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Note

Sum of degrees of irreducible factors for polynomials in several variables

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ABSTRACT

New lower bounds are given for the sum of degrees of simple and distinct irreducible factors of the polynomial $f_1+\cdots+f_n$, where f_i $(1\leqslant i\leqslant n)$ are pairwise relatively prime polynomials of several variables with coefficients in $\mathbb{C}$.

1. Introduction

Let $\mathbb{C}[X]$ be the ring of polynomials in the variable X over the complex number $\mathbb{C}$. For any $f\in\mathbb{C}[X]$, the number of simple and distinct zeros of f, denoted by $n_1(f)$ and $n_0(f)$, respectively, play important roles in mathematical analysis and number theory. In 1986 Brindza [3] derived some lower bounds for the quantities $n_1(f)$ and $n_0(f)$, where f is a special polynomial, see Theorems 3.1 and 3.2. Later, Pintér [5] gave lower bounds for $n_1(f+g)$ and $n_0(f+g)$, where f and g are relatively prime polynomials in $\mathbb{C}[X]$. His result is as follows.

Theorem 1.1 *(Pintér). If f and g are nonconstant relatively prime polynomials in $\mathbb{C}[X]$ with $\deg f>\deg g$, then we have*

$$n_0(f+g)\geqslant\max\{1,\deg(f+g)-n_0(fg)+1\},\tag{1}$$

and

$$n_1(f+g)\geqslant\max\{0,\deg(f+g)-2n_0(fg)+2\}.\tag{2}$$

The main aim here is to generalize the above result to arbitrarily many polynomials of several variables with coefficients in $\mathbb{C}$. In Section 2, we will state and prove the main result of this note. In Section 3, we will use our result to extend Theorems 2 and 3 in [3].

2. The main result

Let $\mathbb{C}[X_1,\dots,X_l]$ be the ring of polynomials in several variables with complex coefficients. Since $\mathbb{C}[X_1,\dots,X_l]$ is a unique factorization domain, for any $f\in\mathbb{C}[X_1,\dots,X_l]$, we have

$$f=\prod_{i=1}^{s}p_i^{\alpha_i},$$

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where p_i's are irreducible, distinct, at most one of them is constant, and the $\alpha_i > 0$ are integers. we define

$$n_0(f) = \deg\left(\prod_{i=1}^{s} p_i\right), \qquad n_1(f) = \deg\left(\prod_{\substack{1\leqslant i\leqslant s\\ \alpha_i=1}} p_i\right).$$

It is easy to see that if $l = 1$, then $n_0(f)$ and $n_1(f)$ are the number of distinct and simple zeros of f, respectively. Now we can state our main result.

Theorem 2.1. *Let $f_1, f_2, \ldots, f_{n-1}$ $(n \geqslant 3)$ be polynomials in $\mathbb{C}[X_1, \ldots, X_l]$, and denote $f_n = f_1 + f_2 + \cdots + f_{n-1}$. Suppose f_i $(1 \leqslant i \leqslant n)$'s are pairwise relatively prime, k out of the n polynomials are constant $(k \leqslant n-2)$, and*

$$\deg f_1 \leqslant \deg f_2 \leqslant \cdots \leqslant \deg f_{n-2} < \deg f_{n-1}$$

then, for $k = 0$, we have

$$n_0(f_1 + f_2 + \cdots + f_{n-1}) \geqslant \max\left\{1, \frac{\deg(f_1 + f_2 + \cdots + f_{n-1})}{n-2} - n_0(f_1 \cdots f_{n-1}) + \frac{n-1}{2}\right\}, \tag{3}$$

$$n_1(f_1 + f_2 + \cdots + f_{n-1}) \geqslant \max\left\{0, \frac{4-n}{n-2}\deg(f_1 + f_2 + \cdots + f_{n-1}) - 2n_0(f_1 \cdots f_{n-1}) + n - 1\right\}, \tag{4}$$

and, when $k \geqslant 1$, we have,

$$n_0(f_1 + f_2 + \cdots + f_{n-1}) \geqslant \max\left\{1, \frac{\deg(f_1 + f_2 + \cdots + f_{n-1})}{n-k-1} - n_0(f_1 \cdots f_{n-1}) + \frac{n-k}{2}\right\}, \tag{5}$$

$$n_1(f_1 + f_2 + \cdots + f_{n-1}) \geqslant \max\left\{0, \frac{3-n+k}{n-k-1}\deg(f_1 + f_2 + \cdots + f_{n-1}) - 2n_0(f_1 \cdots f_{n-1}) + n - k\right\}. \tag{6}$$

Remark. When $l = 1$ and $n = 3$, Theorem 1.1 is a special case of the above theorem.

Our proof of the above theorem is based upon the following lemma which is a several variable extension of Mason's *abc* theorem.

Lemma 2.2. *(Bayat and Teimoori [2].) Let $f_1 + \cdots + f_{n-1} = f_n$, in which the f_i's are pairwise relatively prime in $\mathbb{C}[X_1, \ldots, X_l]$, and k out of the n-polynomials are constant $(k \leqslant n-2)$. Then, for $k = 0$, we have*

$$\max_{1\leqslant i\leqslant n} \deg f_i \leqslant (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}, \tag{7}$$

and when $k \geqslant 1$, we have

$$\max_{1\leqslant i\leqslant n} \deg f_i \leqslant (n-k-1)n_0(f_1 f_2 \cdots f_n) - \frac{(n-k)(n-k-1)}{2}. \tag{8}$$

Now we give a proof of Theorem 2.1.

Proof of Theorem 2.1. We only prove the case when $k = 0$. The proofs for cases $k \geqslant 1$ go in a similar way.

From (7), we have

$$\begin{aligned}\deg(f_1 + \cdots + f_{n-1}) = \max_{1\leqslant i\leqslant n} \deg f_i &\leqslant (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}\\ &= (n-2)\big[n_0(f_1 f_2 \cdots f_{n-1}) + n_0(f_1 + f_2 + \cdots + f_{n-1})\big] - \frac{(n-1)(n-2)}{2},\end{aligned}$$

which yields (1).

Furthermore, by the inequality

$$\begin{aligned}n_0(f_1 + f_2 + \cdots + f_n) &\leqslant n_1(f_1 + f_2 + \cdots + f_{n-1}) + \frac{\deg(f_1 + \cdots + f_{n-1}) - n_1(f_1 + f_2 \cdots + f_{n-1})}{2}\\ &= \frac{\deg(f_1 + \cdots + f_{n-1}) + n_1(f_1 + f_2 \cdots + f_{n-1})}{2}.\end{aligned}$$

We have

$$\deg(f_1+\cdots+f_{n-1}) \leqslant (n-2)\left[n_0(f_1f_2\cdots f_{n-1}) + \frac{\deg(f_1+\cdots+f_{n-1})+n_1(f_1+f_2+\cdots+f_{n-1})}{2}\right] - \frac{(n-1)(n-2)}{2},$$

which yields (2). □

3. An application

In 1986, using some inequalities of Mason [4] and Brownawell and Masser [1] concerning S-unit equations over function fields Brindza [3] derived the following two results. The first gives us a lower bound for the number of distinct zeros. The second gives us a lower bound for the number of simple zeros.

Theorem 3.1 *(Brindza). Let $f_1(X),\dots,f_N(X)$ be nonconstant pairwise relatively prime polynomials in $\mathbb{C}[X]$ and $a_1,\dots,a_N$ be nonzero complex numbers. Suppose that*

$$\mu = \min_{1\leqslant i\leqslant N} k_i > N(N-1).$$

Then the polynomial $Q(X) = a_1 f_1^{k_1}(X)+\cdots+a_N f_N^{k_N}(X)$ has at least $\frac{\mu}{N-1}$ distinct zeros.

Theorem 3.2 *(Brindza). Let $F(X)$ and $G(X)$ be nonconstant relatively prime polynomials in $\mathbb{C}[X]$ and a, b be nonzero complex numbers. Suppose n, m are positive integers, $\frac{1}{n}+\frac{1}{m}\leqslant\frac{1}{2}$ and $n\deg F \geqslant m\deg G$. Then the polynomial $P(X) = aF^n(X)+bG^m(X)$ has at least $2+n(1-\frac{2}{n}-\frac{2}{m})\deg F$ simple zeros.*

As an application of Theorem 2.1, we obtain the following theorem which can be viewed as a generalization of the two results above.

Theorem 3.3. *Let $f_1,\dots,f_N$ be polynomials in $\mathbb{C}[X_1,X_2,\dots,X_l]$ and $a_1,\dots,a_N$ be nonzero complex numbers. Denote $Q = a_1 f_1^{k_1}+\cdots+a_N f_N^{k_N}$. Suppose $f_1,\dots,f_N$ and Q are pairwise relatively prime, k out of the $N+1$ polynomials are constant ($k\leqslant N-1$) and*

$$k_1\deg f_1 \leqslant k_2\deg f_2 \leqslant \cdots \leqslant k_{N-1}\deg f_{N-1} < k_N\deg f_N$$

then, for $k=0$, we have

$$n_0(Q) \geqslant \max\left\{1, \frac{\deg Q}{N-1} - n_0(f_1\cdots f_N) + \frac{N}{2}\right\}, \tag{9}$$

$$n_1(Q) \geqslant \max\left\{0, \frac{3-N}{N-1}\deg(Q) - 2n_0(f_1\cdots f_N) + N\right\}, \tag{10}$$

and, when $k\geqslant 1$,we have,

$$n_0(Q) \geqslant \max\left\{1, \frac{\deg(Q)}{N-k} - n_0(f_1\cdots f_N) + \frac{N-k+1}{2}\right\}, \tag{11}$$

$$n_1(Q) \geqslant \max\left\{0, \frac{2-N+k}{N-k}\deg(Q) - 2n_0(f_1\cdots f_N) + N-k+1\right\}. \tag{12}$$

Proof. Since $Q = a_1 f_1^{k_1}+\cdots+a_N f_N^{k_N}$, $a_i f_i^{k_i}$ $(1\leqslant i\leqslant N)$ and Q are pairwise relatively prime, from Theorem 2.1, we get the results of this theorem. □

Remark. It is easy to see that if $N=2$ in Theorem 3.3, then the condition that f_1, f_2 and Q are pairwise relatively prime is equivalent to that f_1, f_2 are relatively prime.

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