^3\mathbb{C}^3$. Suppose first that $\operatorname{rank} A(\mathcal{S}) = 2$. Then by changing a basis in $\mathbb{C}^3$ we can assume that $\mathcal{S} \in \mathrm{S}^3\mathbb{C}^2$. *2.* yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$.

Suppose that $\operatorname{rank} A(\mathcal{S}) = 3$. If $\operatorname{rank} \mathcal{S} \in \{3,4\}$ then *1.* yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. Suppose now that $\operatorname{rank} \mathcal{S} \geq 5$. The equality $\mu(3,3) = 5$ and (7.1) yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S} = 5$.
5. If $\operatorname{rank} A(\mathcal{S}) = 2$ then *2.* yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. If $\operatorname{rank} A(\mathcal{S}) = 3$ *4.* then yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. If $\operatorname{rank} A(\mathcal{S}) \geq 4$ then *1.* yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$. □

The following proposition gives necessary as sufficient conditions for the equality $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$ for the smallest values of n and d which are not covered by Theorem 7.1:

Proposition 7.2 *The equality* $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$ *holds for all $\mathcal{S} \in \mathrm{S}^d\mathbb{C}^n$ for the following values of d and n if an only if the following conditions hold:*

1. *$n = 2, d = 5$ and $\operatorname{rank} \mathcal{S} = 4$ implies $\operatorname{srank} \mathcal{S} = 4$;*

2. *$n = 4, d = 3$ and $\operatorname{rank} \mathcal{S} = 6$ implies $\operatorname{srank} \mathcal{S} = 6$;*

Proof. It is enough to show that that the above conditions yield the equality $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$.

1. Assume that $n = 2$ and $d = 5$. Corollary 6.2 yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$ if $\operatorname{rank} \mathcal{S} \le 3$. Suppose that $\operatorname{rank} \mathcal{S} = 4$. Then we assume that $\operatorname{srank} \mathcal{S} = 4$. Suppose that $\operatorname{rank} \mathcal{S} \ge 5$. (7.1) and the equality $\mu(5, 2) = 5$ imply that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S} = 5$.

2. *5.* of Theorem 7.1 yields that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S}$ if $\operatorname{rank} \mathcal{S} \le 5$. Suppose that $\operatorname{rank} \mathcal{S} = 6$. Then we assume that $\operatorname{srank} \mathcal{S} = 6$. Suppose that $\operatorname{rank} \mathcal{S} \ge 7$. (7.1) and the equality $\mu(3, 4) = 7$ imply that $\operatorname{srank} \mathcal{S} = \operatorname{rank} \mathcal{S} = 7$. □

References

[1] J. Alexander and A. Hirschowitz, Polynomial interpolation in several variables, *J. Algebraic Geom.*, 4 (1995), 201–222.

[2] G. Blekherman and Z. Teitler, On maximum, typical and generic ranks, Mathematische Annalen, to appear, arXiv:1402.2371.

[3] L. Chiantini, G. Ottaviani and N. Vannieuwenhoven, On generic identifiability of symmetric tensors of subgeneric rank, arXiv:1504.00547.

[4] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, "Symmetric tensors and symmetric tensor rank,*SIAM Journal on Matrix Analysis and Applications*, 30 (2008), no. 3, pp. 1254–1279.

[5] P. Comon and B. Mourrain, Decomposition of quantics in sums of powers of linear forms, *Signal Processing*, Elsevier 53(2), 1996.

[6] S. Friedland, On the generic rank of 3-tensors , *Linear Algebra and its Applications*, 436 (2012) 478–497.

[7] S. Friedland and M. Stawiska, Best approximation on semi-algebraic sets and k-border rank approximation of symmetric tensors, arXiv:1311.1561.

[8] J. Kleppe, Representing a homogenous polynomial as a sum of powers of linear forms, Masters thesis, University of Oslo, 1999, http://folk.uio.no/johannkl/kleppe-master.pdf.

[9] J. B. Kruskal, Three-way arrays: Rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics, *Linear Algebra Applications*, 18 (1977), 95-138.

[10] J.M. Landsberg and Z. Teitler, On the ranks and border ranks of symmetric tensors, *Found. Comp. Math.* 10 (2010), no. 3, 339366.

[11] A. De Paris, A proof that the maximal rank for plane quartics is seven, http://arxiv.org/abs/1309.6475.

[12] J.A. Rhodes, A concise proof of Kruskals theorem on tensor decomposition, *Linear Algebra Appl.* 432 (2010), 1818–1824.

[13] B. Segre, The Non-singular Cubic Surfaces, Oxford University Press, Oxford, 1942.