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MAXIMUM WARING RANKS OF MONOMIALS

ZACH TEITLER

Abstract. We show that monomials in four or more variables have Waring rank less than the generic Waring rank. We compare the maximal rank of monomials in three variables with the currently best known upper bound for Waring rank, and briefly discuss a non-monomial example. We compare ranks of sums of pairwise coprime monomials with the generic Waring rank and conjecture that all such sums have less than generic rank, with exactly three exceptions.

The space of degree d forms in n variables has a basis consisting of powers of linear forms, so every such form F has Waring rank $r(F)$ at most the dimension of the space, $r(F) \leq \binom{d+n-1}{n-1}$. This was improved to $r(F) \leq \binom{d+n-2}{n-1}$ in [BBS08] and then improved further to $r(F) \leq \binom{d+n-2}{n-1} - \binom{d+n-6}{n-3}$ in [Jel13]. But the actual maximum rank is known in only a few cases: binary ($n = 2$) forms of degree d have rank at most d, with $r(xy^{d-1}) = d$; plane cubics ($n = 3$, $d = 3$) have rank at most 5 [Yer32, CM96, LT10] and plane quartics ($n = 3$, $d = 4$) have rank at most 7 [Kle99, BGI11, Par13]. Jelisiejew's upper bound is sharp for plane cubics but not for plane quartics. It would be helpful to find forms of high rank, to provide lower bounds for upper bounds for rank.

The generic Waring rank of a form in n variables of degree d is

$$r_{\text{gen}}(n, d) = \left\lceil \frac{1}{n}\binom{n+d-1}{n-1} \right\rceil,$$

except if $(n, d) = (n, 2), (3, 4), (4, 4), (5, 3), (5, 4)$, by the Alexander–Hirschowitz theorem [AH95, Cha02, Cha01, BO08, Pos12]. In the exceptional cases $r_{\text{gen}}(n, 2) = n$, $r_{\text{gen}}(3, 4) = 6$, $r_{\text{gen}}(4, 4) = 10$, $r_{\text{gen}}(5, 3) = 8$, and $r_{\text{gen}}(5, 4) = 15$. The gap between the generic rank and Jelisiejew's upper bound is not too large:

$$\frac{\left\lceil \frac{1}{n}\binom{d+n-1}{n-1} \right\rceil}{\binom{d+n-2}{n-1} - \binom{d+n-6}{n-3}} \geq \frac{\frac{1}{n}\binom{d+n-1}{n-1}}{\binom{d+n-2}{n-1}} = \frac{d+n-1}{dn}.$$

But, other than the case of binary forms, very few examples are known of forms with greater than generic rank. In fact, it seems that until recently only finitely many such examples were known: just some plane cubics and quartics.

Recently, however, an infinite family of forms was discovered to have greater than generic rank. This family was found by Carlini, Catalisano, and Geramita in their solution of the Waring rank problem for monomials [CCG12]. Let $M = x_1^{a_1} \cdots x_n^{a_n}$ with $0 < a_1 < \cdots < a_n$. The Waring rank of M is

$$r(M) = (a_2 + 1) \cdots (a_n + 1).$$

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Let $d = a_1 + \cdots + a_n = \deg M$. The rank of M is maximized when $a_1 = 1$ and the remaining exponents $a_2, \ldots, a_n$ are as close as possible to being equal. Explicitly, write $d - 1 = q(n-1) + s$ with $0 \leq s < n-1$. Then the maximum rank monomial in n variables of degree d is

$$x_1 x_2^q \cdots x_{n-s}^q x_{n-s+1}^{q+1} \cdots x_n^{q+1}.$$

This has rank approximately $(1 + \frac{d-1}{n-1})^{n-1}$. Asymptotically in d the maximum rank of monomials is $d^{n-1}/(n-1)^{n-1}$, while asymptotically the generic rank is $d^{n-1}/n!$. For $n = 3$, therefore, the maximum rank of monomials is asymptotically $3/2$ the generic rank. In particular there are infinitely many monomials in three variables with greater than generic rank, and in fact it easy to see that they occur in every degree $d \geq 4$. On the other hand for $n \geq 4$ we have $(n-1)^{n-1} > n!$, so for $d \gg 0$, monomials have less than generic rank. This shows that for each $n \geq 4$, there are finitely many monomials with higher than generic rank. All of this was observed in [CCG12].

We show here that in fact, in four or more variables there are *absolutely no* monomials with higher than generic rank. Then we compare the maximum ranks of monomials in three variables with Jelisiejew's upper bound. Next we briefly discuss a non-monomial example suggested by Jarek Buczyński. Finally we consider sums of pairwise coprime monomials, where we conjecture that all such sums have less than generic rank, with exactly three exceptions.

1. Ranks of monomials in four or more variables

All monomials in four or more variables have less than generic rank.

Theorem 1. *Let M be a monomial in $n \geq 4$ variables and let $d = \deg M > 1$. Then $r(M) < r_{gen}(n, d)$.*

That is, the only monomials with greater than generic rank are in three or fewer variables.

We do not assume that M actually involves every variable.

Proof. We assume $n \geq 4$, so we ignore the exceptional case $(n, d) = (3, 4)$. First we dispose of the remaining exceptional cases. If $d = 2$ then $M = x_1^2$ or $x_1 x_2$, so $r(M) = 1$ or 2, while $r_{\text{gen}}(n, 2) = n \geq 4 > r(M)$. The cases $(n, d) = (4, 4), (5, 3)$ are listed in Table 1. For $(n, d) = (5, 4)$, the same monomials appear as in the case $(n, d) = (4, 4)$, and $r_{\text{gen}}(5, 4) = 15$. This takes care of all the exceptional cases.

n	d	M	$r(M)$	generic rank
4	4	$x_1x_2x_3x_4$	8	10
		$x_1x_2x_3^2$	6	10
		$x_1x_2^3$	4	10
		$x_1^2x_2^2$	3	10
		x_1^4	1	10
5	3	$x_1x_2x_3$	4	8
		$x_1x_2^2$	3	8
		x_1^3	1	8

Table 1. Exceptional cases $(4, 4)$ and $(5, 3)$.

Now we consider the nonexceptional case. Say $k \leq n$ of the variables appear in $M = x_1^{a_1} \cdots x_k^{a_k}$, $0 < a_1 < \cdots < a_k$. Write $M = x_1^{a_1} \cdots x_n^{a_n}$, $a_{k+1} = \cdots = a_n = 0$. By the arithmetic-geometric mean inequality,

$$r(M) = (a_2+1)\cdots(a_n+1) \leq \left(\frac{a_2 + \cdots + a_n + n - 1}{n-1}\right)^{n-1}$$
$$= \left(\frac{d+n-1-a_1}{n-1}\right)^{n-1} \leq \left(\frac{d+n-2}{n-1}\right)^{n-1}.$$

We claim

$$\left(\frac{d+n-2}{n-1}\right)^{n-1} < \frac{1}{n}\binom{n+d-1}{n-1}$$

or equivalently

$$\left(\frac{d+n-2}{d+n-1}\right)\cdots\left(\frac{d+n-2}{d+1}\right) < \frac{(n-1)^{n-1}}{n!} = \left(\frac{n-1}{n}\right)\cdots\left(\frac{n-1}{2}\right). \tag{1}$$

First, if $m, c > 0$ then $(c+m)^2(m^2-1) < m^2((c+m)^2-1)$, so

$$\frac{(c+m)^2}{(c+m+1)(c+m-1)} < \frac{m^2}{(m+1)(m-1)}.$$

Substituting $c = d-1$ and $m = n-1$,

$$\left(\frac{d+n-2}{d+n-1}\right)\left(\frac{d+n-2}{d+n-3}\right) < \left(\frac{n-1}{n}\right)\left(\frac{n-1}{n-2}\right).$$

This takes care of the first three factors on each side in (1). (Here we use the hypothesis $n \geq 4$; otherwise $\frac{d+n-2}{d+n-3}$ and $\frac{n-1}{n-2}$ are absent.) For the remaining factors,

$$\frac{(d-1)+(n-1)}{(d-1)+a} < \frac{n-1}{a}$$

for $2 \leq a < n-1$. This proves (1) and completes the proof. □

2. Comparison with upper bound for rank

The maximum rank degree d monomial in three variables is $xy^{(d-1)/2}z^{(d-1)/2}$ if d is odd or $xy^{(d-2)/2}z^{d/2}$ if d is even, with rank either $\left(\frac{d+1}{2}\right)^2$ or $\frac{d(d+2)}{2}$. The generic rank is

$$r_{\text{gen}}(3,d) = \left\lceil \frac{1}{3}\binom{d+2}{2} \right\rceil = \left\lceil \frac{(d+2)(d+1)}{6} \right\rceil$$

so, as noted in [CCG12], the maximum rank of monomials in three variables is asymptotically 3/2 the generic rank. Meanwhile, Jelisiejew's upper bound for rank of forms of degree d in 3 variables is

$$\binom{d+1}{2} - \binom{d-3}{0} = \binom{d+1}{2} - 1.$$

So, asymptotically the maximum rank of monomials in three variables is 1/2 of this upper bound.

3. A non-monomial example

We break briefly from the monomial case to discuss an example of a form in $n = 4$ variables of degree $d = 3$ with higher than generic rank.

Proposition 2. *Let $F = x^2y + y^2z$ be the plane cubic of rank 5. Let $G = F + w^3$. Then $r(G) = 6 > r_{gen}(4, 3) = 5$.*

Proof. Clearly $r(G) \le 6$. The Waring rank of F is its rank as a symmetric tensor. This is equal to the rank of F as a (not necessarily symmetric) tensor by [BL11, Cor. 1.10]. Furthermore the tensor rank of G is greater than or equal to the tensor rank of F plus 1, by [BL13], Proposition 5.5(i) with $p_3 = 0$. Finally the Waring rank of G is at least as great as its tensor rank. This shows $r(G) \ge 6$. □

One expects this to be the first member of a family of such examples.

We thank Jarek Buczyński for suggesting the example and proof.

4. Sums of pairwise coprime monomials

If $M_1, \dots, M_t$ are pairwise coprime monomials, that is, involving pairwise disjoint sets of variables, then $r(M_1 + \cdots + M_t) = \sum r(M_i)$ [CCG12].

Example 3. The form $F = x_1x_2^2 + x_3x_4^2$, with $n = 4$, $d = 3$, has higher than generic rank:

$$r(x_1x_2^2 + x_3x_4^2) = 6 > r_{\text{gen}}(4, 3) = 5. \tag{2}$$

Conjecture 4. *Every sum of pairwise coprime monomials in $n \ge 4$ variables, of degree $d \ge 3$, has rank strictly less than the generic rank, with the following list of exceptions in $(n, d) = (4, 3)$: $x_1x_2^2 + x_3x_4^2$ has rank 6, strictly greater than $r_{gen}(4, 3) = 5$; $x_1x_2x_3 + x_4^3$ and $x_1x_2^2 + x_3^3 + x_4^3$ have rank 5, equal to the generic rank.*

In the remainder of this section we gather some evidence for this conjecture.

We fix notation: $F = M_1 + \cdots + M_t$, where M_i involves exactly s_i variables, $n = \sum s_i$, $s_1 \ge \cdots \ge s_t \ge 1$.

First, we consider some easy extreme cases. Theorem 1 takes care of the case $t = 1$, i.e., a single monomial. At the other extreme, $t = n$, $s_1 = \cdots = s_n = 1$, F is a Fermat polynomial with rank n. It is easy to check $r_{\text{gen}}(n, d) > n$ when $d \ge 3$: perhaps most simply, $r_{\text{gen}}(n, d) > r_{\text{gen}}(n, 2) = n$.

Second, we consider the situation asymptotically. Fix $s_1, \dots, s_t$ and let $d \to \infty$. For each d take F to maximize rank, subject to the choice of s_i and d. Since M_i is a monomial of degree d in s_i variables, $r(M_i) \le \left(1+\frac{d-1}{s_i-1}\right)^{s_i-1}$. So as $d \to \infty$, $r(F)$ is asymptotically $(d/(s_1-1))^{s_1-1}$ (times the number of s_i equal to s_1), while $r_{\text{gen}}(n, d)$ is asymptotically $d^{n-1}/n!$. If $s_1 < n$ then $r(F) < r_{\text{gen}}(n, d)$ for $d \gg 0$.

Third, we consider F of the following special form.

Proposition 5. *Define F as follows. If $n = 2k$ is even,*

$$F = x_1x_2^{d-1} + \cdots + x_{2k-1}x_{2k}^{d-1}.$$

If $n = 2k + 1$ is odd,

$$F = x_1x_2^{d-1} + \cdots + x_{2k-1}x_{2k}^{d-1} + x_{2k+1}^d.$$

Suppose $n \ge 4$ and $d \ge 3$, and $(n, d) \ne (4, 3)$. Then $r(F) < r_{gen}(n, d)$.

Proof. We show

$$\left\lfloor \frac{n}{2} \right\rfloor d + 1 \leq \left\lceil \frac{1}{n}\binom{d+n-1}{n-1} \right\rceil,$$

the left hand side being either $r(F)$ or $r(F)+1$, and the right hand side being either $r_{\text{gen}}(n,d)$ or $r_{\text{gen}}(n,d)-1$ in the exceptional cases. If $n \geq 4$ and $d \geq 3$ then

$$\begin{aligned}
\frac{1}{n}\binom{d+n-1}{n-1} &\geq \frac{1}{n}\binom{d+n-1}{3} \\
&= \frac{(n+d-1)(n+d-2)(n+d-3)}{6n} \\
&= \frac{n^3+3n^2(d-2)+n(3d^2-12d+11)+(d-1)(d-2)(d-3)}{6n} \\
&\geq \frac{n^2+3n(d-2)+(3d^2-12d+11)}{6} \\
&= \frac{nd}{2} + \frac{n(n-6)+3(d-2)^2-1}{6}.
\end{aligned}$$

We have $n(n-6)+3(d-2)^2 \geq 7$ if $n \geq 7$, or if $n = 6$ and $d \geq 4$, or if $n = 5$ and $d \geq 4$, or if $n = 4$ and $d \geq 5$.

If $n > 4$ and $d > 3$ then $\binom{n+d-1}{n-1} > \binom{n+d-1}{3}$. If $n = 4$ and $d > 4$ then $(d-1)(d-2)(d-3) > 0$, so the second inequality (fourth line) in the above string of equations is strict. If $n > 6$ and $d = 3$ then $n(n-6)+3(d-2)^2 > 7$. In all of these cases the strict inequality means that F has strictly less than generic rank.

The only remaining cases are $(n,d) = (6,3), (5,3), (4,4)$, in which one easily checks by hand that F has strictly less than generic rank. Explicitly:

(n,d)	$r(F)$	$\lfloor \frac{n}{2} \rfloor + 1$	$\lceil \frac{1}{n}\binom{d+n-1}{n-1} \rceil$	$r_{\text{gen}}(n,d)$
$(6,3)$	9	10	10	10
$(5,3)$	7	7	7	8
$(4,4)$	8	9	9	10

This completes the proof. □

Fourth, we consider another special form, as follows.

Proposition 6. *Suppose $n = kd$, $d \geq 3$, and define*

$$F = x_1 \cdots x_d + \cdots + x_{n-d+1} \cdots x_n.$$

Then $r(F) = 2^{d-1}k < r_{gen}(n,d)$, except if $d = 3$, $k = 1$.

Proof. If $k = 1$ then F is a monomial and we are done.

If $k = 2$ then $r(F) = 2^d$ and

$$r_{\text{gen}}(2d,d) \geq \frac{1}{2d}\binom{3d-1}{d} > \frac{(2d)^{d-1}}{d!}$$

and $d^{d-1}/d! > 2$ for $d \geq 4$. If $k = 2$ and $d = 3$, $r(F) = 8$ and $r_{\text{gen}}(6,3) = 10$.

Now assume $k \geq 3$. Then

$$r_{\text{gen}}(n,d) \geq \frac{1}{n}\binom{n+d-1}{d} = \frac{1}{kd}\binom{kd+d-1}{d} > \frac{(kd)^{d-1}}{d!} > k^{d-1}$$

and we claim $k^{d-1} \geq 2^{d-1}k$ for $d \geq 4$. This holds if and only if $d \geq 1 + \frac{\log k}{\log k - \log 2}$. This is a decreasing function and $d \geq \log 3/(\log 3 - \log 2) = 2.7095\ldots$, giving the claim.

Finally if $k \geq 3$ and $d = 3$ then $r(F) = 4k$ while

$$r_{\text{gen}}(3k, 3) \geq \frac{1}{3k}\binom{3k+2}{3} = \frac{(3k+2)(3k+1)}{6}$$

and

$$9k^2 + 5k - 4 = (k+1)(9k-4) > 0$$

for $k > 4/9$, showing $(3k+2)(3k+1)/6 > 4k$. □

Fifth, finally, and perhaps most convincingly, it is easy to run a computer program to check, for a given n and d, for all partitions $s_1 + \cdots + s_t = n$, if the maximum rank of F is less than the generic rank. Checking this for, say, $4 \leq n \leq 20$ and $3 \leq d \leq 20$, is very quick (under 2 seconds). See the Appendix for a simple version of such a program.

Appendix

We list some simple `Macaulay2` code to check Conjecture 4.

```
rgen = (n,d) -> (
  if ( n == 1 or d == 1 ) then return 1;
  if ( d == 2 ) then return n;
  if ( {n,d} == {3,4} ) then return 6;
  if ( {n,d} == {5,3} ) then return 8;
  if ( {n,d} == {4,4} ) then return 10;
  if ( {n,d} == {5,4} ) then return 15;
  return ceiling((1/n)*binomial(n+d-1,d));
);

upperRank = (n,d) -> ( -- crude upper bound
  if ( n == 1 ) then return 1;
  if ( d == 1 ) then return 1;
  return (1+(d-1)/(n-1))^(n-1);
);

checkit = (Nmax,Dmax) -> (
  for n from 4 to Nmax do (
    for p in partitions(n) do (
      pparts := toList(p);
      for d from 3 to Dmax do (
        upperRankTotal := sum(pparts, s -> upperRank(s,d));
        if(upperRankTotal >= rgen(n,d)) then (
          << "----------" << endl;
          << "n: " << n << endl;
          << "d: " << d << endl;
          << "p: " << toString(pparts) << endl;
        );
      );
```

```
    );
  );
);

time checkit(20,20); -- 1.46196 seconds on my laptop
```

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Department of Mathematics, 1910 University Drive, Boise State University, Boise, ID 83725-1555, USA

E-mail address: `zteitler@boisestate.edu`