On Function Spaces that are Interpolating at Any k Nodes

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Let M be a topological space, and L be a finite-dimensional vector subspace in the space of continuous real-valued functions on M. The interpolation properties of the space L can depend on the choice of nodes of interpolation. We investigate the dimensions of function spaces interpolating any function on M at any k nodes.

Definition 1. The space L is called *k-interpolating* if any function $M \to \mathbb{R}$ can be interpolated at any k points by an appropriate function from L.

In other words, L is k-interpolating if for any k distinct points $x_1, \dots, x_k$ in M and any real numbers $a_1, \dots, a_k$, there exists a function $F \in L$ taking the value a_i at the point x_i for any $i = 1, \dots, k$.

Denote by $I(M, k)$ the minimal dimension of k-interpolating spaces of functions on M.

The main problem is to find the numbers $I(M, k)$ for all M and k.

Examples. If $M = \mathbb{R}^1$, then $I(M, k) = k$: for a k-interpolating space we can take the space of polynomials of degree $\leq k-1$.

If $M = S^1$, then $I(M, k) = k$ for odd k, and $I(M, k) = k+1$ for even k: in both cases the k-interpolating space is presented by Fourier polynomials of degree $\leq [k/2]$.

Consider the first nontrivial case, $M = \mathbb{R}^2$.

Theorem 1. *The number $I(\mathbb{R}^2, k)$ satisfies the inequality*

$$2k - d(k) \leq I(\mathbb{R}^2, k) \leq 2k - 1,$$

where $d(k)$ is the number of ones in the binary representation of k. In particular, $I(\mathbb{R}^2, k) = 2k-1$ if k is a power of 2.

Note that in the first questionable case (when $k = 3$) the complicated lower estimate turns out to be realistic, and not the simple upper one: the 4-dimensional space spanned by the functions 1, x, y, $x^2 + y^2$ is 3-interpolating for $\mathbb{R}^2$.

Theorem 2. *For any n and for any n-dimensional manifold M we have*

$$I(M, k) \leq k(n+1).$$

Conjecture 1. *If n is a power of 2, then*

$$I(\mathbb{R}^n, k) \geq k + (n-1)(k - d(k)).$$

This conjecture is closely related to certain conjectures on the multiplicative structure in the cohomology of configuration spaces; see Conjecture 2 below.

Definition 2. A space L of functions $M \to \mathbb{R}$ is called *k-distinguishing* if the linear span of L and the function identically equal to 1 is a k-interpolating space.

This notion has an obvious geometrical interpretation: a set of functions $f_1, \dots, f_N$ forms a basis of a k-distinguishing space iff the map $M \to \mathbb{R}^N$ defined by them takes any k different points from M to the vertices of some $(k-1)$-dimensional simplex in $\mathbb{R}^N$. In particular, the calculation of the minimal dimension $D(M, k)$ of k-distinguishing spaces involves (for $k = 2$) the problem of imbedding of manifolds into Euclidean spaces. On the other hand, $I(M, k) = D(M \cup \{\text{a point}\}, k)$.

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Theorem 3. *Theorems 1, 2 remain valid if we replace the numbers* $I(M, k)$ *in their statements by* $D(M, k)+1$.

Proof of Theorem 1. The upper estimate $I(\mathbb{R}^2, k) \leq 2k-1$ is provided by the functions 1, $\operatorname{Re}(z^t)$, $\operatorname{Im}(z^t)$, $t=1, \ldots, k-1$, where z is the complex coordinate on $\mathbb{R}^2$.

Now, for an arbitrary topological space M, consider the configuration space $B(M, k)$, i.e., the space of all subsets of cardinality k in M with the obvious topology. Consider the k-dimensional vector bundle $T(M, k)$ over $B(M, k)$ whose fiber over the point $\{x_1, \ldots, x_k\}$ is the space of real-valued functions on the set $\{x_1, \ldots, x_k\}$. For any N-dimensional space L of functions on M, consider also the trivial N-dimensional vector bundle $\{L\} \equiv L \times B(M, k)$ over $B(M, k)$. There exists a natural homomorphism Restr: $\{L\} \to T(M, k)$: over any point $\{x_1, \ldots, x_k\} \in B(M, k)$, to any function $F \in L$ there corresponds its restriction to the set $\{x_1, \ldots, x_k\}$. Obviously, the space L is k-interpolating iff this morphism is epimorphic, and hence the $(N-k)$-dimensional vector bundle Ker Restr is well-defined. This implies

Theorem 4. *For any* M *and* k, $I(M, k)-k \geq \deg w^{-1}(T(M, k))$, *where* $w(E)$ *is the total Stiefel–Whitney class of the vector bundle* E *(see [1]),* $w^{-1}(E)$ *is its inverse in the multiplicative group of the ring* $H^*(B(M, k), Z_2)$, *and* deg *is the grading of the highest nonzero homogeneous part of an element of a graded ring.*

For example, put $M=\mathbb{R}^2$. The rings $H^*(B(\mathbb{R}^2, k), Z_2)$, as well as the Stiefel–Whitney classes of the bundles $T(\mathbb{R}^2, k)$, were calculated in [2]. In particular, it follows from [2] that $\deg w(T(\mathbb{R}^2, k)) = k-d(k)$. In [2] it was also proved that $h^2=0$ for any element h of positive dimension in the ring $H^*(B(\mathbb{R}^2, k), Z_2)$. Hence, $w^{-1}(E)=w(E)$ for any bundle E over $B(\mathbb{R}^2, k)$, and $\deg w^{-1}(T(\mathbb{R}^2, k)) = k-d(k)$. This proves the lower estimate in Theorem 1.

The estimate $I(S^1, k) \geq k+1$ for even k is proved similarly.

Theorem 2 immediately follows from Thom's multijet transversality theorem; see [3]. Indeed, let $F(M, k)$ be the space of ordered subsets of cardinality k in M. Then the set of real-valued functions $f_1, \ldots, f_N$ on M defines a map of $F(M, k)$ into the space of $k \times N$-matrices. The set of matrices of nonmaximal rank has codimension $N-k+1$ in this space, so, if $N-k+1 > k \dim M$, then the image of $F(M, k)$ under the map defined by a generic set $\{f_1, \ldots, f_N\}$ does not intersect this set.

Remark. The estimate of Theorem 2 is not realistic even in the case $k=2$: the strong Whitney imbedding theorem implies that $I(M^n, 2) \leq 2n+1$.

Conjecture 2. *If* n *is a power of* 2, *then the* n*th power of any element of positive dimension in* $H^*(B(\mathbb{R}^n, k), Z_2)$ *equals zero. In particular,* $[w(T(\mathbb{R}^n, k))]^n = 1$.

Conjecture 1 follows (in the same way as Theorem 1) from the latter conjecture and from the fact that if k is a power of 2, then the class $[w_{k-1}(T(\mathbb{R}^n, k))]^{n-1}$ is nontrivial.

References

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