Geometric Aspects of Polynomial Interpolation in More Variables and of Waring's Problem

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Abstract. In this paper I treat the problem of determining the dimension of the vector space of homogeneous polynomials in a given number of variables vanishing with some of their derivatives at a finite set of general points in projective space. I will illustrate the geometric meaning of this problem and the main results and conjectures about it. I will finally point out its connection with the so-called Waring's problem for forms, of which I will also indicate the geometric meaning.

1. Introduction

The classical polynomial interpolation theory of functions in numerical analysis is based on the elementary fact, recalled in §2, that a polynomial of degree d in one variable is uniquely determined by its zeroes with their multiplicities. In the present expository paper I will deal with an extension of this property to polynomials in more variables. This extension has to do with linear systems of hypersurfaces in projective space with finitely many assigned base points of given multiplicities. The general setting is presented in §3, where I will also state the main problems. Among these, I will mainly discuss the *general dimensionality problem*, which, roughly speaking, can be stated as follows: given a linear system of hypersurfaces in projective space and finitely many general points with assigned multiplicities at each point, what is the dimension of the subsystem of the given one formed by all hypersurfaces having at the given points at least the assigned multiplicities? This problem, which is elementary to state and somehow basic in algebraic geometry, has been considered, in one form or another, since the beginnings of this discipline. It can be traced back in Bezout's work in the XVIII century, it is present in Plücker, Cremona, M. Noether, Bertini, C. Segre etc. in the XIX century and contributions have been given in the XX century by Castelnuovo, Enriques, Severi, Terracini, among the others. Its relations with other topics have been considered by several authors, like [63] and [57]. So-far however the problem is still unsolved in its generality. I will discuss here what is known about it and what are the techniques involved, what are the conjectures, the open problems and the connections of this problem with others.

In §4 I will concentrate on the case of linear systems of plane curves. I will state and explain the main conjectures on the general dimensionality problem and in §5 I will illustrate the main results about it. In §6 I turn to the higher dimensional case and I will state and discuss the main result, which is a theorem of Alexander and Hirschowitz. In the last section §7 I will show the connection of Alexander-Hirschowitz's theorem with a famous algebraic problem, the Waring's problem, of which I will illustrate the geometric meaning. This will lead us to the geometry of secant varieties and to the classification of defective varieties, the ones whose secant varieties have dimension smaller than expected, about which I will recall the main known results.

This is a survey paper, which contains and expands the material covered by my talk at the ECM of Barcelona 2000. For other interesting surveys on the subject I refer the reader to [38] and [61].

2. Polynomial Interpolation

A polynomial $f(x) = a_0 + a_1x + \cdots + a_dx^d \in K[x]$ of degree at most d over a field K depends on $d+1$ parameters, namely its coefficients $a_0, a_1, \ldots, a_d$. If we fix $d+1$ distinct points $x_0, \ldots, x_d \in \mathbf{A}^1_K$ on the affine line over K and set the values:

$$f(x_i) = f_i \in K, \quad i = 0, \ldots, d \tag{1}$$

then, by linear algebra, there is some polynomial $f(x)$ satisfying the conditions (1). Moreover this polynomial is unique. The reason for this is that there is no non-zero polynomial of degree d with zeros at $x_0, \ldots, x_d$.

More generally, if we fix distinct points $x_1, \ldots, x_d \in \mathbf{A}^1_K$ and positive integers $m_1, \ldots, m_h$ such that $m_1 + \cdots + m_h = d+1$ and set the values of the derivatives:

$$f^{(j-1)}(x_i) = f_{i,j}, \quad i = 1, \ldots, h, \quad j = 1, \ldots, m_i \tag{2}$$

again there is a unique polynomial $f(x)$ satisfying the conditions (2). This is because there is no non-zero polynomial of degree d with zeros of multiplicities $m_1, \ldots, m_h$ at $x_0, \ldots, x_h$.

In particular, the following happens. Let $F(x)$ be a differentiable function of a real variable. Fix $x_1, \ldots, x_h$ distinct points where $F(x)$ is defined, and positive integers $m_1, \ldots, m_h$ such that $m_1 + \cdots + m_h = d+1$. Then there is a unique polynomial $f(x)$ of degree d satisfying (2) with $f_{i,j} = F^{(j-1)}(x_i)$, $i = 1, \ldots, h$, $j = 1, \ldots, m_i$. The polynomial $f(x)$ approximates $F(x)$ and of course the approximation is better and better as d increases. This approximating procedure of differentiable functions is called *polynomial interpolation*.

What is the situation in $n \geq 2$ variables? A polynomial $f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ of degree at most d depends on $N_{n,d} + 1 := \binom{d+n}{n}$ parameters, namely its coefficients. Again we may fix points $p_i = (x_{1i}, \ldots, x_{ni}) \in \mathbf{A}^n_K$, $i =$

$1,\dots,h$, in the n-dimensional affine space over the field K and integers $m_1,\dots,m_h$ such that:

$$\sum_{i=1}^{h}\binom{m_i+n-1}{n}=N_{n,d}+1 \tag{3}$$

and we may impose $D^{(j-1)}f(x_i)=0$, $i=1,\dots,h$, $j=1,\dots,m_i$, where $D^{(k)}$ is any derivative of order k. Notice that, according to (3), the number of conditions imposed equals the number of parameters on which the polynomials depend. In analogy with the one-variable case we may then ask: *is the resulting polynomial f identically zero?*

This is the question I will mainly deal with in this talk. Interestingly enough, there is yet no general answer to it and I will try to indicate the classical and recent developments on the subject and the techniques involved. It is useful to address the above question in a more general setting. This is what I will do next.

3. Linear Systems with Multiple Base Points

Let X be a smooth, irreducible, projective, complex variety of dimension n. Let $\mathcal{L}$ be a complete linear system of divisors on X. I will often abuse notation and denote with the same letter a linear system $\mathcal{L}$ and the corresponding line bundle on X. Fix $p_1,\dots,p_h$ distinct points on X and $m_1,\dots,m_h$ positive integers. I will denote by $\mathcal{L}(-\sum_{i=1}^{h}m_ip_i)$ the sublinear system of $\mathcal{L}$ formed by all divisors in $\mathcal{L}$ having multiplicity at least m_i at p_i, $i=1,\dots,h$.

Having a point of multiplicity m at a fixed point p imposes $\binom{m+n-1}{n}$ linear conditions on the divisors of $\mathcal{L}$. Indeed this translates, for the equation $f(x_0,\dots,x_n)=0$ of a divisor of $\mathcal{L}$ in local coordinates $(x_0,\dots,x_n)$ centered at p, in the vanishing of all monomials of degree at most $m-1$ appearing in the Taylor expansion of $f(x_0,\dots,x_n)$. Thus, it makes sense to define the *expected dimension* of $\mathcal{L}(-\sum_{i=1}^{h}m_ip_i)$ as:

$$\operatorname{expdim}(\mathcal{L}(-\sum_{i=1}^{h}m_ip_i)):=\max\{\dim(\mathcal{L})-\sum_{i=1}^{h}\binom{m_i+n-1}{n},-1\}$$

and one clearly has:

$$\dim(\mathcal{L}(-\sum_{i=1}^{h}m_ip_i))\geq \operatorname{expdim}(\mathcal{L}(-\sum_{i=1}^{h}m_ip_i))\,. \tag{4}$$

The system $\mathcal{L}(-\sum_{i=1}^{h}m_ip_i)$ is said to be *non-special* if the equality holds in (4). Otherwise it is said to be *special*. Notice that, by definition, a system which is empty is non-special. For a not empty system instead non-speciality means that the imposed conditions are independent. It is natural to expect that most systems are non-special. The *dimensionality problem* can be posed as follows: *classify all special systems.*

Put in this way, the problem is too complicated. Indeed one moment of reflection shows that the answer depends not only on the numerical data involved in it, but also on the position of the points $p_1, \dots, p_h$ on X. However, $\dim(\mathcal{L}(-\sum_{i=1}^h m_i p_i))$ is an *upper-semicontinuous* function in the position of the points $p_1, \dots, p_h$, hence it reaches its minimum for $p_1, \dots, p_h$ in *general position* on X.

When the points $p_1, \dots, p_h$ are sufficiently general on X, we set:

$$\mathcal{L}(-\sum_{i=1}^h m_i p_i) := \mathcal{L}(m_1, \dots, m_h)$$

or equivalently:

$$\mathcal{L}(m_1, \dots, m_h) := \mathcal{L}(m_1^{l_1}, \dots, m_t^{l_t})$$

if $l_1 + \cdots + l_t = h$ and m_i is repeated l_i times. The case $t = 1$ is called *homogeneous.* The case $t = 2$, $l_1 = 1$ is called *quasi-homogeneous.*

Then we define the *general dimension* of the linear system $\mathcal{L}(-\sum_{i=1}^h m_i p_i)$ as:

$$\operatorname{gendim}(\mathcal{L}(-\sum_{i=1}^h m_i p_i)) := \dim(\mathcal{L}(m_1, \dots, m_h))\,.$$

With this definition in mind, the dimensionality problem splits as:

(i) the *general dimensionality problem*: is $\operatorname{gendim}(\mathcal{L}(-\sum_{i=1}^h m_i p_i))$ equal to the expected dimension? Or rather, put in a different, but equivalent way: classify all systems $\mathcal{L}(m_1, \dots, m_h)$ which are special;

(ii) the *hard dimensionality problem*: describe the stratification of X^h determined by the closures of the loci of h-tuples of points where

$$\dim(\mathcal{L}(-\sum_{i=1}^h m_i p_i)) > \operatorname{gendim}(\mathcal{L}(-\sum_{i=1}^h m_i p_i))\,.$$

The hard dimensionality problem has never been systematically explored. In the rest of this paper I will therefore stick to the general dimensionality problem.

Of course one can also ask more refined questions about systems of the type $\mathcal{L}(m_1, \dots, m_h)$, like: describe its base locus, its general element, etc. Very little is known about this kind of questions and we will only touch upon some of them later on.

Notice that the general dimensionality problem is equivalent to the problem of determining the *Hilbert function* of the 0-dimensional subscheme of $\mathbf{P}^2$ given by the union of h general *fat points* of given multiplicities. One may of course ask for more refined questions like: what is the resolution of the ideal sheaf of this 0-dimensional scheme? This is also an active field of research, which I will not report on here, referring the reader, for example, to the recent paper [34] for information and references.

As in the interpolation problem, the general dimensionality problem is trivial in one variable, namely in the case of curves, i.e. if $n = \dim(X) = 1$. In this case all systems $\mathcal{L}(m_1, \ldots, m_h)$ are non-special. Another case which never causes speciality is when all the points have multiplicity one, i.e. $m_1 = \cdots = m_h = 1$: imposing to the divisors of a not empty linear system to contain a general point of the variety (e.g. a non base point of the system) certainly imposes one condition. By contrast, the problem becomes quite difficult in more variables, namely as soon as $n = \dim(X) \geq 2$, and higher multiplicities, namely $m_1, \ldots, m_h \geq 2$, the situation which we will consider from now on.

A first wise reduction of the problem is to consider, for the time being, particular varieties X and linear systems $\mathcal{L}$ on them. From this viewpoint, the first obvious choice it to take $X = \mathbf{P}^n$ and $\mathcal{L} = \mathcal{L}_{n,d} := |\mathcal{O}_{\mathbf{P}^n}(d)|$ the system of all hypersurfaces of degree d in $\mathbf{P}^n$. It should be clear to the reader that, in this setting, the problem essentially coincides with the original *interpolation problem* for polynomials in more variables considered in §2.

In this case

$$\operatorname{expdim}(\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)) = \max\{\operatorname{virtdim}(\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)), -1\}$$

where:

$$\operatorname{virtdim}(\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)) := \binom{d+n}{n} - 1 - \sum_{i=1}^{h} \binom{m_i + n - 1}{n}$$

is the so-called *virtual dimension* of $\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i p_i)$.

4. The Planar Case

I will consider now the case $X = \mathbf{P}^2$, in which there is a very precise conjecture about the general dimensionality problem, namely the Harbourne-Hirschowitz conjecture 4.8. This section is devoted to state this important conjecture.

The prototype of this conjecture goes back to B. Segre [78] who apparently has been the first one to stress that speciality yields reducibility and even non reducedness of the general curve of the involved linear systems with general multiple base points.

Conjecture 4.1. (B. Segre, 1961) *If a linear system of plane curves with general multiple base points $\mathcal{L}_{2,d}(m_1, \ldots, m_h)$ is special, then its general member is non reduced, namely the linear system has, according to Bertini's theorem, some multiple fixed component.*

B. Segre's conjecture has been made more precise by A. Gimigliano [37] in 1987, on the basis of various examples, which we will partly mention in a moment:

Conjecture 4.2. (A. Gimigliano, 1987) *Consider a linear system of plane curves with general multiple base points* $\mathcal{L}_{2,d}(m_1,\ldots,m_h)$. *Then one has the following possibilities:*

(i) *the system is non-special and its general member is irreducible;*
(ii) *the system is non-special, its general member is reduced, reducible, its fixed components are all rational curves, but at most one (this may occur only if the system has dimension* 0*), the general member of its movable part is either irreducible or composed of rational curves in a pencil;*
(iii) *the system is non-special of dimension* 0 *and it consists of a unique multiple elliptic curve;*
(iv) *the system is special and it has some multiple rational curve as a fixed component.*

I want now to state the Harbourne-Hirschowitz conjecture. I will later explain, in §5, the relations among the various conjectures. In order to do so, let us consider the blow-up $\pi\colon \tilde{\mathbf{P}}^2 \to \mathbf{P}^2$ of the plane $\mathbf{P}^2$ at $p_1,\ldots,p_h$. Let $E_1,\ldots,E_h$ be the exceptional divisors corresponding to the blown-up points $p_1,\ldots,p_h$ and let H be the pull-back of a general line of $\mathbf{P}^2$ via π. The strict transform of the system $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_i p_i)$ is the system $\tilde{\mathcal{L}} = |dH - \sum_{i=1}^h m_i E_i|$.

Consider two linear systems of this type $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_i p_i)$ and $\mathcal{L}' := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_i' p_i)$. We define their *intersection product* by using the intersection product of their strict transforms on $\tilde{\mathbf{P}}^2$, i.e. we set:

$$\mathcal{L}\cdot\mathcal{L}' := \tilde{\mathcal{L}}\cdot\tilde{\mathcal{L}}' = dd' - \sum_{i=1}^h m_i m_i'$$

as dictated by the classical Bezout's theorem. Also consider the *anticanonical class* $-K := -K^2_{\tilde{\mathbf{P}}}$ of $\tilde{\mathbf{P}}^2$ corresponding to the linear system $\mathcal{L}_{2,3}(-\sum_{i=1}^h p_i)$, which, by abusing notation, we also denote by $-K$.

The adjunction formula tells us that the arithmetic genus $p_a(\tilde{\mathcal{L}})$ of a curve in $\tilde{\mathcal{L}}$ is:

$$p_a(\tilde{\mathcal{L}}) = \frac{\mathcal{L}\cdot(\mathcal{L}+K)}{2} + 1 = \binom{d-1}{2} - \sum_{i=1}^h \binom{m_i}{2}$$

which one defines to be:

$$g_{\mathcal{L}} = \textit{the geometric genus of}\, \mathcal{L}\,.$$

This is the classical *Clebsch's formula*. The theorem of Riemann-Roch then says that:

$$\begin{aligned}\dim(\mathcal{L}) = \dim(\tilde{\mathcal{L}}) &= \frac{\mathcal{L}\cdot(\mathcal{L}-K)}{2} + h^1(\tilde{\mathbf{P}}^2,\tilde{\mathcal{L}}) - h^2(\tilde{\mathbf{P}}^2,\tilde{\mathcal{L}}) = \\ &= \mathcal{L}^2 - g_{\mathcal{L}} + 1 + h^1(\tilde{\mathbf{P}}^2,\tilde{\mathcal{L}}) = \mathrm{virtdim}(\mathcal{L}) + h^1(\tilde{\mathbf{P}}^2,\tilde{\mathcal{L}})\end{aligned} \tag{5}$$

because clearly $h^2(\tilde{\mathbf{P}}^2, \tilde{\mathcal{L}}) = 0$. Hence:

$$\mathcal{L} \textit{ is non-special} \Leftrightarrow h^0(\tilde{\mathbf{P}}^2, \tilde{\mathcal{L}}) \cdot h^1(\tilde{\mathbf{P}}^2, \tilde{\mathcal{L}}) = 0\,. \tag{6}$$

According to a recent terminology, this is expressed by saying that $\mathcal{L}$, or rather $\tilde{\mathcal{L}}$, has *natural cohomology*, meaning with this that at most one of the cohomology groups of the line bundle $\tilde{\mathcal{L}}$ is different from zero.

Now we can see how, in this setting, special systems naturally arise. Indeed let us look for an irreducible curve C on $\tilde{\mathbf{P}}^2$, corresponding to a linear system $\mathcal{L}$ on $\mathbf{P}^2$, which is *expected to exist* but *its double is not expected to exist*. However crazy this requirement may appear, it translates in the following set of inequalities:

$$\begin{aligned} \mathrm{virtdim}(\mathcal{L}) &\geq 0 \\ g_{\mathcal{L}} &\geq 0 \\ \mathrm{virtdim}(2\mathcal{L}) &\leq -1\,. \end{aligned} \tag{7}$$

This system is equivalent to:

$$\begin{aligned} C^2 - C \cdot K &\geq 0 \\ C^2 + C \cdot K &\geq -2 \\ 2C^2 - C \cdot K &\leq 0 \end{aligned} \tag{8}$$

and it has the only solution:

$$C^2 = C \cdot K = -1$$

which makes all the inequalities in (7) and (8) equalities. Accordingly C is a *rational curve*, i.e. a curve of genus 0, with *self-intersection* -1. Surface theorists call these curves (-1)-*curves*. A famous theorem of Castelnuovo's (see [10, pg. 27]) says that these are the only curves that can be contracted to smooth points via a birational morphism of the surface on which they lie to another surface. By abusing terminology the curve $\Gamma \subset \mathbf{P}^2$ corresponding to C is also called a (-1)-curve.

Example 4.3. (i) *A line through two points $\mathcal{L}_{2,1}(-p-q)$ is a (-1)-curve. Hence $\mathcal{L}_{2,2}(-2p-2q)$, a conic with two double points, is special. Its virtual dimension is -1, namely it is expected not to exist, however it exists, and consists precisely of the line through the two points counted twice.*

(i') *More generally $\mathcal{L} := \mathcal{L}_{n,2}(-\sum_{i=1}^{h} 2p_i)$ is special if $h \leq n$. Actually, quadrics in $\mathbf{P}^n$ singular at h independent points $p_1, \ldots, p_h$ are cones with vertex the $\mathbf{P}^{h-1}$ spanned by $p_1, \ldots, p_h$. Therefore the system is empty as soon as $h \geq n+1$, whereas, if $h \leq n$ one easily computes:*

$$\dim(\mathcal{L}) = \mathrm{virtdim}(\mathcal{L}) + \binom{h}{2}\,.$$

(ii) *A conic through five general points $\mathcal{L}_{2,2}(-\sum_{i=1}^{5} p_i)$ is a (-1)-curve. Hence the system $\mathcal{L}_{2,4}(-\sum_{i=1}^{5} 2p_i)$ of quartics singular at five general points is special. Again its virtual dimension is -1, but it is not empty, consisting of the conic through the five points counted twice.*

(ii') *Similarly for* $\mathcal{L} := \mathcal{L}_{n,4}(-\sum_{i=1}^{h} 2p_i)$. *Speciality arises if:*

$$\text{virtdim}(\mathcal{L}) < 0, \quad \text{virtdim}(\mathcal{L}_{n,2}(-\sum_{i=1}^{h} p_i)) \geq 0$$

which occurs if and only if (n, h) *is one of the pairs* $(2,5)$, $(3,9)$, $(4,14)$.

More generally, one has special linear systems in the following situation. Let $\mathcal{L}$ be a linear system on $\mathbf{P}^2$ which is not empty, let C be a (-1)-curve on $\tilde{\mathbf{P}}^2$ corresponding to a curve Γ on $\mathbf{P}^2$, such that $\tilde{\mathcal{L}} \cdot C = -N < 0$. Then C [resp. Γ] splits off with multiplicity N as a fixed component from all curves of $\tilde{\mathcal{L}}$ [resp. $\mathcal{L}$], and one has:

$$\tilde{\mathcal{L}} = NC + \tilde{\mathcal{M}}, \quad [\text{resp. } \mathcal{L} = N\Gamma + \mathcal{M}]$$

where $\tilde{\mathcal{M}}$ [resp. $\mathcal{M}$] is the residual linear system. Then one computes:

$$\dim(\mathcal{L}) = \dim(\mathcal{M}) \geq \text{virtdim}(\mathcal{M}) = \text{virtdim}(\mathcal{L}) + \binom{N}{2}$$

and therefore, if $N \geq 2$, then $\mathcal{L}$ is special.

Example 4.4. *One immediately finds examples of special systems of this type by starting from the* (-1)*-curves of example 4.3. For instance consider* $\mathcal{L} := \mathcal{L}_{2,2d}(-\sum_{i=1}^{5} dp_i)$ *which is not empty, consisting of the conic* $\mathcal{L}_{2,2}(-\sum_{i=1}^{d} p_i)$ *counted* d *times, though it has virtual dimension* $-\binom{d}{2}$.

Even more generally, consider a linear system $\mathcal{L}$ on $\mathbf{P}^2$ which is not empty, $C_1, \ldots, C_k$ some (-1)-curves on $\tilde{\mathbf{P}}^2$ corresponding to curves $\Gamma_1, \ldots, \Gamma_k$ on $\mathbf{P}^2$, such that $\tilde{\mathcal{L}} \cdot C_i = -N_i < 0$, $i = 1, \ldots, k$. Then:

$$\mathcal{L} = \sum_{i=1}^{k} N_i \Gamma_i + \mathcal{M}, \quad \tilde{\mathcal{L}} = \sum_{i=1}^{k} N_i C_i + \tilde{\mathcal{M}}$$

and $\tilde{\mathcal{M}} \cdot C_i = 0$, for $i = 1, \ldots, k$. As before, $\mathcal{L}$ is special as soon as there is an $i = 1, \ldots, k$ such that $N_i \geq 2$. Furthermore $C_i \cdot C_j = \delta_{ij}$, because the union of two (-1)-curves meeting moves, according to Riemann-Roch theorem, in a linear system of positive dimension on $\tilde{\mathbf{P}}^2$, and therefore it cannot be fixed for $\tilde{\mathcal{L}}$. In this situation, the reducible curve $C := \sum_{i=1}^{k} C_i$ [resp. $\Gamma := \sum_{i=1}^{k} N_i \Gamma_i$] is called a (-1)-*configuration* on $\tilde{\mathbf{P}}^2$ [resp. on $\mathbf{P}^2$].

Example 4.5. *Consider* $\mathcal{L} := \mathcal{L}_{2,d}(-m_0 p_0 - \sum_{i=1}^{h} m_i p_i)$, *with* $m_0 + m_i = d + N_i$, $N_i \geq 1$. *Let* Γ_i *be the line joining* p_0, p_i. *It splits off* N_i *times from* $\mathcal{L}$. *Hence* $\mathcal{L} = \sum_{i=1}^{h} N_i \Gamma_i + \mathcal{L}_{2,d-\sum_{i=1}^{h} N_i}(-(m_0 - \sum_{i=1}^{h} N_i)p_0 - \sum_{i=1}^{h}(m_i - N_i)p_i)$. *If we require the latter system to have non negative virtual dimension, e.g.* $d \geq \sum_{i=1}^{h} m_i$ *if* $m_0 = d$, *and some* $N_i > 1$ *we have as many special systems as we want.*

With all this in mind we can now give a definition:

Definition 4.6. *A linear system $\mathcal{L}$ on $\mathbf{P}^2$ is* (-1)-reducible *if* $\tilde{\mathcal{L}} = \sum_{i=1}^k N_i C_i + \tilde{\mathcal{M}}$, *where* $C = \sum_{i=1}^k C_i$ *is a* (-1)*-configuration,* $\tilde{\mathcal{M}} \cdot C_i = 0$*, for all* $i = 1, \dots, k$*, and* $\mathrm{virtdim}(\mathcal{M}) \geq 0$.

The system $\mathcal{L}$ is called (-1)-special *if, in addition, there is an* $i = 1, \dots, k$ *such that* $N_i > 1$.

Remark 4.7. *This is an effective definition once one has an algorithm to produce* (-1)*-curves. We do not discuss this aspect here but refer the reader to* [25] *and* [26]*, where the question is treated.*

We are finally ready to state Harbourne-Hirschowitz conjecture (see also [40, 41, 45, 25] and [26]), which, with the terminology we just introduced, sounds very simple.

Conjecture 4.8. (Harbourne-Hirschowitz, 1989) *A linear system of plane curves* $\mathcal{L} := \mathcal{L}_{2,d}(m_1, \dots, m_h)$ *with general multiple base points is special if and only if it is* (-1)*-special.*

This is a rather bold conjecture, whose basic motivation lies in the fact that, in more than a century of research on the subject, no special system has been discovered except (-1)-special systems. On the other hand, as we shall see in the next section, there are several recent results that make the conjecture rather plausible.

We can complement it and Gimigliano's one with another conjecture, mentioned in [58] and also attributed to Hirschowitz:

Conjecture 4.9. (Hirschowitz) *Consider a linear system of plane curves* $\mathcal{L} := \mathcal{L}_{2,d}(m_1, \dots, m_h)$ *with general multiple base points* $p_1, \dots, p_h$ *which is not empty and non-special, with* $g_{\mathcal{L}} \geq 0$*. Then the general curve* $C \in \mathcal{L}$ *is irreducible, smooth off the multiple base points* $p_1, \dots, p_h$ *where it has ordinary singularities of multiplicities exactly* $m_1, \dots, m_h$*, unless* $d = 3m$*,* $h = 9$*,* $m_1 = \cdots = m_9 = m \geq 2$*, in which case* $\mathcal{L}$ *consists of the unique cubic through* $p_1, \dots, p_9$*, counted with multiplicity* m*.*

I close this section with the following useful remark:

Remark 4.10. *Suppose that the Harbourne-Hirshowitz conjecture holds. Let* $p_1, \dots, p_h$ *be general points of* $\mathbf{P}^2$ *and let* C *be an irreducible curve on the blow-up* $\tilde{\mathbf{P}}^2$ *at those points. Then one has* $C^2 \geq p_a(C) - 1 \geq -1$ *and* $C^2 = -1$ *if and only if* C *is a* (-1)*-curve. This is an immediate consequence of the Riemann-Roch theorem (5).*

5. Results on the Harbourne-Hirschowitz Conjecture

In this section I will present what is known about the conjectures introduced in §4, and I will try to briefly explain what are the techniques involved in the proofs. I will mainly follow the chronological development of the subject. As the reader will see, though the subject is so classical, almost all results do not go back more than twenty years and most of them are very recent.

As shown by (6), a not empty linear system $\mathcal{L}$ in the plane $\mathbf{P}^2$ is non-special if and only if the corresponding linear system $\tilde{\mathcal{L}}$ on $\tilde{\mathbf{P}}^2$ has $h^1(\tilde{\mathbf{P}}^2, \tilde{\mathcal{L}}) = 0$. Any algebraic geometer knows that the literature is full of *vanishing theorems* for cohomology spaces. Think about Kodaira's or Kawamata-Vieweg's vanishing theorem (see [48]) and, in the surface case we are specifically dealing with, Mumford's and Franchetta-Ramanujam's ones (see [62, 70, 71]). If one tries to apply one of these theorems to the general dimensionality problem, one, unfortunately, does not go too far. It turns out that, in order to usefully apply them, one needs $-K$ to be effective on $\tilde{\mathbf{P}}^2$, hence all the points we blow-up have to lie on a cubic. If we blow-up h general points, this means that $h \leq 9$. In this way one proves a result already known to Castelnuovo [16], and later rediscovered by several authors like Nagata [64], Gimigliano [37] and Harbourne [41].

Theorem 5.1. (Castelnuovo, 1891; Nagata, 1960; Gimigliano, Harbourne, 1986) *The Harbourne-Hirschowitz conjecture holds for all linear systems with $h \leq 9$ general multiple base points.*

Recall that we are considering linear systems $\mathcal{L}_{2,d}(m_1, \ldots, m_h)$ such that $m_1, \ldots, m_h \geq 2$. The simplest case to look at is therefore the homogeneous case $m_1 = \cdots = m_h = 2$. This case was classically examined by Campbell [15], Palatini [66] and Terracini [84] in a wider context, as we will see later in §7. In recent times it has been reconsidered by Arbarello and Cornalba [7] in 1981. Their approach relies on the use of a classical *infinitesimal deformation technique* consisting in moving the base points of the system and computing the first order deformation of a curve which moves keeping its singularities. Arbarello-Cornalba's result, in the case they consider, is half way between Harbourne-Hirschowitz conjecture 4.8 and Hirschowitz conjecture 4.9.

Theorem 5.2. (Arbarello-Cornalba, 1981) *Consider $\mathcal{L} := \mathcal{L}_{2,d}(2^h)$. Assume:*

(i) $\frac{d(d+3)}{2} \geq 3h$, *i.e.* $\mathrm{virtdim}(\mathcal{L}) \geq 0$*;*
(ii) $\binom{d-1}{2} \geq h$, *i.e.* $g_{\mathcal{L}} \geq 0$.

Then $\mathcal{L}$ is non-special, and $C \in \mathcal{L}$ general is irreducible, with nodes at the imposed general double points $p_1, \ldots, p_h$, and elsewhere smooth, except for $\mathcal{L}_{2,6}(2^9)$ which is a double cubic.

The infinitesimal deformation computation performed by Arbarello and Cornalba is a particular case of a lemma which goes back to Terracini and we will come back to it later (see lemma 6.3 and [82]). Unfortunately it works well only in the case of double points. In the higher multiplicity case infinitesimal deformation techniques have never been successfully used in this problem. However work in progress by C. Ciliberto, H. Clemens and R. Miranda [22], suggests that there are some chances in this direction. In particular they have been able to examine the case of triple points with infinitesimal deformation techniques.

A conceptually opposite approach, which is also natural to the problem, is to argue by *degeneration*, meaning with this that one specializes the base points of the linear system in order to be able to better compute the dimension of the linear system. Recall that the dimension of $\mathcal{L} := \mathcal{L}_{n,2}(-\sum_{i=1}^{h} m_i p_i)$ is upper-semicontinuous in the position of the points $p_1, \dots, p_h$. Therefore if one finds a *particular* set of points $q_1, \dots, q_h$ such that $\mathcal{L}_0 := \mathcal{L}_{n,2}(-\sum_{i=1}^{h} m_i q_i)$ is non-special, then also $\mathcal{L}$ is non-special. Unfortunately, this is often too naive: as soon as one puts the points $p_1, \dots, p_h$ in a particular position, e.g. one puts them on some curve on which they should not lie, then the dimension of $\mathcal{L}$ tends to increase, and the method, in this crude form, does not work. However, there is still something which one can do: even if the dimension of $\mathcal{L}$ increases, one can actually compute the *limit* of $\mathcal{L}$ when $p_1, \dots, p_h$ approach $q_1, \dots, q_h$. There is no time here to enter in any detail about this idea, which is the one elaborated by A. Hirschowitz in his paper [44]. He called his degeneration technique *la méthode d'Horace*, i.e. the *Horace's method*, consisting in successive specializations of the multiple base points on particular curves. Exploiting it, he has been able to prove the:

Theorem 5.3. (Hirschowitz, 1985) *The Harbourne-Hirschowitz conjecture holds in the homogeneous case* $\mathcal{L}_{2,d}(m^h)$, $m \leq 3$.

The application of the Horace's method usually requires a deep geometric understanding of the problem and a special capability of guessing the right specializations to be performed. The unfortunate circumstance of this is that the Horace's method seldom appears to be systematic, rather it seems ingenuous but too ad hoc to become a theory.

More recently a different specialization technique has been introduced, and successfully used, in this problem by Ciliberto and Miranda [25, 26]. The idea, which I will explain in some detail a few lines below, basically consists in using a degeneration technique worked out by Z. Ran [72] mainly for studying enumerative problems of families of plane nodal curves. It consists in *degenerating the plane* to a reducible surface and in following the linear system in the degeneration. The restriction of the limit linear system to the components of the reducible limit surface are *easier* than the system one starts with, so that one can hope to successfully use induction. The outcome of this method is the following substantial improvement of Hirschowit's theorem 5.3:

Theorem 5.4. (Ciliberto-Miranda, 1998) *The Harbourne-Hirschowitz conjecture holds in the quasi-homogeneous cases* $\mathcal{L}_{2,d}(n, m^h)$, $m \leq 3$ *and in the homogeneous cases* $\mathcal{L}_{2,d}(m^h)$, $m \leq 12$.

Remark 5.5. *It is worth mentioning, along the same lines, a recent result independently proved, with similar techniques, by J. Seibert* [79] *and A. Laface* [50]*, to the effect that the Harbourne-Hirschowitz conjecture holds in the quasi-homogeneous case* $\mathcal{L}_{2,d}(n, 4^h)$*, and a result of Laface's* [50]*, who proves that a suitable version of the Harbourne-Hirschowitz conjecture holds in the homogeneous case* $\mathcal{L}(m^h)$,

$m \leq 3$, for any linear system $\mathcal{L}$ on a Hirzebruch surface $\mathbf{F}_n$. Recall that $\mathbf{F}_n$ is the unique minimal rational ruled surface with an irreducible curve of self-intersection $-n$ (see [10, Chapter III]*).*

Note that a contribution to the case $\mathcal{L}_{2,d}(4^h)$, with different combinatorial methods coming from numerical analysis, is due to G. Lorentz and R. Lorentz [55] *(see also* [53, 54]*). These methods however can be interpreted as an application of degeneration techniques rather similar to those used by Evain in* [31]*, on which I will come back later on.*

Let us now go back to the proof of theorem 5.4. As promised, I want to give some details of the ideas involved into it. For more information, I refer to the original papers [25, 26].

First, let me describe Z. Ran's degeneration of the plane. Let Δ be a disc in $\mathbf{C}$ with centre the origin. Let $p\colon X \to \Delta$ be the flat family obtained by blowing-up $\Delta \times \mathbf{P}^2$ along a line L in the fibre of $0 \in \Delta$. The general fibre of the family is $X_t = p^{-1}(t) = \mathbf{P}^2$, for $t \neq 0$, whereas the *central fibre* is $X_0 = p^{-1}(0) = \mathbf{P} \cup \mathbf{F}$, where $\mathbf{P} = \mathbf{P}^2$, $\mathbf{F} = \mathbf{F}_1$ is the exceptional divisor of the blow-up, and $\mathbf{P} \cap \mathbf{F} = L$. Notice that X_0 thus appears as a flat limit of $\mathbf{P}^2$.

Next one takes to the limit the linear system. The natural map $\pi\colon X \to \mathbf{P}^2$ endowes X with a line bundle $\mathcal{O}_X(d) := \pi^*(\mathcal{O}_{\mathbf{P}^2}(d))$ for any integer d. Of course $\mathcal{O}_X(d)_{|X_t} \simeq \mathcal{O}_{\mathbf{P}^2}(d)$ for all $t \neq 0$. But, for any integer k, one has also $\mathcal{O}_{\mathbf{P}^2}(d) = \mathcal{O}_X(d) \otimes \mathcal{O}_X(k\mathbf{P})_{|X_t}$. Hence each one of the line bundles $\mathcal{O}_{X_0}(d,k) := \mathcal{O}_X(d) \otimes \mathcal{O}_X(k\mathbf{P})_{|X_0}$ is *a limit* of $\mathcal{O}_{\mathbf{P}^2}(d)$ on the limit, reducible surface X_0. The failure of the uniqueness of the limit line bundle plays in our favour, inasmuch as the presence of the parameter k gives us more freedom in the numerical choices we will have to do next.

Send now $b < h$ of the h limiting points $q_1, \ldots, q_h$ on X_0 to $\mathbf{F}$ as general points, the remaining $h-b$ to $\mathbf{P}$ as general point, and consider the linear system $\mathcal{L}_0$ of all divisors in the linear system associated to $\mathcal{O}_{X_0}(d,k)$ having multiplicty at least m_i at q_i, $i = 1, \ldots, h$. This is a limit linear system of $\mathcal{L}_{2,d}(m_1, \ldots, m_h)$, which is called a (k,b)-*degeneration* $\mathcal{L}_0$ of $\mathcal{L}_{2,d}(m_1, \ldots, m_h)$. The usual upper-semicontinuity argument tells us that if the dimension of $\mathcal{L}_0$ equals the expected dimension of $\mathcal{L}$, then $\mathcal{L}$ is non-special.

Notice now that the two components $\mathbf{P}$ and $\mathbf{F}$ of X_0 are a plane and a plane blown-up at a point. Hence the restrictions of $\mathcal{L}_0$ to the two components $\mathbf{P}$ and $\mathbf{F}$ of X_0 are basically again linear systems of plane curves with general multiple base points. Thus one is in a position to use induction in order to estimate the dimension of $\mathcal{L}_0$. A basic ingredient in this computation is a *transversality lemma*, which, roughly speaking, tells us that the restrictions of $\mathcal{L}_0$ to $\mathbf{P}$ and $\mathbf{F}$ in turn restrict to $L = \mathbf{P} \cap \mathbf{F}$ in the most general possible way. A systematic use of (m,b)-degenerations leads then to the following result of independent interest:

Proposition 5.6. *There is a function $D(m) = \frac{m^2}{3} + o(m)$ such that if the Harbourne-Hirschowitz conjecture holds for every homogenous system $\mathcal{L}_{2,d}(m^h)$ with $d \leq$*

$D(m)$, then the same conjecture holds for all homogenous systems of the form $\mathcal{L}_{2,d}(m^h)$.

The final part of the strategy is to try to prove the Harbourne-Hirschowitz conjecture for all homogenous systems $\mathcal{L}_{2,d}(m^h)$, with $d \leq D(m)$, i.e. with d small with respect to m. In order to do so, one uses other (k,b)-degenerations, with other k's, but this does not work in all cases. For example Dixmier's example $\mathcal{L}_{2,19}(6^{10})$ worked out by Hirschowitz in [44] with the Horace's method, cannot be attacked with (k,b)-degenerations. So one has to use also ad hoc geometric arguments or rely on the help of suitable computer programs. This is what is done in [26] in the cases $m \leq 12$.

Remark 5.7. *As a concluding remark on the proof of theorem 5.4, I want to stress that a (k,b)-degeneration can be seen, ultimately, as a way of degenerating the set of points $p_1, \dots, p_h$ by putting b of them on a line, and of letting the line split from the curves of the linear system k times. Thus, in principle, there is not so great a difference with the Horace's method. However this approach seems quite systematic and has given so far very good results. Indeed, in principle, there is no reason why it should not work for higher values of m and in fact there is promising work in progress with F. Cioffi, R. Miranda and F. Orecchia on the algorithmic side mentioned a few lines above, in order to improve the bound $m \leq 12$ in theorem 5.4. So far we have been able to work out a computer program which verifies the Harbourne-Hirschowitz conjecture for $\mathcal{L}_{2,d}(m^h)$. We tested the program and we have been able in this way to prove the conjecture for $m \leq 20$.*

I strongly believe that the method of (k,b)-degenerations can be still pushed further, to give better and better results along these lines.

Another aspect of the results in [25, 26] to be mentioned is the full classification of homogenous (-1)-special systems, which is rather interesting and surprising in its own and plays an important role in the induction process described before.

First a little combinatorial analysis leads to the following:

Proposition 5.8. (Classification of homogenous (-1)-configurations) *The only homogeneous linear systems $\mathcal{L}_{2,d}(m^h)$ which are (-1)-configurations are:*

$$\begin{aligned}
\mathcal{L}_{2,1}(1^2):&\ \textit{a line through 2 points}\\
\mathcal{L}_{2,2}(1^5):&\ \textit{a conic through 5 points}\\
\mathcal{L}_{2,3}(2^3):&\ \textit{3 lines each through 2 of 3 points}\\
\mathcal{L}_{2,12}(5^6):&\ \textit{6 conics each through 5 of 6 points}\\
\mathcal{L}_{2,21}(8^7):&\ \textit{7 cubics each through 6 points, double at another}\\
\mathcal{L}_{2,48}(17^8):&\ \textit{8 sextics double at 7 points, triple at another}.
\end{aligned}$$

This leads to the following:

Theorem 5.9. (Classification of homogenous (-1)-special systems) *The only homogeneous linear systems $\mathcal{L}_{2,d}(m^h)$ which are (-1)-special are:*

$$\begin{aligned} &\mathcal{L}_{2,d}(m^2) \ \textit{with} \quad m \le d \le 2m-2 \\ &\mathcal{L}_{2,d}(m^3) \ \textit{with} \quad \frac{3m}{2} \le d \le 2m-2 \\ &\mathcal{L}_{2,d}(m^5) \ \textit{with} \quad 2m \le d \le \frac{5m-2}{2} \\ &\mathcal{L}_{2,d}(m^6) \ \textit{with} \quad \frac{12m}{5} \le d \le \frac{5m-2}{2} \\ &\mathcal{L}_{2,d}(m^7) \ \textit{with} \quad \frac{21m}{8} \le d \le \frac{8m-2}{3} \\ &\mathcal{L}_{2,d}(m^8) \ \textit{with} \quad \frac{48m}{17} \le d \le \frac{17m-2}{6}\,. \end{aligned}$$

As a remarkable consequence we have that the Harbourne-Hirschowitz conjecture for homogenous system takes the form:

Conjecture 5.10. *Every homogenous system of the form $\mathcal{L}_{2,d}(m^h)$ with $h \ge 10$ is non-special.*

It is probably this the right moment for recalling another famous conjecture concerning singular plane curves. In [63] Nagata showed a counterexample to the fourteenth problem of Hilbert. In his construction, he proved that if the linear system $\mathcal{L}_{2,d}(m^{k^2})$ is not empty for a integer $k \ge 4$, then one has $d > km$. He also conjectured that a similar result should hold for any, not necessarily a square, number of points in general position, namely he fomulated the following:

Conjecture 5.11. (Nagata, 1960) *$\mathcal{L}_{2,d}(m^h)$ is empty as soon as $h \ge 10$ and $d \le \sqrt{h} \cdot m$.*

It is worth pointing out the following fact:

Remark 5.12. *Harbourne-Hirschowitz conjecture 4.8 or 5.10, implies Nagata's conjecture 5.11. Indeed, let $\mathcal{L} := \mathcal{L}_{2,d}(m^h)$, $h \ge 10$, be not empty and let C be an irreducible component of the strict transform of the general element of $\mathcal{L}_{2,d}(m^h)$ on $\tilde{\mathbf{P}}^2$. By remark 4.10 we have $C^2 \ge p_a(C) - 1$. On the other hand one cannot have $C^2 = -1, p_a(C) = 0$, since, by proposition 5.8, there is no (-1)-configuration for $h \ge 10$. Thus $C^2 \ge 0$. Hence $\mathcal{L}^2 \ge 0$, which reads $d^2 \ge hm^2$.*

Before going back to our main topic, I cannot resist indicating the following connection of Nagata's conjecture, hence of Harbourne-Hirschowitz conjecture, with another interesting subject.

Remark 5.13. *Let C be a curve of genus g. Curves on the product $C \times C$ are* correspondences *of the curve into itself. Similarly curves on the symmetric product $C(2)$ are symmetric correspondences of C. A version of* Petri's problem *(see* [8]*)* for correspondences *is:* describe the effective cone of the symmetric product $C(2)$ when C is a general curve of genus g. *It is known that, if C is general of* *genus g, then $NS(C(2)) \simeq \mathbf{Z}\langle x, \frac{\delta}{2}\rangle$, where x is the class of the curve and δ is the diagonal. The structure of the effective cone of $C(2)$ for C general of genus g is known when $g \leq 3$. Ciliberto and Kouvidakis' paper* [24] *(see also* [49]*), suggests the following conjecture:* if $g \geq 4$ there is no curve of negative self-intersection on $C(2)$ except the diagonal. *This conjecture would imply that the effective cone of $C(2)$ is bounded by the line spanned by the diagonal and the line of slope $-\frac{1}{\sqrt{g}-1}$ in the $(x, \frac{\delta}{2})$-plane, which is an open boundary line as soon as $g \geq 5$. One of the results in* [24] *is that:* Nagata's conjecture implies Ciliberto-Kouvidakis' conjecture. *The rather unespected connection is provided by the fact that one may degenerate C to a rational g-nodal curve so that the curves on $C(2)$ degenerate to suitable plane curves.*

For more information on Nagata's conjecture and recent results on the subject, see [42, 74, 31].

Going back to the Harbourne-Hirschowitz conjecture, the following recent results are worth to be mentioned.

Theorem 5.14. (A. Bruno [14]**, 1998)** *$\mathcal{L} = \mathcal{L}(m_1, \dots, m_h)$ is non-special if* $\mathrm{virtdim}(\mathcal{L}) \geq 0$ *and $g_{\mathcal{L}} \geq 0$ and the general curve in $\mathcal{L}$ has ordinary m_i-tuple points at p_i, $i = 1, \dots, h$.*

This result is quite interesting, though the hypothesis about the general curve in $\mathcal{L}$ is certainly too strong. The proof uses a bit of deformation theory, which reappears here after Arbarello-Cornalba's theorem 5.2. However, the main tool in Bruno's proof is the use of the moduli space of curves, of stable reduction, and of the theory of limit linear series on reducible curves (for a general introduction to these ideas, see Harris-Morrison's book [43]). This is a really new idea in this setting and may possibly give further good results in the future.

The following theorem is due to T. Mignon in his thesis [58, 59] and it is based on the use of the Horace's method:

Theorem 5.15. (T. Mignon, 1998) *Let $\mathcal{L} = \mathcal{L}(m_1, \dots, m_h)$. Then:*

(i) *if $m_i \leq 4$ Harbourne-Hirschowitz conjecture 4.8 holds;*
(ii) *if $g_{\mathcal{L}} \leq 4$ and* $\mathrm{virtdim}(\mathcal{L}) \geq 0$ *then Harbourne-Hirschowitz conjecture 4.8 and Hirschowitz conjecture 4.9 both hold;*
(iii) *if $m_i \leq 3$, $d \geq 33$,* $\mathrm{virtdim}(\mathcal{L}) \geq 0$ *and $g_{\mathcal{L}} \geq 0$ then Harbourne-Hirschowitz conjecture 4.8 and Hirschowitz conjecture 4.9 both hold.*

The interest of the next theorem, due to L. Evain [32], resides in the fact that it is the only evidence, so far, that the Harbourne-Hirschowitz conjecture holds for $\mathcal{L}_{2,d}(m^h)$ for infinitely many values of h.

Theorem 5.16. (L. Evain, 1998) *$\mathcal{L}_{2,d}(m^h)$ is never special if h is of the form $h = 4^k$.*

The proof uses a suitable version of the Horace's method. Evain lets all the multiple points come together in a smart way in a unique singular point which gives independent conditions to curves of any degree. Some of these techniques go back to Hirschowitz [44] and to Caporaso-Harris (1996, unpublished) where they prove the Harbourne-Hirschowitz conjecture in the homogeneous case $m \leq 6$ under some restrictive hypotheses. Using Evain's ideas one can probably prove that $\mathcal{L}_{2,d}(m^h)$ is non-special in other situations, e.g. if $h = k^2 + l$, $l \leq 2k$ and either $d \leq km$ or $d \geq km + m + l - 3$, giving more information on Nagata's conjecture. Also other cases like $h = 4^k l^2$, $h = 9^k$ etc. can probably be analysed in the same way. There is also work in progress by Ciliberto and Miranda, who are able to give a rather easy proof of Evain's theorem using a Z. Ran's type of degeneration of the plane, like for the proof of theorem 5.4. This also gives some hope of further extensions to other values of h.

A further important application of a refined version of the Horace's method (what the authors call the *differential Horace method*, see [6]) is to get asymptotic results confirming the Harbourne-Hirschowitz conjecture. The prototype of results of this sort is the following theorem of Hirschowitz [45]:

Theorem 5.17. *The system $\mathcal{L}_{2,d}(m_1, \dots, m_h)$ is non-special as soon as $[\frac{(d+3)^2}{4}] > \sum_{i=1}^{h} \binom{m_i+2}{2}$.*

A much deeper results is the following theorem of Alexander-Hirschowitz [6]:

Theorem 5.18. (Alexander-Hirschowitz, 1998) *Given any projective, reduced variety X and an ample line bundle $\mathcal{L}$ on it, there is a function $d(m)$ such that if $m_i < m$, $i = 1, \dots, h$, and $d > d(m)$ then $\mathcal{L}^{\otimes d}(m_1, \dots, m_h)$ is non-special.*

Note the independence of $d(m)$ by h the number of points: this makes theorem 5.18 stronger than theorem 5.17. More precise results about the function $d(m)$ in the planar case are due to other authors (see [9, 60, 39, 85]).

I will finish this section discussing the relations between the various conjectures 4.1, 4.2, 4.8. While it is clear that both, the Harbourne-Hirschowitz conjecture and Gimigliano's conjecture, imply Segre's one, it is rather surprising that the three conjectures are essentially equivalent. This is the content of the following theorem, whose proof will appear in a paper [22] in preparation:

Theorem 5.19. (Ciliberto-Clemens-Miranda, 2000) *Segre's conjecture 4.1 implies both Gimigliano's conjecture 4.2 and the Harbourne-Hirschowitz conjecture 4.8. In particular, given a linear system $\mathcal{L} := \mathcal{L}_{2,d}(m_1, \dots, m_h)$ of plane curves with general multiple base points $p_1, \dots, p_h$, if Segre's conjecture holds then:*

- (i) *$\mathcal{L}$ is special if and only if it is (-1)-special;*
- (ii) *if $\mathcal{L} \neq \emptyset$, then $C \in \mathcal{L}$ general has multiplicity m_i at p_i, $i = 1, \dots, h$;*
- (iii) *if $\mathcal{L}$ is non-special, then either $C \in \mathcal{L}$ general is irreducible, or $\mathcal{L}$ is (-1)-reducible, or $\mathcal{L}$ consists of a unique, may be multiple, elliptic curve, or $\mathcal{L}$ is composed of a pencil of rational curves.*

The surprisingly easy proof is based on standard surface theory. I want also to mention, from the same paper [22], the following result, which goes in the direction of Hirschowitz conjecture 4.9:

Corollary 5.20. (Ciliberto-Clemens-Miranda, 2000) *Suppose Segre's conjecture is true. Consider a non-special, not (-1)-reducible, linear system $\mathcal{L}:=\mathcal{L}_{2,d}(m_1,\ldots,m_h)$ of plane curves with general multiple base points such that the general curve $C \in \mathcal{L}$ is reducible. Then there is a Cremona transformation sending $\mathcal{L}$ to one of the two systems:*

$$\begin{aligned} \mathcal{L}_{2,d}(d), &\quad \textit{d-tuples of lines through a point } p \\ \mathcal{L}_{2,3d}(d^9), &\quad \textit{the cubic through 9 general points, counted d-times}. \end{aligned}$$

The proof is a consequence of theorem 5.19 and of the following classical fact:

Lemma 5.21. (Noether's lemma) *Let $\mathcal{L}$ be as in the statement of corollary 5.20, let $C \in \mathcal{L}$ general be irreducible of genus $g \leq 1$. Then there is a Cremona transformation sending $\mathcal{L}$ to one of these systems:*

$$\begin{array}{rll} \textit{a linear system of lines}, & \dim(\mathcal{L}) \leq 2, & g=0 \\ \mathcal{L}_{2,2}, & \dim(\mathcal{L}) = 5, & g=0 \\ \mathcal{L}_{2,d}(d-1), & \dim(\mathcal{L}) = 2d, & g=0 \\ \mathcal{L}_{2,d}(1,d-1), & \dim(\mathcal{L}) = 2d-1, & g=0 \\ \mathcal{L}_{2,3}(1^h), h \leq 9, & \dim(\mathcal{L}) = 9-h, & g=1 \\ \mathcal{L}_{2,4}(2^2), & \dim(\mathcal{L}) = 8, & g=1\,. \end{array}$$

6. Interpolation in More Variables

Little is known about the general dimensionality problem for linear systems in $\mathbf{P}^n$, $n \geq 3$. That little is mostly concentrated in the following beautiful result of Alexander-Hirschowitz which classifies the special linear systems $\mathcal{L}_{n,d}(2^h)$:

Theorem 6.1. (Alexander-Hirschowitz, 1996) $\mathcal{L}_{n,d}(2^h)$ *is non-special unless:*

$$\begin{array}{cccccc} n & \textit{any} & 2 & 3 & 4 & 4 \\ d & 2 & 4 & 4 & 4 & 3 \\ h & 2,\ldots,n & 5 & 9 & 14 & 7\,. \end{array} \tag{9}$$

Remark 6.2. *The statement of Alexander-Hirschowitz theorem was divined by Bronowski in* [11]*, but he had only a plausibility argument rather than a proof of it. Terracini* [83] *instead has a proof for the case* $n=3$.

Almost all the special systems shown in table (9) have been met already in examples 4.3, (i') and (i"). The only new one is $\mathcal{L}_{4,3}(2^7)$ *whose virtual dimension is* -1 *whereas it is not empty. In fact there is a unique rational normal quartic curve* Γ *through* 7 *general points* $p_1,\ldots,p_7$ *in* $\mathbf{P}^4$*. The secant variety of* Γ*, i.e. the variety described by all lines meeting* Γ *at two points, is a hypersurface of degree* 3 *and it is singular along* Γ*, hence it is singular at* $p_1,\ldots,p_7$*, thus it sits in* $\mathcal{L}_{4,3}(2^7)$*. This examples was well know to Terracini* [82] *and it has been more recently rediscovered by Ciliberto-Hirschowitz* [23].

The original proof of the theorem of Alexander-Hirschowitz requires the full strengh of the Horace's method and it is long and difficult. Indeed it occupies a whole series of papers [1]–[5]. An easier proof has been recently provided by K. Chandler [18]. It still uses the Horace's method but in a much simpler way, by subsequent specializations of part of the general double points of the linear system to a hyperplane. Work in progress by Ciliberto and Miranda indicates that an alternative and quite simple proof can be also obtained by using suitable Z. Ran's type of degenerations of $\mathbf{P}^n$. In essence, this approach is not so different from Chandler's one, but, again, it looks more systematic and it gives some hope to extend the analysis to higher multiplicities.

The important feature of the special systems appearing in table (9) is the following: *for each special $\mathcal{L}_{n,d}(2^h)$, the general member $D \in \mathcal{L}_{n,d}(2^h)$ is singular along a positive dimensional variety containing the general double base points of the system $\mathcal{L}_{n,d}(2^h)$*. Roughly speaking the phenomenon of speciality is not concentrated at the base points but somehow *propagates in space*! At least for double base points, this observation has a quite general meaning, and goes back to Terracini [81] (for a modern version see [23]):

Lemma 6.3. (Terracini, 1915; Ciliberto-Hirschowitz, 1991) *Let X be any projective variety, let $\mathcal{L}$ be a linear system on X, let $p_1, \dots, p_h$ be general points of X. If $\mathcal{L}(-\sum_{i=1}^h 2p_i)$ is special then every $D \in \mathcal{L}(-\sum_{i=1}^h 2p_i)$ is singular along a positive dimensional variety containing $p_1, \dots, p_h$.*

The proof of the lemma is based on an easy first order infinitesimal computation, to the effect that any first order deformation of a singular hypersurface D which preserves the singularities of D is a hypersurface D' containing the singular locus of D. As already mentioned before, this computation is basically the one needed for the proof of Arbarello-Cornalba's theorem 5.2.

It is quite natural to conjecture that the phenomenon of *propagation in space* of speciality of linear systems with multiple general base points should take place for higher multiplicities too. This is the content of the following conjecture, which I share with R. Miranda:

Conjecture 6.4. (Ciliberto-Miranda) *Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_h)$ be a linear system with multiple base points at $p_1, \dots, p_h$. If the general member $D \in \mathcal{L}$ has isolated singularities at $p_1, \dots, p_h$, then $\mathcal{L}$ is non-special.*

Remark 6.5. *Notice that the converse of the conjecture certainly does not hold. If $\mathcal{L}$ is non-special then the general member $D \in \mathcal{L}$ may very well have non isolated singularities containing $p_1, \dots, p_h$. An example is $\mathcal{L}_{3,4}(2^8)$, which, according to Alexander-Hirschowitz's theorem, is non-special, hence of dimension 2. Notice that there is a pencil of quadrics $\mathcal{P} = \mathcal{L}_{3,2}(1^8)$ through the 8 base points, having a base locus Γ which is an elliptic quartic curve. Then $\mathcal{L}_{3,4}(2^8)$ is composed of all pairs of elements of $\mathcal{P}$ and therefore the general element of $\mathcal{L}_{3,4}(2^8)$ is singular along Γ.*

Conjecture 6.4 would be in perfect analogy with Segre's conjecture 4.1. However one can be bolder, and try to make a conjecture which parallels Harbourne-Hirschowitz's one. Let us try to do it now.

Let me start with a definition. Recall, in the course of it, that a theorem of Bellatalla-Grothendieck [10, pg. 43], asserts that any vector bundle on $\mathbf{P}^1$ splits in a unique way as a direct sum of line bundles.

Definition 6.6. *Let X be a smooth, projective variety of dimension n, let C be a smooth, irreducible curve on X and let $\mathcal{N}_{C|X}$ be the normal bundle of C in X. We will say that C is a* negative curve *if there is a line bundle $\mathcal{N}$ of negative degree and a surjective map $\mathcal{N}_{C|X} \to \mathcal{N}$.*

The curve C is called a (-1)-curve of size a, with $1 \leq a \leq n-1$, on X if $C \simeq \mathbf{P}^1$ and $\mathcal{N}_{C|X} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus a} \oplus \mathcal{N}$, where $\mathcal{N}$ has no summands of negative degree.

We are now ready to make our general conjecture:

Conjecture 6.7. *Let X be the blow-up of $\mathbf{P}^n$ at general points $p_1, \ldots, p_h$ and let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_h)$ be a linear system with multiple base points at $p_1, \ldots, p_h$. Then:*

- (i) *the only negative curves on X are (-1)-curves;*
- (ii) *$\mathcal{L}$ is special if and only if there is a (-1)-curve C on X corresponding to a curve Γ on $\mathbf{P}^n$ containing $p_1, \ldots, p_h$ such that the general member $D \in \mathcal{L}$ is singular along Γ;*
- (iii) *if $\mathcal{L}$ is special, let B be the component of the base locus of $\mathcal{L}$ containing Γ according to Bertini's theorem. Then the codimension of B in $\mathbf{P}^n$ is equal to the size of C and B appears multiply in the base locus scheme of $\mathcal{L}$.*

Remark 6.8. *Of course, the above conjecture concides with conjecture 4.8 for dimension $n = 2$ (see also remark 4.10 for part (i) of the conjecture).*

As a general warning, I should stress that there is not too much evidence for conjecture 6.7, to the extent that I do not even know whether it is true for general multiple double points. In particular the case $\mathcal{L}_{4,4}(2^{14})$ remains rather difficult to attack, whereas in the other cases in table (9) the conjecture holds.

For instance, in the case $\mathcal{L}_{3,4}(2^7)$, the rational quartic curve Γ through the 7 double base points $p_1, \ldots, p_7$, corresponds to a curve $C \simeq \mathbf{P}^1$ on the blow-up of $\mathbf{P}^4$ at $p_1, \ldots, p_7$, whose normal bundle is $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 3}$, hence C is a (-1)-curve of size 3. This fits with conjecture 6.7.

The other case in table (9) is $\mathcal{L}_{3,4}(2^9)$ which consists of the unique quadric $B \in \mathcal{L}_{3,2}(1^9)$ counted twice. On B there is a unique rational quintic curve Γ through the nine base points, which corresponds to a curve C on the blow-up $X = \tilde{\mathbf{P}}^3$. Since the normal bundle to $\Gamma \simeq \mathbf{P}^1$ is $\mathcal{O}_{\mathbf{P}^1}(8) \oplus \mathcal{O}_{\mathbf{P}^1}(10)$, then the normal bundle to C in X is $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. Hence C is a (-1)-curve of size 1 on X, which again fits with the conjecture.

It goes without saying that, at the present moment, I have no idea about the possible relations between the two conjectures 6.4 and 6.7 above. Another related *interesting question would be:* describe the set of (-1)-curves on the blow-up X of $\mathbf{P}^n$ at general points $p_1, \ldots, p_h$. *If $n = 2$, for instance, it is known that this set is finite if and only if $n \leq 8$ and its behaviour under the action of the so-called* Kantor group *is known also in the infinite case $n \geq 9$ (see [30]). Is there any extension of this to the case $n \geq 3$?*

7. The Waring's Problem and Defective Varieties

The birthplace of the Waring's problem is number theory. It can be stated as follows:

Problem 7.1. (Waring's problem) *Given positive integers d, h, may we write any positive integer as a sum of h non-negative d-th powers?*

One of the first instances of this problem is the famous *Gauss' four squares theorem* to the effect that: *every positive integer is a sum of* 4 *squares.* A theorem which is sharp, inasmuch as there are integers, like 7, which cannot be written as a sum of 3 squares.

I will not be interested here in the original, number theoretic problem 7.1, but rather in the following transposition of it to the realm of polynomials:

Problem 7.2. (Waring's problems for forms) *Given positive integers d, h, n, may we write any homogeneous polinomial $f(x_0, \ldots, x_n)$ of degree d as a sum of h d-th powers of linear forms $l_i(x_0, \ldots, x_n)$, $i = 1, \ldots, h$, i.e. as $f(x_0, \ldots, x_n) = \sum_{i=1}^h l_i(x_0, \ldots, x_n)^d$?*

The relations of this problem with the general dimensionality problem introduced in §3 will be clear in a while, once we give a geometric interpretation of it based on *secant varieties* and *Terracini's lemma*, ideas which have been developed by the classical Italian school of algebraic geometry. One word of warning before doing that: an equivalent interpretation can be given in a slightly different way using other concepts like differential operators and *inverse systems*. This approach, classically developed by Macaulay [56], has been in recent times taken up by various authors, staring with Iarrobino [46] and [47]. Via the classical and well known notion of *apolarity*, the two approachs are basically equivalent, so that I feel free of presenting here only the first one, referring the reader to [46] or to the nice expository paper [61] for the other.

First I recall a basic definition:

Definition 7.3. *Let $X \subseteq \mathbf{P}^N$ be an irreducible, non-degenerate projective variety of dimension n and let k be a positive integer. Take $k+1$ independent points $p_0, \ldots, p_k$ of X. The span $\langle p_0, \ldots, p_k \rangle$ is a subspace of $\mathbf{P}^N$ of dimension k which is called a $(k+1)$-*secant $\mathbf{P}^k$ *of X. By* $\mathrm{Sec}_k(X)$ *we denote the closure of the union of all $(k+1)$-secant $\mathbf{P}^k$'s of X. This is an irreducible algebraic variety which is called the k-*th secant variety *of X.*

A basic parameter count shows that:

$$\dim(\mathrm{Sec}_k(X)) \leq \min\{(n+1)(k+1)-1, N\} \tag{10}$$

the right hand side of (10) is the expected dimension *of* $\mathrm{Sec}_k(X)$. *If the strict inequality holds in (10), then X is said to be* k-defective, *and its* k-defect *is* $\delta_k(X) := \min\{(n+1)(k+1)-1, N\} - \dim(\mathrm{Sec}_k(X))$.

Let now $\mathbf{P} := \mathbf{P}^{N_{n,d}}$ be the projective space $\mathbf{P}(\mathbf{C}[x_0,\dots,x_n]_d)$ associated to the vector space $\mathbf{C}[x_0,\dots,x_n]_d$ of all complex homogeneous polynomial of degree d in the variables $x_0,\dots,x_n$. Recall that $N_{n,d} = \binom{d+n}{n} - 1$. A remarkable subvariety of $\mathbf{P}$ is the so called *dual d-th Veronese of* $\mathbf{P}^n$, namely the set $\mathcal{V}_{n,d}$ of all d-th powers of linear forms in $x_0,\dots,x_n$. Of course $f(x_0,\dots,x_n) \in \mathbf{C}[x_0,\dots,x_n]_d$ is a sum of h d-th powers of linear forms if and only if $[f] \in \mathbf{P}$ is contained in a h-secant $\mathbf{P}^{h-1}$ to $\mathcal{V}_{n,d}$. Thus, given the positive integers d, h, n, if the Waring's problem has an affirmative answer for all forms $f(x_0,\dots,x_n) \in \mathbf{C}[x_0,\dots,x_n]_d$ then $\mathrm{Sec}_{h-1}(\mathcal{V}_{n,d}) = \mathbf{P}^{N_{n,d}}$, which implies that $h(n+1)-1 \geq N_{n,d}$, i.e. one has $h \geq \lceil \frac{1}{n+1} \cdot \binom{d+n}{n} \rceil$.

The main question is then: does the converse hold? It turns out that the answer is provided by Alexander-Hirschowitz's theorem 6.1 In order to explain why this is the case, we need another bit of geometric information, namely the following result (see [81, 27]) which is the basic tool for understanding defective varieties:

Lemma 7.4. (Terracini's lemma) *Let $X \subseteq \mathbf{P}^N$ be an irreducible, non-degenerate projective variety of dimension n and let $k \leq N$ be a positive integer. Take $k+1$ general points $p_0,\dots,p_k$ of X and let $p \in \langle p_0,\dots,p_k\rangle$ be a general point in* $\mathrm{Sec}_k(X)$. *Then the tangent space $T_{\mathrm{Sec}_k(X),p}$ to* $\mathrm{Sec}_k(X)$ *at p is given by $T_{\mathrm{Sec}_k(X),p} = \langle \cup_{i=0}^k T_{X,p_i}\rangle$.*

Hence, assume that $h \geq \lceil \frac{1}{n+1} \cdot \binom{d+n}{n} \rceil$ and that Waring's problem has no solution for $f(x_0,\dots,x_n) \in \mathbf{C}[x_0,\dots,x_n]_d$ general. Then $\mathrm{Sec}_{h-1}(\mathcal{V}_{n,d})$ is not equal to $\mathbf{P}^{N_{n,d}}$ and therefore, for $p_1,\dots,p_h \in \mathcal{V}_{n,d}$ general, one has that the subspace $\langle \cup_{i=1}^h T_{\mathcal{V}_{n,d},p_i}\rangle$ must be contained in a hyperplane of $\mathbf{P}^{N_{n,d}}$.

To connect all this with the general dimensionality problem and Alexander-Hirschowitz theorem, dualize and look at $\mathcal{V}_{n,d}$ in $\mathbf{P}^{N_{n,d}}$ as $\mathbf{P}^n$ embedded via the complete linear system $\mathcal{L}_{n,d}$. Thus a hyperplane H in $\mathbf{P}^{N_{n,d}}$ cuts out on $\mathcal{V}_{n,d}$ a divisor whose pull-back to $\mathbf{P}^n$ is hypersurface D_H of degree d, i.e. a member of $\mathcal{L}_{n,d}$. If p is a point in $\mathcal{V}_{n,d}$, we abuse notation and denote by p also the corresponding point in $\mathbf{P}^n$. The hyperplane H is tangent to $\mathcal{V}_{n,d}$ at p, i.e. it contains $T_{\mathcal{V}_{n,d},p}$, if and only if the corresponding hypersurface D_H is singular at p.

In conclusion, if Waring's problem has no solution for a general $f(x_0,\dots,x_n) \in \mathbf{C}[x_0,\dots,x_n]_d$, then for $p_1,\dots,p_h \in \mathbf{P}^n$ general points, there is a hypersurface $D \in \mathcal{L}_{n,d}$ singular at $p_1,\dots,p_h$. Since $h(n+1)-1 \geq N_{n,d}$ is equivalent to $\mathrm{virtdim}(\mathcal{L}_{n,d}(2^h)) \leq -1$, we see that the cases in which Waring's problem has no solution for $f(x_0,\dots,x_n) \in \mathbf{C}[x_0,\dots,x_n]_d$ general and $h(n+1)-1 \geq N_{n,d}$ are exactly those listed in table (9) from Alexander-Hirschowitz theorem 6.1.

Remark 7.5. *The above cases of failure of a positive answer to Waring's problem were well known to the old geometers and invariant theorists. Besides the case of quadrics, which is trivial, the case of plane quartics was known to Clebsch (see the citation in* [83]*), who proved that in that case the sum of* 6*, instead of* 5*, as expected, fourtuple powers of linear forms is needed to obtain a general form of degree* 4 *in* 3 *variables. The remaining three cases where known to Palatini* [69] *and Terracini* [83]*, who also proved that one more summand than expected is needed to represent a general form in each of the cases in question.*

In the old times an interesting extension of the Waring's problem has also been considered. It goes back to Darboux [29], Reye [73], London [52], Palatini [65], Bronowski [12], Terracini [82] and is the following:

Problem 7.6. (Extended Waring's problem) *Given positive integers* d*,* h*,* n*,* s*, may we write any* s *homogeneous polinomial* $f_j(x_0,\dots,x_n)$*,* $j=1,\dots,s$*, of degree* d *as linear combinations of the* same h d*-th powers of linear forms* $l_i(x_0,\dots,x_n)$*,* $i=1,\dots,h$*?*

This leads right away to the following quite intriguing geometric problem:

Problem 7.7. *What is the dimension of the variety* $\mathrm{Sec}_{l,k}(X)$ *described by all* $\mathbf{P}^l$*'s contained in* $(k+1)$*-secant* $\mathbf{P}^k$*'s to a variety* X *of dimension* n *in* $\mathbf{P}^N$*?*

The expected dimension of $\mathrm{Sec}_{l,k}(X)$, which is contained in the grassmannian $\mathbf{G}(l,N)$, is $\min\{(l+1)(N-l),(k+1)n+(l+1)(k-l)\}$, but its actual dimension may be smaller. Thus a refined, grassmannian version of the concept of defect arises. Unfortunately there is no Terracini type lemma which helps in this situation. However there is recent interesting work of Chiantini-Coppens [19] on the case $n=2$, $N=5$, $l=1$, $k=2$ for problem 7.7. It should also be noticed that Terracini claims in [82] that for $n=s=2$ he has a complete solution to problem 7.6: the only exception to the expected answer is for $d=3$, i.e. the lines contained in a 5-secant $\mathbf{P}^4$ to $\mathcal{V}_{2,3}\subset\mathbf{P}^9$ are not all the lines of $\mathbf{P}^9$ as a parameter count suggests.

Like the original Waring's problem is related to the Alexander-Hirschowitz's theorem, the extended Waring's problem 7.6 might lead to interesting extensions of the Alexander-Hirschowitz's theorem. I believe this is an open, promising field of research.

In the same circle of ideas presented in this section, one is lead in a natural way to the problem of the *classification of defective varieties*, which thus appears as another geometric counterpart of the general dimensionality problem introduced in §3. This is a classical, basic question concerning the extrinsic geometry of projective varieties, which has also applications to other branches of mathematics. For instance, computation of defects of Segre varieties, i.e. the products of projective spaces, is related to the linear algebra problem of determining the *rank* of a general tensor, which again is a relevant question in numerical analysis (see [17, 33, 51] and [80])

Among the main tools here there are the two Terracini's lemmas 6.3 and 7.4. It is not possible to enter now in too many details, but, before finishing, I will briefly recall, without any pretense of being exhaustive, some of the main results on the subject.

As in the general dimensionality problem, curves are never defective. Surfaces instead, can be defective. The subject of defective surfaces has been considered classically by Palatini [67] and [68], whose classification theorem contained a serious gap, and Terracini [84], who completed Palatini's classification (see also Scorza's [76] and Bronowski's [13] papers on the subject). Both Palatini and Terracini's papers are quite obscure and difficult to read. In more recent times Palatini-Terracini's classification of defective surfaces has been reconsidered, rediscovered and worked out again by M. Dale [28].

As for higher dimensional defective varieties, we are rather far away from a classification. In Zak's book [86] one finds several general properties. In particular a smooth, irreducible, non-degenerate, 1-defective variety $X \subset \mathbf{P}^N$ of dimension n is such that $N+1 \leq \binom{n+1}{2}$ and if the equality holds then $X = \mathcal{V}_{n,2}$ is the Veronese variety of quadrics of $\mathbf{P}^n$ in which case the defect is 1 (see [86, Theorem 2.1, pg. 126]). Zak has similar, equally beautiful, theorems for 1-defective varieties with higher defect (see [86, Chapter VI]).

A weaker concept than the one of a k-defective variety, is the concept of a *k-weakly defective* variety. This is a variety $X \subset \mathbf{P}^N$ such that the general hyperplane which is tangent at $k+1$ general points $p_0, \ldots, p_k$, is tangent along a positive dimensional variety contanining $p_0, \ldots, p_k$. Recalling lemma 6.3, it is clear that a k-defective variety is also k-weakly defective. The converse does not hold in general. It turns out that the classification of weakly defective varieties of dimension smaller than n matters in the classification of defective varieties of dimension n.

A full classification of weakly defective surfaces, which extends previous partial results by Terracini [84], has been recently obtained by Chiantini-Ciliberto [20]. This can be also seen as a wide extension of Arbarello-Cornalba's theorem 5.2. In addition this gives some hope for the complete classification of defective threefolds. This is a subject which has classically been studied by Scorza [75], who claims to have a classification of all 1-defective 3-folds. He also studied defective 4-fold in [77]. In more recent times the subject has been reconsidered by other authors, e.g. Zak [86], Fujita-Roberts [36] and Fujita [35], who essentially consider the case of smooth threefolds. For any threefold, without any smoothness assumption, Ciliberto-Chiantini [21] have reworked Scorza's classification of 1-defective threefolds. Their approach, easier and faster than Scorza's original one, is essentially based on a refined version of lemma 6.3. The result is the following:

Theorem 7.8. *An irreducible, non-degenerate, projective 3-fold $X \in \mathbf{P}^N$ is 1-defective if and only if it is of one of the following types:*

(i) *X is a cone;*
(ii) *X sits in a 4-dimensional cone over a curve;*
(iii) *$N = 7$ and X is contained in a 4-dimensional cone over the Veronese surface $\mathcal{V}_{2,2}$ in $\mathbf{P}^5$;*

(iv) *X is the 2-Veronese embedding $\mathcal{V}_{3,3}$ of $\mathbf{P}^3$ in $\mathbf{P}^9$ or a projection of it in $\mathbf{P}^8$;*

(v) *$N = 7$ and X is a hyperplane section of the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$ in $\mathbf{P}^8$.*

Work in progress by Ciliberto-Chiantini indicates that along the same lines one may possibly obtain the full classification of defective threefolds.

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