

**THE ASYMPTOTIC LEADING TERM FOR MAXIMUM
RANK OF TERNARY FORMS OF GIVEN DEGREE**

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ABSTRACT. Let $r_{\max}(n, d)$ be the maximum Waring rank for the set of *all* homogeneous polynomials of degree d in n indeterminates with coefficients in an algebraically closed field of characteristic zero. We prove that $r_{\max}(3, d) = d^2/4 + O(d)$, as a consequence of the upper bound $\lfloor (d^2 + 6d + 1)/4 \rfloor$.

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1. INTRODUCTION

A natural kind of Waring problem asks for the least of the numbers r such that every homogeneous polynomial of degree $d > 0$ in n indeterminates can be written as a sum of r d -th powers of linear forms. For instance, when $(n, d) = (3, 4)$ (and the coefficients are taken in an algebraically closed field of characteristic zero) the answer is 7 and it was found for the first time in [10].

Every power sum decomposition gives rise to a set of points in the projectivized space of linear forms, and in [6] we showed that for ternary quartics we can always obtain a power sum decomposition by considering 7 points arranged along three lines. In [7], considering sets of points arranged along four lines, we proved that every ternary quintic is a sum of 10 fifth powers of linear forms. Soon after, ternary quintics without power sum decompositions with less than 10 summands were exhibited in [3]. Hence the answer in the case $(n, d) = (3, 5)$ is 10. For general information on the topic see, e.g., [11, Introduction].

In the present paper, we test ‘at infinity’ the technique of arranging decompositions of ternary forms along lines. More precisely, let $r_{\max}(n, d)$ denote the desired answer to the mentioned Waring problem. Elementary considerations show that $r_{\max}(n, d) = O(d^{n-1})$, and if $r_{\max}(n, d) = c_n d^{n-1} + O(d^{n-2})$ for some constant c_n (as it is reasonable to expect), then it must be $1/n! \leq c_n \leq 1/(n-1)!$. The best general upper bound on $r_{\max}(n, d)$ to our knowledge before the present paper was established in [1]. It implies that the constant c_n (if it exists) is at most $2/n!$. Using [4, Proposition 4.1] (see also [2, Theorem 7]), we deduce $r_{\max}(3, d) \geq \lfloor (d-1)^2/4 \rfloor$, hence it must be $c_3 \geq 1/4$. In the present work, for all ternary forms of degree d we obtain power sum decompositions by considering $\lfloor (d^2 + 6d + 1)/4 \rfloor$ points arranged along d lines. Hence we have $r_{\max}(3, d) = d^2/4 + O(d)$.

The upper bound we are proving lowers the general upper bound of [1] in the special case $n = 3$ and for $d \geq 6$. Nevertheless, it is not the best we can achieve, because our purpose was to determine the asymptotic leading term as simply as we could. In this respect we should mention that when the present paper was in preparation, a suggestion changed our mind on what is likely to expect for

$r_{\max}(3, d)$, and led us to some significant simplifications (to give a more precise acknowledgement here would be difficult).

2. PREPARATION

We work over an algebraically closed field \mathbb{K} of characteristic zero and fix two symmetric \mathbb{K} -algebras $S^\bullet = \text{Sym}^\bullet S^1$, $S_\bullet = \text{Sym}^\bullet S_1$; we shall keep this notation throughout the paper. We also assume that an *apolarity pairing* between S^\bullet, S_\bullet is given. It is naturally induced by a perfect pairing $S^1 \times S_1 \rightarrow \mathbb{K}$ (for more details see [6, Introduction]). This amounts to say that S^\bullet, S_\bullet are rings of polynomials in a finite and the same number of indeterminates, acting on each other by constant coefficients partial differentiation. For each $x \in S^\bullet$ and $f \in S_\bullet$ we shall denote by $\partial_x f$ the apolarity action of x on f . For each form (homogeneous polynomial) $f \in S_{d+\delta}$, we shall denote by $f_{\delta,d}$ the *partial polarization* map $S^\delta \rightarrow S_d$ defined by $f_{\delta,d}(x) := \partial_x f$. The *apolar ideal* of $f \in S_d$ is the set of all $x \in S^\bullet$ such that $\partial_x f = 0$. We also define the evaluation of a homogeneous form $x \in S^d$ on a linear form $v \in S_1$, by setting

$$x(v) := \frac{\partial_x v^d}{d!}.$$

The (Waring) *rank* of $f \in S_d$, denoted by $\text{rk } f$, is the least of the numbers r such that f can be written as a linear combination of r d -th powers of forms in S_1 ⁽¹⁾; $r_{\max}(n, d)$ is the maximum of the ranks of all such f when $\dim S_1 = n$. The span of v_1, \dots, v_r in some vector space V will be denoted by $\langle v_1, \dots, v_r \rangle$, and we formally define the projective space $\mathbb{P}V$ as the set of all one-dimensional subspaces $\langle v \rangle \subseteq V$, $v \neq 0$. By a *morphism of projective spaces* we mean a map $\mathbb{P}\varphi : \mathbb{P}V \setminus \mathbb{P} \text{Ker } \varphi \rightarrow \mathbb{P}W$, determined by a linear map $\varphi : V \rightarrow W$ by setting $\mathbb{P}\varphi(\langle v \rangle) := \langle \varphi(v) \rangle$. The sign \perp will refer to orthogonality with respect to the apolarity pairing $S^d \times S_d \rightarrow \mathbb{K}$, when some degree d is fixed (sometimes implicitly).

In [8, Sec. 1.3], building on classical results due to Sylvester, the authors deal with binary forms, $\dim S_1 = 2$ in our notation. They show that power sum decompositions are closely related with the initial degree of the (homogeneous) apolar ideal, that is, the least degree of a nonzero homogeneous element of that ideal. That is the notion of *length* of a binary form (see [8, Def. 1.32 and Lemma 1.33]), which can be generalized in various ways for forms in more indeterminates: see [8, Def. 5.66]. Nowadays, terms related to length are replaced by similar terms related with rank, probably because of the renewed interest in the interplay with the rank of tensors. In the present paper we need that notion only when the form is essentially binary, and what we really use is only its algebraic property of being the initial degree of the apolar ideal in a ring of binary forms. Note that a form $f \in S_d$ belongs to some subring $T_\bullet = \text{Sym}^\bullet T_1$ with $\dim T_1 = 2$, if and only if $\text{Ker } f_{1,d-1}$ has codimension at most 2 in S^1 (it suffices to take a two-dimensional $T_1 \supseteq \text{Ker } f_{1,d-1}^\perp$). Moreover, f belongs to more than one of such subrings if and only if $\text{Ker } f_{1,d-1}$ has codimension at most 1, in which case the initial degree of the apolar ideal of f in each of the subrings T_\bullet , whatever dual ring T^\bullet one chooses,

¹Since the Waring problem is about *sums* of powers, Waring rank is frequently defined using sums instead of linear combinations. But, since we are assuming that \mathbb{K} is algebraically closed, when $d > 0$ a form $f \in S_d$ is a sum of r d -th powers of linear forms if and only if it is a linear combination of r d -th powers of linear forms. Using linear combinations allows one to define Waring rank in degree 0 as well, and of course it is 1 for every nonzero constant.

is always the same (and equal to the codimension). This allows us to state the following definition.

Definition 2.1. Let $f \in S_d$. If f belongs to some ring $T_\bullet = \text{Sym}^\bullet T_1$, contained or containing S_\bullet (with the ring structures preserved), with $\dim T_1 = 2$, then we define the *binary length of f* as the initial degree of its apolar ideal, and we denote it by $\text{bl } f$.

The following definition is also useful.

Definition 2.2. Given $x \in S^\bullet$ and $f \in S_\bullet$, an *x -antiderivative of f* is a polynomial $F \in S_\bullet$ such that $\partial_x F = f$, and when x, f are nonzero and homogeneous we sometimes also say that $\langle F \rangle$ is a $\langle x \rangle$ -*antiderivative of $\langle f \rangle$* . Moreover, still in the homogeneous case $x \in S^\delta$, $f \in S_d$, if a decomposition

$$(1) \quad f = \lambda_1 v_1^d + \cdots + \lambda_r v_r^d, \quad \lambda_1, \dots, \lambda_r \in \mathbb{K}, v_1, \dots, v_r \in S_1,$$

is given and x vanishes on no one of v_1, \dots, v_r , then we define *the x -antiderivative of f* ⁽²⁾ *relative to (1)* as the form

$$F := \frac{d! \lambda_1}{(d + \delta)! x(v_1)} v_1^{d+\delta} + \cdots + \frac{d! \lambda_r}{(d + \delta)! x(v_r)} v_r^{d+\delta};$$

when the powers v_1^d, \dots, v_r^d are linearly independent we also say that the above x -antiderivative is *relative to v_1, \dots, v_r* .

Note that, in the above notation, the x -antiderivative relative to v_1, \dots, v_r is the unique x -antiderivative of f that lies in $\langle v_1^{d+\delta}, \dots, v_r^{d+\delta} \rangle$.

Now we explicitly point out two basic facts that are probably well-known, but for which we are not aware of a direct reference ⁽³⁾.

Remark 2.3. Let $h \in S^d$, $h' \in S^{d'}$ and $e \geq d$. Then h' divides h if and only if $S^e \cap \text{Ker } \partial_{h'} \subseteq S^e \cap \text{Ker } \partial_h$. One implication immediately follows from $\partial_{ph'} = \partial_p \circ \partial_{h'}$, and by the same reason we have $S^e \cap \text{Ker } \partial_{h'} \subseteq (h' S^{e-d'})^\perp$. When $h' \neq 0$, since $\partial_{h'}$ maps S^e onto $S^{e-d'}$ we have $\dim S^e \cap \text{Ker } \partial_{h'} = \dim S^e - \dim S^{e-d'}$. Since the apolarity pairing is nondegenerate in fixed degree, we also have $\dim (h' S^{e-d'})^\perp = \dim S^e - \dim S^{e-d'}$ when $h' \neq 0$. Hence $S^e \cap \text{Ker } \partial_{h'} = (h' S^{e-d'})^\perp$ (even for $h = 0$) and, similarly, $S^e \cap \text{Ker } \partial_h = (h S^{e-d})^\perp$. Now, to show the converse implication, let us suppose that $S^e \cap \text{Ker } \partial_{h'} \subseteq S^e \cap \text{Ker } \partial_h$, that is, $(h' S^{e-d'})^\perp \subseteq (h S^{e-d})^\perp$. Again because apolarity is nondegenerate in fixed degree, we deduce that $h S^{e-d} \subseteq h' S^{e-d'}$. Choosing $x \in S^1$ that does not divide h' (we can assume $h' \not\subseteq 0$ and $\dim S_1 \geq 2$, since the proof is trivial in the opposite case), we have that h' divides $x^{e-d} h$, hence h' divides h .

Remark 2.4. Let $f \in S^d$ with $d \geq 3$ and $\dim S_1 = 3$. Then f is a d -th power if and only if for each $\langle x \rangle \in \mathbb{P}S^1$, $\partial_x f$ is a $(d-1)$ -th power. One implication is immediate. Conversely, suppose that for each $\langle x \rangle \in \mathbb{P}S^1$, $\partial_x f$ is a $(d-1)$ -power of

²Or also the $\langle x \rangle$ -antiderivative of $\langle f \rangle$, if $f \neq 0$.

³Basic facts like these are heavily scattered in the literature, and we easily overlook some reference. For instance, [7, Remark 3.6] could have been avoided by using the original reference [5, Theorem 2] instead of [9, 9.2.2.1]. We take this occasion for apologizing for that.

some linear form. Let $x, y \in S^1$ be linearly independent. According to [7, Prop. 4.1] we have that $\partial_l f = 0$ for some nonzero $l \in \langle x, y \rangle$ ⁽⁴⁾. Repeating the argument with another two-dimensional subspace of S^1 that does not contain l , we get $m \in S^1$ such that $\partial_m = 0$ and with l, m linearly independent. Hence $\text{Ker } f_{1,d-1} \geq 2$, and since $\dim S_1 = 3$ we have that f is a d -th power of a linear form.

Moreover, let $f \in S_d$, $d \geq 3$, be not a d -th power. The set of all $\langle v^{d-1} \rangle$ with $\langle v \rangle \in \mathbb{P}S_1$ is an algebraic (Veronese) variety in $\mathbb{P}S_{d-1}$, and $\mathbb{P}f_{1,d-1}$ is an algebraic morphism from a nonempty Zariski open subset of $\mathbb{P}S^1$ to $\mathbb{P}S_{d-1}$. Then $\partial_x f$ is not a $(d-1)$ -th power for all $\langle x \rangle$ in a nonempty open Zariski subset of $\mathbb{P}S^1$.

We end this section with two technical lemmas.

Lemma 2.5. *Let $\langle f \rangle \in \mathbb{P}S_d$ and $\langle x \rangle \in \mathbb{P}S^1$, with $\dim S_1 = 2$. Let I be the apolar ideal of f and set $\ell := \text{bl } f$, $\ell' := d + 2 - \ell$,*

$$(2) \quad W := S_{d+1} \cap \partial_x^{-1}(\langle f \rangle), \quad H := S^{\ell'} \cap I, \quad K := S^{\ell'} \cap xI.$$

Finally, let X be the locus of all $\langle h \rangle \in \mathbb{P}H$ such that h is not squarefree and set $\langle v_\infty \rangle := \langle x \rangle^\perp$, so that

$$\langle v_\infty^{d+1} \rangle = S_{d+1} \cap \text{Ker } \partial_x \subset W.$$

Then

- *there exists an epimorphism of projective spaces*

$$\omega : \mathbb{P}H \setminus \mathbb{P}K \rightarrow \mathbb{P}W, \quad \omega(\langle h \rangle) =: \langle w_h \rangle,$$

such that $\partial_h w_h = 0$ for all $\langle h \rangle$;

- *for all $\langle w \rangle \in \mathbb{P}W \setminus \langle v_\infty^{d+1} \rangle$ but at most one, we have*

$$\text{bl } w = \min\{\ell + 1, \ell'\};$$

- $X \subsetneq \mathbb{P}H$;
- *for each projective line $\mathbb{P}L \subseteq \mathbb{P}H$ that does not meet $\mathbb{P}K$, the restriction $\mathbb{P}L \rightarrow \mathbb{P}W$ of ω is an isomorphism of projective spaces, and if the line $\mathbb{P}L$ is not contained in X then there exists a cofinite subset $U \subset \mathbb{P}L$ such that for each $\langle h \rangle \in U$ we have*
 - h has distinct roots $\langle v_1 \rangle, \dots, \langle v_{\ell'} \rangle \in \mathbb{P}S_1$;
 - $f \in \langle v_1^d, \dots, v_{\ell'}^d \rangle$;
 - x vanishes on no one of $v_1, \dots, v_{\ell'}$ and $\langle w_h \rangle$ is the $\langle x \rangle$ -antiderivative of $\langle f \rangle$ relative to $v_1, \dots, v_{\ell'}$;

Proof. For each $h \in H$ and $w \in W$ we have $\partial_{xh} w = 0$, because $\partial_x w \in \langle f \rangle$ and $h \in I$; hence $\partial_h w \in S_{\ell-1} \cap \text{Ker } \partial_x = \langle v_\infty^{\ell-1} \rangle$. Thus we have a bilinear map

$$\beta : H \times W \rightarrow \langle v_\infty^{\ell-1} \rangle, \quad \beta(h, w) := \partial_h w.$$

If $h \in K$, then $h = xh'$ for some $h' \in I$, hence for all $w \in W$ we have $\partial_h w = \partial_{h'} \partial_x w = 0$, because $\partial_x w \in \langle f \rangle$. This shows that K is contained in the left kernel of β . Conversely, if h is in the left kernel, then ∂_h vanishes on W , and in particular on $\langle v_\infty^{d+1} \rangle \subset W$. Hence $h = xh'$ for some $h' \in S^d$, by Remark 2.3 ⁽⁵⁾. Choosing

⁴In the statement of the cited result, by mistake, one finds $f \in S_d$ instead $f \in S_{d+1}$ and the condition $d \geq 2$ is omitted; cf. [7, Rem. 2.2].

⁵Alternatively, one may observe that $0 = \partial_h(v_\infty^{d+1}) = \frac{(d+1)!}{\ell!} h(v_\infty) v_\infty^{\ell-1}$. Hence h vanishes on the root v_∞ of x , that is, h is divisible by x .

an x -antiderivative w of f , we have $0 = \partial_h w = \partial_{h'} f$, and thus $h' \in I$. We conclude that K is the left kernel of β .

Let

$$\bar{\beta} : H \rightarrow \text{Hom}(W, \langle v_\infty^{\ell-1} \rangle), \quad \bar{\beta}(h)(w) := \beta(h, w) = \partial_h w,$$

be the map induced by β , and let $\iota : W \rightarrow \text{Hom}(W, \langle v_\infty^{\ell-1} \rangle)$ be an isomorphism such that $\iota(w)(w) = 0$ for all $w \in W$ (in other words, ι is the map induced by a nondegenerate bilinear alternating map on W with values in $\langle v_\infty^{\ell-1} \rangle$, which certainly exists because $\dim W = 2$). Then $\varphi := \iota^{-1} \circ \bar{\beta} : H \rightarrow W$ is a linear map with kernel K such that $\partial_h(\varphi(h)) = 0$ for all $h \in H$. This shows that $\omega := \mathbb{P}\varphi$ is a morphism of projective spaces such that $\partial_h w_h = 0$ (under the notation $\langle w_h \rangle := \omega(\langle h \rangle) = \langle \varphi(h) \rangle$). We have to check that ω is surjective.

According to [8, Theorem 1.44(iv)], I is generated by two homogeneous forms $l \in S^\ell, h^0 \in S^{\ell'}$ (hence $h^0 \in H$). Recall also that $\ell \leq \ell'$ because $\ell = \text{bl } f$. Therefore

$$(3) \quad H = lS^{\ell'-\ell} + \langle h^0 \rangle, \quad K = lxS^{\ell'-\ell-1}.$$

Since $S^{d+1} \subset I$, we have that l, h^0 are coprime, and therefore $h^0 \notin lS^{\ell'-\ell}$. Since $\dim \varphi(lS^{\ell'-\ell}) = 1$ we have that φ is surjective, and hence ω is surjective as it was to be shown.

Let $\varphi(lS^{\ell'-\ell}) =: \langle w_l \rangle \in \mathbb{P}W$ (possibly $\langle w_l \rangle = \langle v_\infty^{d+1} \rangle$). Since $\partial_{l_p} w_l = 0$ for all $p \in S^{\ell'-\ell}$, we have $\partial_l w_l = 0$. Note also that $w_l = w_{pl}$ for all $p \in S^{\ell'-\ell} \setminus xS^{\ell'-\ell-1}$. Moreover,

$$(4) \quad \langle h \rangle, \langle h' \rangle \in \mathbb{P}H \setminus \mathbb{P}K, \langle w_{h'} \rangle \neq \langle w_h \rangle \implies \partial_h w_{h'} \neq 0,$$

because $\partial_h w_{h'} = 0$ would imply that ∂_h vanishes on $\langle w_h, w_{h'} \rangle = W$ ($\dim W = 2$), and this is excluded since $h \notin K$. Since ω is surjective, we conclude that $\partial_l w \neq 0$ for each $\langle w \rangle \in \mathbb{P}W \setminus \{\langle w_l \rangle\}$. On the other hand, if $\langle w \rangle \in \mathbb{P}W \setminus \langle v_\infty^{d+1} \rangle$, then $\langle \partial_x w \rangle = \langle f \rangle$, and hence the apolar ideal of w is contained in I and contains xI . Thus

$$\ell \leq \text{bl } w \leq \ell + 1, \quad \forall \langle w \rangle \in \mathbb{P}W \setminus \langle v_\infty^{d+1} \rangle.$$

Now, if $\ell' \geq \ell + 1$, then for each $\langle w \rangle \in \mathbb{P}W \setminus \{\langle v_\infty^{d+1} \rangle, \langle w_l \rangle\}$ we have $\text{bl } w = \ell + 1 = \min\{\ell + 1, \ell'\}$. To deal with the case $\ell' = \ell$, notice that for each $w \in W$ we have $\partial_h w = 0$ for some $\langle h \rangle \in H$, because φ is surjective; hence $\text{bl } w \leq \ell'$. Thus if $\ell = \ell'$, then for each $\langle w \rangle \in \mathbb{P}W \setminus \langle v_\infty^{d+1} \rangle$ we have $\text{bl } w = \ell' = \min\{\ell + 1, \ell'\}$ ⁽⁶⁾.

Since l, h^0 are coprime, taking into account (3) and Bertini's theorem (see also [10, Lemma 1.1, Remark 1.1.1]), we have that X is a proper subset of $\mathbb{P}H$.

Finally, let $\mathbb{P}L \subseteq \mathbb{P}H \setminus \mathbb{P}K$ be a projective line. The restriction $\mathbb{P}L \rightarrow \mathbb{P}W$ of ω is an isomorphism simply because $\mathbb{P}W$ is a projective line as well, and $\mathbb{P}L \cap \mathbb{P}K = \emptyset$. Since the proper subset $X \subsetneq \mathbb{P}H$ is algebraic, with equation given by the discriminant of degree ℓ' forms (inside $\mathbb{P}H$), we have that $U := \mathbb{P}L \setminus (X \cup \omega^{-1}(\langle v_\infty^{d+1} \rangle))$ is a cofinite subset of $\mathbb{P}L$. Since each $\langle h \rangle \in U$ is outside X , h is squarefree, that is, it has distinct roots $\langle v_1 \rangle, \dots, \langle v_{\ell'} \rangle \in \mathbb{P}S_1$. For such $h, v_1, \dots, v_{\ell'}$, according to [8, Lemma 1.31], we have $f \in \langle v_1^d, \dots, v_{\ell'}^d \rangle$ as required. By the same reason, we have $w_h \in \langle v_1^{d+1}, \dots, v_{\ell'}^{d+1} \rangle$, and since $\langle w_h \rangle \neq \langle v_\infty^{d+1} \rangle$, $\langle w_h \rangle$ is a $\langle x \rangle$ -antiderivative of $\langle f \rangle$. Moreover, x vanishes on no one of $v_1, \dots, v_{\ell'}$ by (4), and

⁶The equality $\text{bl } w = \min\{\ell + 1, \ell'\}$ we have just proved for all $\langle w \rangle \in \mathbb{P}W \setminus \{\langle v_\infty^{d+1} \rangle, \langle w_l \rangle\}$, in other terms, says that $\text{bl } w = \ell + 1$ unless d is even, $d = 2s$, and ℓ is the maximum allowed for that degree, that is, $s + 1$.

$v_1^d, \dots, v_{\ell'}^d$ are linearly independent because $\ell' \leq d+1$. The above said suffices to prove that $\langle w_h \rangle$ is the $\langle x \rangle$ -antiderivative of $\langle f \rangle$ relative to $v_1, \dots, v_{\ell'}$. \square

Lemma 2.6. *Let $\langle g' \rangle \in \mathbb{P}S_d$ with $\dim S_1 = 3$, $d = 2s + \varepsilon$, $\varepsilon \in \{0, 1\}$ and s integer. Let $\langle l^0 \rangle, \dots, \langle l^t \rangle \in \mathbb{P}S^1$ be distinct and such that $\partial_{l^0} g' = 0$, and for each $i \in \{1, \dots, t\}$ let g_i be an l^i -antiderivative of g' . If*

$$\text{bl } g' = \text{bl } \partial_{l^0} g_1 = \dots = \text{bl } \partial_{l^0} g_t = s + 1$$

(⁷), then there exists a power sum decomposition

$$(5) \quad g' = v_1^d + \dots + v_r^d$$

such that: $r \leq s + 1 + \varepsilon$ and for each $i \in \{1, \dots, t\}$

- l^i vanishes on no one of v_1, \dots, v_r ;
- denoting by F_i the l^i -antiderivative relative to (5), $\text{bl } (g_i - F_i) = s + 1 + \varepsilon$.

Proof. For each $i \in \{0, \dots, t\}$, let $R_i^\bullet := S^\bullet / (l^i)$, $R_{i,\bullet} := \text{Ker } \partial_{l^i} \subset S_\bullet$, with the apolarity pairing induced by the one between S^\bullet and S_\bullet . Let $I \subset R_0^\bullet$ be the apolar ideal of $g' \in R_{0,d}$, set

$$H := R_0^{s+1+\varepsilon} \cap I,$$

for each $i \in \{1, \dots, t\}$ set

$$W_{0,i} := R_{0,d+1} \cap \partial_{l^i}^{-1}(\langle g' \rangle)$$

and when $\varepsilon = 1$, also $\langle k_i \rangle := R_0^{s+2} \cap l^i I$. For each $i \in \{1, \dots, t\}$, let us exploit Lemma 2.5 with R_0^\bullet , $R_{0,\bullet}$, g' , $l^i + (l^0)$ in place of S^\bullet , S_\bullet , f , x . We get projective epimorphisms

$$\omega_i : \mathbb{P}H \rightarrow \mathbb{P}W_{0,i}.$$

Moreover, we can fix a projective line $\mathbb{P}L \subseteq \mathbb{P}H$ ($\mathbb{P}L = \mathbb{P}H$ when $\varepsilon = 0$) not contained in the singular locus X (which does not depend on i) and passing through no one of $\langle k_1 \rangle, \dots, \langle k_t \rangle$ (when $\varepsilon = 1$). Hence, the restriction $\varrho_i : \mathbb{P}L \rightarrow \mathbb{P}W_{0,i}$ of ω_i is a projective isomorphism for each i , and we also have cofinite subsets $U_{0,i} \subset \mathbb{P}L$ that fulfill the properties listed by the end of the statement of Lemma 2.5.

Now, for each i we have $\partial_{l^0} g_i \neq 0$ because $\text{bl } \partial_{l^0} g_i = s + 1 > 0$. Hence the vector space

$$W_{i,0} := R_{i,d+1} \cap \partial_{l^0}^{-1}(\langle \partial_{l^0} g_i \rangle)$$

is two-dimensional. Since $W_{0,i} = R_{0,d+1} \cap \partial_{l^i}^{-1}(\langle g' \rangle)$, for all $w \in W_{0,i}$ we have

$$\partial_{l^i} w = \lambda_i(w) g'$$

for some scalar $\lambda_i(w)$, and therefore $\lambda_i(w) g_i - w \in W_{i,0}$. This defines a map $W_{0,i} \rightarrow W_{i,0}$ and to check that it is a vector space isomorphism is easy (take into account that $\partial_{l^0}(\lambda_i(w) g_i - w) = \lambda_i(w) \partial_{l^0} g_i$). Therefore we have projective isomorphisms

$$\tau_i : \mathbb{P}W_{0,i} \rightarrow \mathbb{P}W_{i,0}, \quad \langle w \rangle \mapsto \langle \lambda_i(w) g_i - w \rangle.$$

According to Lemma 2.5, we have cofinite subsets $U'_{i,0} \subset \mathbb{P}W_{i,0}$ such that

$$(6) \quad \text{bl } w = s + 1 + \varepsilon, \quad \forall \langle w \rangle \in U'_{i,0}$$

(more precisely, $\sharp(\mathbb{P}W_{i,0} \setminus U'_{i,0}) \leq 2$).

⁷For each i , $\partial_{l^0} g_i$ is annihilated by l^i .

Let $U_{i,0} := \varrho_i^{-1}(\tau_i^{-1}(U'_{i,0}))$ for each i , which is obviously a cofinite subset as well. Now, let us pick $\langle h \rangle$ in the nonempty intersection

$$U_{0,1} \cap \cdots \cap U_{0,t} \cap U_{1,0} \cap \cdots \cap U_{t,0},$$

and let $\langle v_1 \rangle, \dots, \langle v_{s+1+\epsilon} \rangle$ be its roots, which are distinct because $\langle h \rangle \in U_{0,i}$ (whatever i one chooses). For each i , l^i vanishes on no one of $v_1, \dots, v_{s+1+\epsilon}$, because $\langle h \rangle \in U_{0,i}$. Since $g' \in \langle v_1^d, \dots, v_{s+1+\epsilon}^d \rangle$, for an appropriate choice of the representatives $v_1, \dots, v_{s+1+\epsilon}$ one gets (5). Since F_i is the l^i -antiderivative of g' relative to (5), that is, relative to $v_1, \dots, v_{s+1+\epsilon}$, we have $\langle F_i \rangle = \omega_i(\langle h \rangle)$. Since F_i is an l^i -antiderivative of g' , we have $\lambda_i(F_i) = 1$, and hence

$$\tau_i(\omega_i(\langle h \rangle)) = \langle g_i - F_i \rangle.$$

Since $\langle h \rangle \in U_{i,0}$ for each i , we have $\langle g_i - F_i \rangle \in U'_{i,0}$, and therefore $\text{bl}(g_i - F_i) = s + \epsilon + 1$ by (6). \square

3. THE UPPER BOUND

Proposition 3.1. *Let $f \in S_d$, with $\dim S_1 = 3$ and $d \geq 2$, let $f_1, \dots, f_a \in S_\bullet$ be homogeneous forms with degrees $\geq d$, and let $\langle x^1 \rangle, \dots, \langle x^b \rangle \in \mathbb{P}S^1$. If f, f_1, \dots, f_a are not powers of linear forms, then there exist distinct*

$$\langle l^1 \rangle, \dots, \langle l^d \rangle \in \mathbb{P}S^1 \setminus \{ \langle x^1 \rangle, \dots, \langle x^b \rangle \}$$

such that

$$\partial_{l^1 \dots l^d} f = 0, \quad \partial_{l^1 \dots \widehat{l^i} \dots l^d} f \neq 0 \quad \forall i \in \{1, \dots, d\}, \quad \partial_{l^1 \dots l^d} f_j \neq 0 \quad \forall j \in \{1, \dots, a\},$$

where the hat denotes omission.

Proof. Let us first suppose that $d = 2$. For each $i \in \{1, \dots, a\}$, since f_i is not a power of a linear form, we may have $\partial_l f_i = 0$ for at most one $\langle l \rangle \in \mathbb{P}S^1$. Therefore there exists a finite subset $\Sigma \subset \mathbb{P}S^1$ such that for all $\langle l \rangle \in \mathbb{P}S^1 \setminus \Sigma$ and $i \in \{1, \dots, a\}$ we have $\partial_l f_i \neq 0$. By hypothesis, f is not a square, hence $\partial_l f = 0$ for at most one $\langle l \rangle \in \mathbb{P}S^1$. Let us fix $\langle l^2 \rangle \in \mathbb{P}S^1$ such that $f' := \partial_{l^2} f \neq 0$. Since $\text{Ker } f'_{1,d-1} = \langle f' \rangle^\perp$ is an infinite set, we can pick out $\langle l^1 \rangle \in \mathbb{P} \langle f' \rangle^\perp$ such that

- $\partial_{l^1} f \neq 0$;
- $\langle l^1 \rangle \notin \{ \langle l^2 \rangle, \langle x^1 \rangle, \dots, \langle x^b \rangle \} \cup \Sigma$.

It is immediate to check that $\langle l^1 \rangle, \langle l^2 \rangle$ fulfill all the requirements in the statement.

Now, let us assume $d \geq 3$. According to Remark 2.4, we can fix

$$\langle l^d \rangle \in \mathbb{P}S^1 \setminus \{ \langle x^1 \rangle, \dots, \langle x^b \rangle \}$$

such that $f' := \partial_{l^d} f$, $f'_1 := \partial_{l^d} f_1, \dots, f'_a := \partial_{l^d} f_a$ are not powers of linear forms. By induction on d , we can assume that the statement holds with f' in place of f , with f, f'_1, \dots, f'_a in place of f_1, \dots, f_a and with l^d, x^1, \dots, x^b in place of x^1, \dots, x^b . This gives linear forms l^1, \dots, l^{d-1} that, together with l^d , fulfill all the requirements. \square

Proposition 3.2. *Let $f \in S_d$ with $\dim S_1 = 3$, $e \in \{0, \dots, d\}$, $e = 2s + \epsilon$, with $\epsilon \in \{0, 1\}$ and s integer, and let*

$$\langle l^1 \rangle, \dots, \langle l^d \rangle \in \mathbb{P}S^1$$

be distinct and such that

$$\partial_{l^1 \dots l^d} f = 0; \quad \partial_{l^1 \dots \widehat{l^i} \dots l^d} f \neq 0 \quad \forall i.$$

Then there exists a power sum decomposition

$$(7) \quad \partial_{l^{e+1}\dots l^d} f = v_1^e + \dots + v_r^e$$

such that:

- $r \leq s^2 + 3s + \varepsilon(s + 2)$;
- for each $i \in \{e + 1, \dots, d\}$, l^i vanishes on no one of v_1, \dots, v_r and denoting by F_i the l^i -antiderivative relative to (7), we have

$$\text{bl}(\partial_{l^{e+1}\dots \widehat{l^i}\dots l^d} f - F_i) = s + 1 + \varepsilon.$$

Proof. When $e = 0$ it suffices to define (7) as the decomposition of 0 with no summands. By induction, we can assume that $e \geq 1$ and that the proposition holds with $e - 1$ in place of e . Therefore we get a decomposition

$$(8) \quad \partial_{l^e\dots l^d} f = v_1'^{e-1} + \dots + v_{r'}'^{e-1}$$

such that

$$(9) \quad r' \leq s^2 + 2s - 1 + \varepsilon(s + 1)$$

and each of l^e, \dots, l^d vanishes on no one of $v_1', \dots, v_{r'}'$. We can also consider for each $i \in \{e, \dots, d\}$ the l^i -antiderivative relative to (8), which we denote by G_i' , and set

$$g_i' := \partial_{l^e\dots \widehat{l^i}\dots l^d} f - G_i',$$

so that

$$\text{bl} g_i' = s + 1.$$

For each $i \in \{e + 1, \dots, d\}$, let G_i be the $l^e l^i$ -antiderivative relative to (8) and set

$$g_i := \partial_{l^{e+1}\dots \widehat{l^i}\dots l^d} f - G_i,$$

so that

$$\partial_{l^e} g_i = g_i', \quad \partial_{l^i} g_i = g_e'.$$

By the above construction, we can exploit Lemma 2.6 with g_e', l^e, \dots, l^d in place of g', l^0, \dots, l^t . We get a decomposition

$$(10) \quad g_e' = v_1^e + \dots + v_{r''}^e$$

such that

$$(11) \quad r'' \leq s + 1 + \varepsilon,$$

each of l^{e+1}, \dots, l^d vanishes on no one of $v_1, \dots, v_{r''}$ and denoting by H_i the l^i -antiderivative relative to (10), we have

$$(12) \quad \text{bl}(g_i - H_i) = s + 1 + \varepsilon.$$

Since we defined G_e' as the l^e -antiderivative relative to (8), by taking suitable multiples of $v_1', \dots, v_{r'}'$ and calling them $v_{r''+1}, \dots, v_{r''+r'}$, respectively, we have

$$G_e' = v_{r''+1}^e + \dots + v_r^e,$$

with $r := r'' + r'$. By definition of g_e' and by (10) we conclude that

$$\partial_{l^{e+1}\dots l^d} f = g_e' + G_e' = v_1^e + \dots + v_r^e.$$

To show that the above is the required decomposition (7), first note that (9) and (11) give

$$r \leq s^2 + 3s + \varepsilon(s + 2),$$

as it was to be shown. Moreover, since each of l^e, \dots, l^d vanishes on no one of $v'_1, \dots, v'_{r'}$, which are proportional to $v_{r''+1}, \dots, v_r$, and each of l^{e+1}, \dots, l^d vanishes on no one of $v_1, \dots, v_{r''}$, we have that for each $i \in \{e+1, \dots, d\}$, l^i vanishes on no one of v_1, \dots, v_r . Finally, for the l^i -antiderivatives F_i s we have

$$F_i = G_i + H_i = \partial_{l^{e+1} \dots \widehat{l^i} \dots l^d} f - g_i + H_i,$$

hence

$$\partial_{l^{e+1} \dots \widehat{l^i} \dots l^d} f - F_i = g_i - H_i$$

and the last condition to be shown follows from (12). \square

Proposition 3.3. *When $\dim S_1 = 3$, for all $f \in S^d$ we have*

$$\text{rk } f \leq \left\lfloor \frac{d^2 + 6d + 1}{4} \right\rfloor.$$

Proof. If f is a d -th power then $\text{rk } f \leq 1$ and the result trivially follows. Hence we can assume that f is not a d -th power. Exploiting Proposition 3.1 with $a = b = 0$, we get $\langle l^1 \rangle, \dots, \langle l^d \rangle \in \mathbb{P}S^1$ such that

$$\partial_{l^1 \dots l^d} f = 0, \quad \partial_{l^1 \dots \widehat{l^i} \dots l^d} f \neq 0 \quad \forall i \in \{1, \dots, d\}.$$

Now the result immediately follows from Proposition 3.2 with $e := d$. \square

Proposition 3.4. *We have $r_{\max}(3, d) = d^2/4 + O(d)$.*

Proof. An immediate consequence of Proposition 3.3, together with [4, Proposition 4.1] (see also [2, Theorem 7]) or [3, Theorem 1]. \square

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