Curves and semigroups

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Even if one is interested in singularities of algebraic curves, one is lead to analytic functions. The standard example is the irreducible curve

$$y^2 - x^2 - x^3 = 0.$$

This has a double point in the origin, and in a neighbourhood it seems that the curve has two branches.

In the ring of formal power series $\mathbb{C}[[x, y]]$, or in the ring of germs of analytic functions around the origin $\mathbb{C}\langle x, y\rangle$,

$$y^2 - x^2 - x^3 = (y - x\sqrt{1 + x})(y + x\sqrt{1 + x}).$$

Thus, here we see the two branches. If f is an irreducible element in $\mathbb{C}[[x, y]]$, then f (or $\mathbb{C}[[x, y]]/(f)$) is called an *algebroid plane branch*. If f is an irreducible element in $\mathbb{C}\langle x, y\rangle$, then f (or $\mathbb{C}\langle x, y\rangle/(f)$) is called an *analytic plane branch*.

More general, a 1-dimensional domain

$$\mathbb{C}[[x_1, \dots, x_n]]/P$$

($\mathbb{C}\langle x_1, \dots, x_n\rangle/P$, resp.) is an algebroid branch (an analytic branch, resp.).

An element $f \in \mathbb{C}[[x_1, \dots, x_n]]$ (or $f \in \mathbb{C}\langle x_1, \dots, x_n\rangle$) is said to be **general** in x_n if $f(0, \dots, 0, x_n) \neq 0$. The multiplicity of $\mathbb{C}[[x_1, \dots, x_n]]/I$ is the smallest order of an element in I.

If $f \in \mathbb{C}[[x_1, \dots, x_n]]$ (or $f \in \mathbb{C}\langle x_1, \dots, x_n\rangle$) and $o(f) = k$, there is a transformation

$$\begin{cases} x_i = y_i + c_i y_n, & i = 1, \dots, n \\ x_n = y_n & \end{cases}$$

such that $f(X(Y)) \in \mathbb{C}[[y_1, \dots, y_n]]$
(or $f \in \mathbb{C}\langle x_1, \dots, x_n\rangle$) is general in y_n of order k.

Weierstrass' Preparation theorem

If $g \in \mathbb{C}[[x_1, \dots, x_n]]$ (or $g \in \mathbb{C}\langle x_1, \dots, x_n\rangle$) is general of order k, there exist a unit $\alpha(x_1, \dots, x_n)$ and a polynomial in x_n

$$p = x_n^k + a_1(x_1, \dots, x_{n-1})x_n^{k-1} + \cdots + a_n(x_1, \dots, x_{n-1}),$$

so $p \in \mathbb{C}[[x_1, \dots, x_{n-1}]][x_n]$ such that $g = \alpha(x_1, \dots, x_n)p$. Here p is called a Weierstrass polynomial, and p has the same zeros as g.

If $f \in \mathbb{C}\langle x_1, \dots, x_n\rangle$, then $f = f_1^{n_1} \cdots f_r^{n_r}$, where f_i are irreducible for all i, and the zero set of f is the union of the zero sets of the f_i's, $V(f) = V(f_1) \cup \cdots \cup V(f_r)$. The rings $\mathbb{C}[[y_1, \dots, y_n]]$ and $\mathbb{C}\langle x_1, \dots, x_n\rangle$ are in many respects similar to the polynomial ring. They are Noetherian UFD's.

The implicit function theorem

Suppose $f \in \mathbb{C}\langle x_1, \dots, x_n, y\rangle$, $f(\mathbf{0}) = 0$, and $\partial f/\partial y(\mathbf{0}) \neq 0$. Then there exists $\phi(x_1, \dots, x_n)$ such that $f(x_1, \dots, x_n, \phi(x_1, \dots, x_n)) = 0$ in a neighbourhood of $\mathbf{0}$. We get a parametrization

$$\begin{cases} x_1 &= & t_1 \\ & \vdots & \\ x_n &= & t_n \\ y &= & \phi(t_1, \dots, t_n) \end{cases}$$

Now $f = x^3 - y^2$ is an irreducible power series, so if we could paramertize $f = 0$ as

$$\begin{cases} x &= & t \\ y &= & \phi(t) \end{cases}$$

we would have $t^3 - (\phi(t))^2 = 0$, which is impossible, but we can write

$$\begin{cases} x &= & t \\ y &= & t^{3/2} \end{cases}$$

or

$$\begin{cases} x &= & t^2 \\ y &= & t^3 \end{cases}$$

Parametrization in Puiseux series

Theorem Suppose $f \in \mathbb{C}[[x, y]]$ (or $f \in \mathbb{C}\langle x, y\rangle$) is general in y of order $k \geq 1$. Then there exist $n \geq 0$ and $\phi(t) \in \mathbb{C}[[t]]$ such that $\phi(0) = 0$ and $f(t^n, \phi(t)) = 0$.

Let $\mathcal{O} = \mathbb{C}[[t^n, \sum a_i t^i]]$ be a plane branch. Then the integral closure of $\mathcal{O}$ is $\bar{\mathcal{O}} = \mathbb{C}[[t]]$, which is a discrete valuation ring. $(v(f) = o(f).)$ Thus $\mathcal{O}$ and $\bar{\mathcal{O}}$ have the same fraction field. Thus there are elements $f_1, f_2 \in \mathcal{O}$ such that $f_1/f_2 = t$, so $v(f_1) = 1 + v(f_2)$, so $v(f_1)$ and $v(f_2)$ are relatively prime. This gives that the set of values is a numerical semigroup. that $\mathcal{O}$ is defined by a Weierstrass polynomial $f = y^n + \cdots$. The blowup of f (or quadratic transform) is defined as f' from

$$f(x', x'y') = (x')^n f'(x', y').$$

As an example, if

$$\mathcal{O} = \mathbb{C}[[t^4, t^6 + t^7]] =$$
$$\mathbb{C}[[x, y]]/(y^4 - 2x^3y^2 + x^6 - 4x^5y - x^7),$$

then the blowup of $y^4 - 2x^3y^2 + x^6 - 4x^5y - x^7$ is $f' = y^4 - 2xy^2 + x^2 - 4x^2y - x^3$ so the blowup ring is $\mathbb{C}[[x, y]]/f'$ or $\mathbb{C}[[t^4, t^2 + t^3]]$.

Successive blowups give less and less singular rings, and after a finite number of blowups, we get a regular ring. The multiplicity sequence is the sequence of multiplicities of the successive blowups. This is a decreasing sequence. In the example above, it is $4, 2, 2, 1, \ldots$.

Two algebroid plane curves are defined to be equisingular (Zariski) if they have the same multiplicity sequence. For analytic curves C, C' this means that they are topologically equivalent, i.e. there is a homeomorphism between neighbourhoods of the respective origins such that C is mapped onto C'. It is known that any analytic branch is equivalent to an algebraic branch.

The Apery set $\mathrm{Ap}(S; s)$ with respect to $s \in S$ of a semigroup S is the set of smallest representatives in S of the congruence classes $\pmod{s}$. If we order $\mathrm{Ap}(S; e)$ (e the multiplicity or smallest positive element in S) as $0 = a_0 < a_1 < \cdots < a_{e-1}$, then the ordered Apery set with respect to e of the blowup is $0 < a_1 - e < a_2 - 2e < \cdots < a_{e-1} - (e-1)e$ (Apery).

Let $\mathcal{O} = \mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \ldots$ be the sequence of blowups, and let $e_0, e_1, \ldots, e_k = 1$ be the corresponding multiplicity sequence. Then $v(\mathcal{O}^{(k)}) = \mathbb{N}$, which has ordered Apery set $\{0, 1, \ldots, e_{k-1} - 1\}$. This gives the Apery sequence of $\mathcal{O}^{(k-1)}$, and thus its semigroup.

Conclusion: Two plane curves are equisingular if and only if they have the same semigroup.

Let S be a semigroup minimally generated by $a_0 < a_1 \cdots < a_k$, and let $d_i = \gcd(a_0, \ldots, a_i)$. Then S is a semigroup of a plane curve if and only if
(1) $d_0 > d_1 > \cdots > d_k = 1$
(2) $a_i > \mathrm{lcm}(d_{i-2}, a_{i-1})$

Example $S = \langle 30, 42, 280, 855 \rangle$ is the semigroup of $\mathbb{C}[[t^{30}, t^{42} + t^{112} + t^{127}]]$. The semigroup of $\mathbb{C}[[t^8, t^{12} + t^{14} + t^{15}]]$ is $\langle 8, 12, 26, 55 \rangle$.

Question 1 Can these semigroups be characterized in some other way? E.g., are they special in the semigroup tree?

The moduli problem for plane branches.

Recall that two plane branches C and C' are topologically equivalent if there is a homeomorphism between neighbourhoods of the respective origins such that C is mapped onto C'. They are called analytically equivalent if there is such an analytic isomorphism. The moduli space of an equisingular class is the quotient space of this equivalence relation. (We want to know which curves have the same semigroup, but we consider two curves with isomorphic rings equal.)

This is a hard problem, because the answer is not an algebraic variety in the coefficients. As an example, the rings with semigroup $\langle 4, 6, 13 \rangle$ is either isomorphic to $\mathbb{C}[[t^4, t^6, t^{13}]]$ (not plane) or to
$\mathbb{C}[[t^4, t^6 + ct^7 + dt^9]]$ with $c \neq 0$.

One question in this vein is the following. For which semigroups have only the semigroup ring in its class?

Answer (Pfister-Steenbrink,Micale): A curve with semigroup $S = \langle a_1, \cdots, a_k \rangle$ ($a_1 < \cdots < a_k$) is isomorhic to $\mathbb{C}[[t^{a_1}, \ldots, t^{a_k}]]$ if and only if it is in one of the following classes:
(1) The only elements below the conductor are multiples of a_1.

(2) $x \notin S$ for only one $x > a_1$.
(3) The only elements greater than a_1 that are not in S are $a_1 + 1$ and $2a_1 + 1$, $a_1 \geq 3$.

Question 2 Can one characterize these semigroups in some other way? E.g., are they special in the semigroup tree?

There is a hierarchy of local domains: (A) regular, (B) complete intersections, (C) Gorenstein, (D) Cohen-Macaulay. For curves this means:

(A) $\mathbb{C}[[x]]$,
(B) $\mathbb{C}[[x_1, \ldots, x_k]]/(f_1, \ldots, f_{k-1})$, where $f_1, \ldots, f_{k-1}$ is a regular sequence,
(C) $\mathcal{O}/(f)$ has a 1-dimensional socle for each $f \neq 0$
(D) all.

For the corresponding semigroup rings $\mathbb{C}[[S]]$, this means:
(A) $S = \mathbb{N}$
(B) Characterized by Delorme
(C) S symmetric
(D) all

For a local ring (A, m), the associated graded ring (or the tangent cone) $\mathrm{gr}(A) = \oplus_{i \geq 0} m^i/m^{i+1}$, is an important invariant. The local ring itself is always "better" than the associated graded ring. There has been a lot of work on when the associated ring of a local (mostly 1-dimensional) ring is Cohen-Macaulay or Gorenstein.

Semigroup rings have been used to get examples and counterexamples in local algebra, because they are more accessible than general rings. For an element s in a semigroup $S = \langle a_1, \ldots, a_k \rangle$, let $o(s) = \max\{\sum n_i; s = \sum n_i a_i\}$, and let $r(S) = \min\{k; a_1 + kM = (k+1)M\}$. These numbers corresponds to $o(t^s)$ and the reduction number with respect to t^{a_1} of the maximal ideal, i.e. $\min\{k; t^{a_1} m^k = m^{k+1}\}$.

There are two general results for the CM-ness of the associated graded of semigroup rings:

(Garcia) $\mathrm{gr}(A)$ is CM iff $o(s + a_1) = 1 + o(s)$ for all $s \in S$, and iff $o(\omega + ka_1) = o(\omega) + k$ for all $\omega \in \mathrm{Ap}(S, a_1)$.

(Barucci-F) $\mathrm{gr}(A)$ is CM iff $\mathrm{Ap}(B(S), a_1) = \{\omega - o(\omega); \omega \in \mathrm{Ap}(S, a_1)\}$, where $B(S)$ is the blowup of S.

(Bryant) The associated graded to a semigroup ring is Gorenstein iff $o(\omega_i) + o(\omega_j) = o(\omega_{a_1 - 1}) = r(S)$ when $i + j = a_1 - 1$.

Question 3 For which semigroup rings is $\mathrm{gr}A$ a complete intersection?