BRIEF COMMUNICATIONS

On Function Spaces that are Interpolating at Any \( k \) Nodes

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Let \( M \) be a topological space, and \( L \) be a finite-dimensional vector subspace in the space of continuous real-valued functions on \( M \). The interpolation properties of the space \( L \) can depend on the choice of nodes of interpolation. We investigate the dimensions of function spaces interpolating any function on \( M \) at any \( k \) nodes.

**Definition 1.** The space \( L \) is called \( k \)-interpolating if any function \( M \to \mathbb{R} \) can be interpolated at any \( k \) points by an appropriate function from \( L \).

In other words, \( L \) is \( k \)-interpolating if for any \( k \) distinct points \( x_1, \ldots, x_k \) in \( M \) and any real numbers \( a_1, \ldots, a_k \), there exists a function \( F \in L \) taking the value \( a_i \) at the point \( x_i \) for any \( i = 1, \ldots, k \).

Denote by \( I(M, k) \) the minimal dimension of \( k \)-interpolating spaces of functions on \( M \).

The main problem is to find the numbers \( I(M, k) \) for all \( M \) and \( k \).

**Examples.** If \( M = \mathbb{R}^1 \), then \( I(M, k) = k \): for a \( k \)-interpolating space we can take the space of polynomials of degree \( \leq k - 1 \).

If \( M = S^1 \), then \( I(M, k) = k \) for odd \( k \), and \( I(M, k) = k + 1 \) for even \( k \): in both cases the \( k \)-interpolating space is presented by Fourier polynomials of degree \( \leq \lfloor k/2 \rfloor \).

Consider the first nontrivial case, \( M = \mathbb{R}^2 \).

**Theorem 1.** The number \( I(\mathbb{R}^2, k) \) satisfies the inequality
\[
2k - d(k) \leq I(\mathbb{R}^2, k) \leq 2k - 1,
\]
where \( d(k) \) is the number of ones in the binary representation of \( k \). In particular, \( I(\mathbb{R}^2, k) = 2k - 1 \) if \( k \) is a power of 2.

Note that in the first questionable case (when \( k = 3 \)) the complicated lower estimate turns out to be realistic, and not the simple upper one: the 4-dimensional space spanned by the functions 1, \( x \), \( y \), \( x^2 + y^2 \) is 3-interpolating for \( \mathbb{R}^2 \).

**Theorem 2.** For any \( n \) and for any \( n \)-dimensional manifold \( M \) we have
\[
I(M, k) \leq k(n + 1).
\]

**Conjecture 1.** If \( n \) is a power of 2, then
\[
I(\mathbb{R}^n, k) \geq k + (n - 1)(k - d(k)).
\]

This conjecture is closely related to certain conjectures on the multiplicative structure in the cohomology of configuration spaces; see Conjecture 2 below.

**Definition 2.** A space \( L \) of functions \( M \to \mathbb{R} \) is called \( k \)-distinguishing if the linear span of \( L \) and the function identically equal to 1 is a \( k \)-interpolating space.

This notion has an obvious geometrical interpretation: a set of functions \( f_1, \ldots, f_N \) forms a basis of a \( k \)-distinguishing space iff the map \( M \to \mathbb{R}^N \) defined by them takes any \( k \) different points from \( M \) to the vertices of some \( (k - 1) \)-dimensional simplex in \( \mathbb{R}^N \). In particular, the calculation of the minimal dimension \( D(M, k) \) of \( k \)-distinguishing spaces involves (for \( k = 2 \)) the problem of imbedding of manifolds into Euclidean spaces. On the other hand, \( I(M, k) = D(M \cup \{ \text{a point} \}, k) \).


Theorem 3. Theorems 1, 2 remain valid if we replace the numbers \(I(M, k)\) in their statements by \(D(M, k) + 1\).

Proof of Theorem 1. The upper estimate \(I(\mathbb{R}^2, k) \leq 2k - 1\) is provided by the functions \(1, \text{Re}(z^t), \text{Im}(z^t), t = 1, \ldots, k - 1\), where \(z\) is the complex coordinate on \(\mathbb{R}^2\).

Now, for an arbitrary topological space \(M\), consider the configuration space \(B(M, k)\), i.e., the space of all subsets of cardinality \(k\) in \(M\) with the obvious topology. Consider the \(k\)-dimensional vector bundle \(T(M, k)\) over \(B(M, k)\) whose fiber over the point \(\{x_1, \ldots, x_k\}\) is the space of real-valued functions on the set \(\{x_1, \ldots, x_k\}\). For any \(N\)-dimensional space \(L\) of functions on \(M\), consider also the trivial \(N\)-dimensional vector bundle \(\{L\} \cong L \times B(M, k)\) over \(B(M, k)\). There exists a natural homomorphism \(\text{Restr}: \{L\} \to T(M, k)\): over any point \(\{x_1, \ldots, x_k\} \in B(M, k)\), to any function \(F \in L\) there corresponds its restriction to the set \(\{x_1, \ldots, x_k\}\). Obviously, the space \(L\) is \(k\)-interpolating iff this morphism is epimorphic, and hence the \((N-k)\)-dimensional vector bundle \(\ker\text{Restr}\) is well-defined. This implies

Theorem 4. For any \(M\) and \(k\), \(I(M, k) - k \geq \deg w^{-1}(T(M, k))\), where \(w(E)\) is the total Stiefel-Whitney class of the vector bundle \(E\) (see [1]), \(w^{-1}(E)\) is its inverse in the multiplicative group of the ring \(H^*(B(M, k), \mathbb{Z}_2)\), and \(\deg\) is the grading of the highest nonzero homogeneous part of an element of a graded ring.

For example, put \(M = \mathbb{R}^2\). The rings \(H^*(B(\mathbb{R}^2, k), \mathbb{Z}_2)\), as well as the Stiefel–Whitney classes of the bundles \(T(\mathbb{R}^2, k)\), were calculated in [2]. In particular, it follows from [2] that \(\deg w(T(\mathbb{R}^2, k)) = k - d(k)\). In [2] it was also proved that \(h^2 = 0\) for any element \(h\) of positive dimension in the ring \(H^*(B(\mathbb{R}^2, k), \mathbb{Z}_2)\). Hence, \(w^{-1}(E) = w(E)\) for any bundle \(E\) over \(B(\mathbb{R}^2, k)\), and \(\deg w^{-1}(T(\mathbb{R}^2, k)) = k - d(k)\). This proves the lower estimate in Theorem 1.

The estimate \(I(S^1, k) \geq k + 1\) for even \(k\) is proved similarly.

Theorem 2 immediately follows from Thom's multijet transversality theorem; see [3]. Indeed, let \(F(M, k)\) be the space of ordered subsets of cardinality \(k\) in \(M\). Then the set of real-valued functions \(f_1, \ldots, f_N\) on \(M\) defines a map of \(F(M, k)\) into the space of \(k \times N\)-matrices. The set of matrices of nonmaximal rank has codimension \(N - k + 1\) in this space, so, if \(N - k + 1 > k \dim M\), then the image of \(F(M, k)\) under the map defined by a generic set \(\{f_1, \ldots, f_N\}\) does not intersect this set.

Remark. The estimate of Theorem 2 is not realistic even in the case \(k = 2\): the strong Whitney imbedding theorem implies that \(I(\mathbb{R}^n, 2) \leq 2n + 1\).

Conjecture 2. If \(n\) is a power of \(2\), then the \(n\)th power of any element of positive dimension in \(H^*(B(\mathbb{R}^n, k), \mathbb{Z}_2)\) equals zero. In particular, \([w(T(\mathbb{R}^n, k))]^n = 1\).

Conjecture 1 follows (in the same way as Theorem 1) from the latter conjecture and from the fact that if \(k\) is a power of \(2\), then the class \([w_{k-1}(T(\mathbb{R}^n, k))]^{n-1}\) is nontrivial.

References


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