“Fingerprinting” the Lemniscates Stockholm University

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2. Conformal Welding

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Introduction

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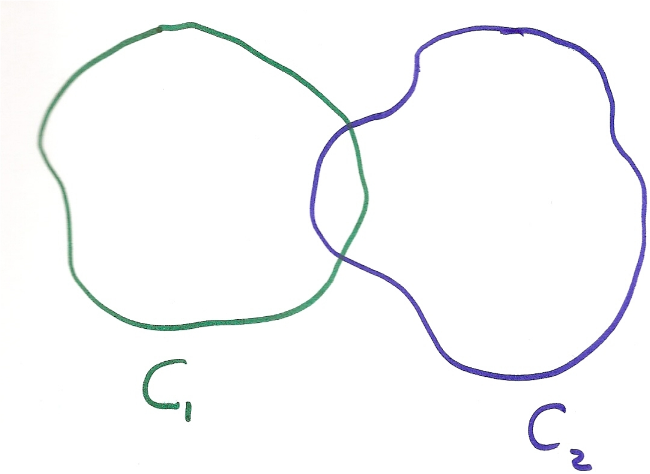

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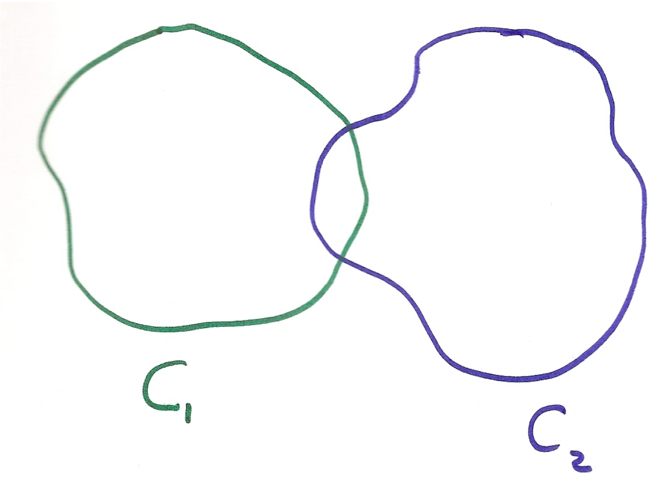

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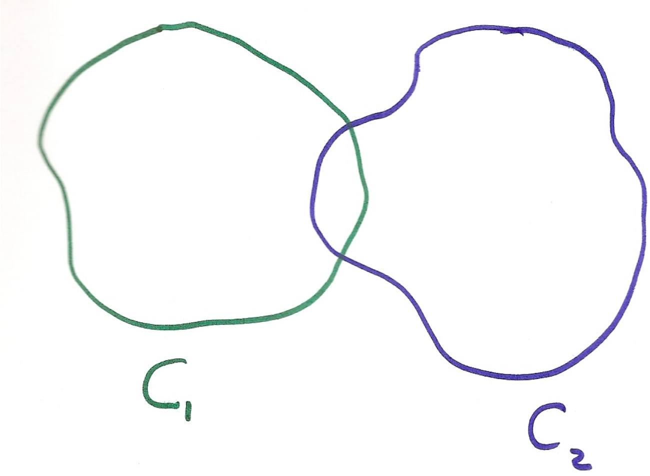

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$\rightsquigarrow$ an orientation preserving diffeo of the circle $\mathbb{T}$.

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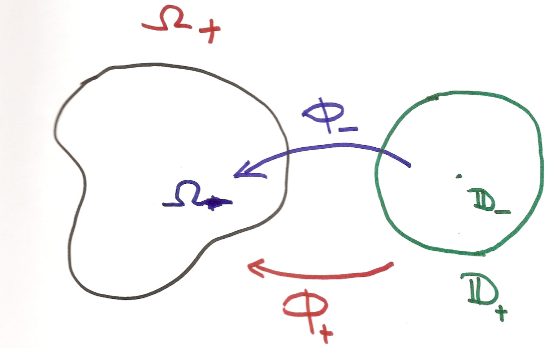

$$\Phi_+(\infty) = \infty; \quad \Phi_+'(\infty) > 0.$$

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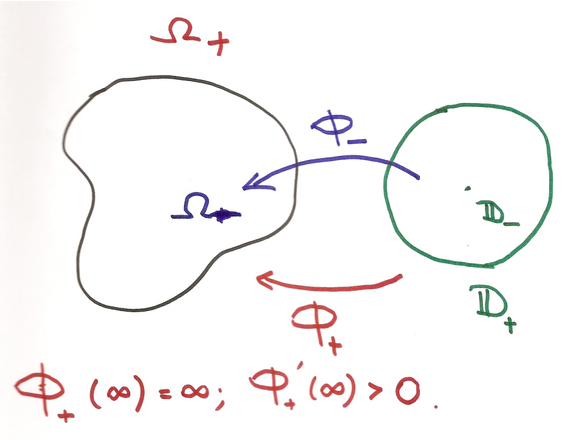

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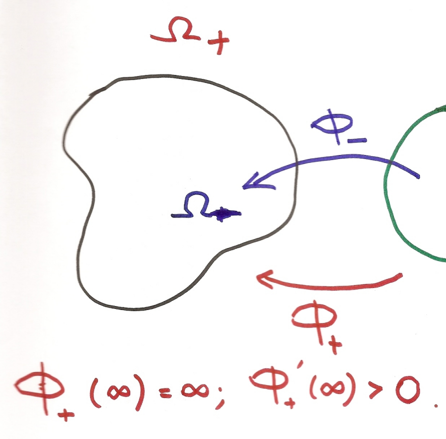

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$k = e^{i\psi}$, $\psi(\theta + 2\pi) = \psi(\theta) + 2\pi$, $\psi' > 0$.

Kirillov's Theorem

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Note: The statement is false if we replace $\mathrm{Diff}_+(\mathbb{T})$ by $\mathrm{Homeo}_+(\mathbb{T})$, ($\mathfrak{F}$ is neither 1-1 , nor onto).

D. Mumford - E. Sharon, 2004

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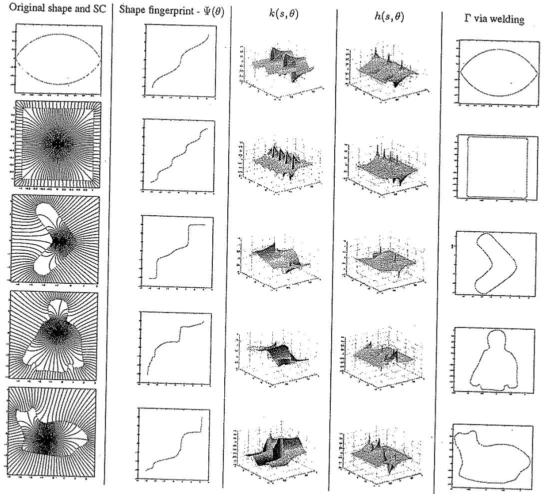
- For $\mathfrak{F}$, $\Phi_{-,+}$ are approximated by the Schwarz - Christoffel integrals.
- For $\mathfrak{F}^{-1}$, $\Phi_{-,+}$ are found via a series of renormalizations and by solving a Riemann - Hilbert type problem.

Mumford - Sharon Data, Examples



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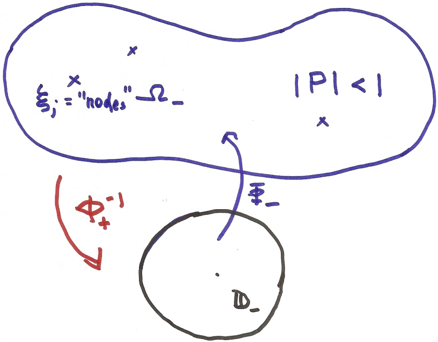

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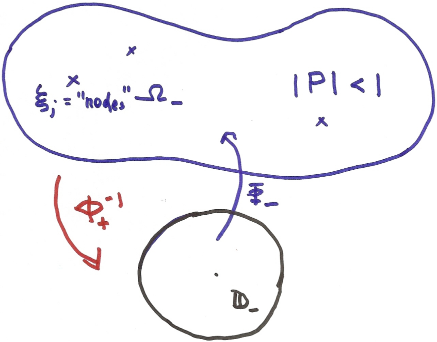

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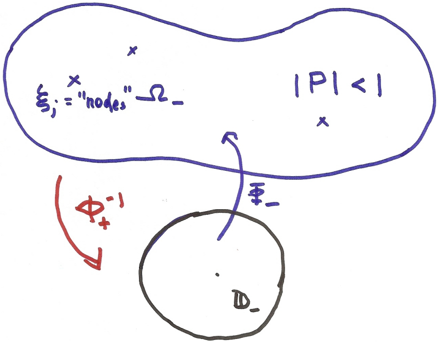
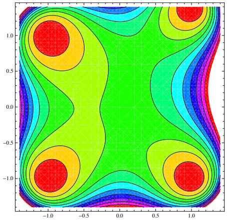


- Ω_- is connected
- All zeros ξ_j, $j = 1, \ldots, n$ and critical points of P lie inside Ω_-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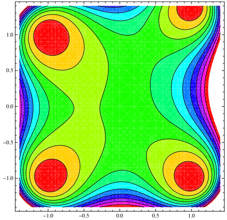
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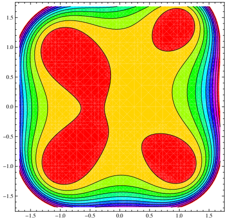
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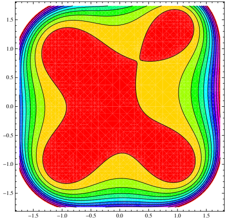
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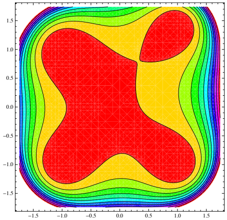
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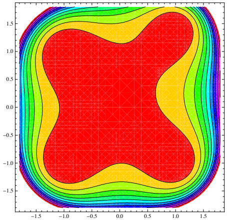
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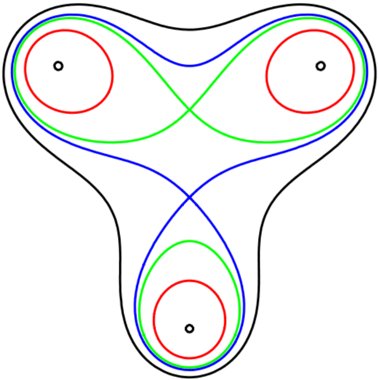
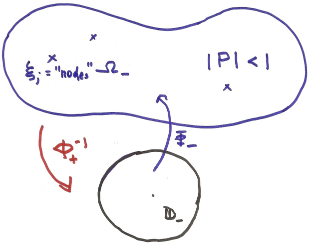


Fingerprints of Lemniscates



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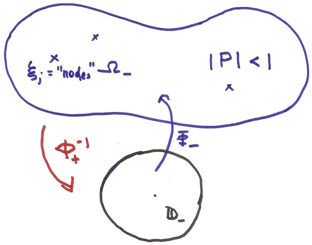

$B_1 := P \circ \Phi_- : \mathbb{D} \to \mathbb{D}$. This is n to 1 map.

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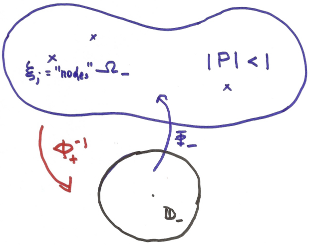

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$$
\begin{aligned}
B_1 &= e^{i\theta} \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \\
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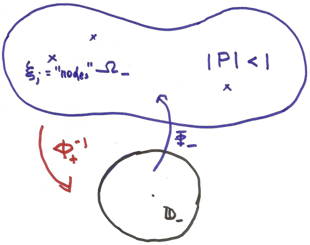

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Moreover, $\Phi_+^{-1}(w) = \sqrt[n]{P(w)}$ and

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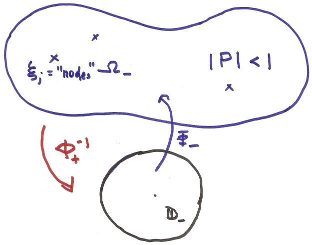

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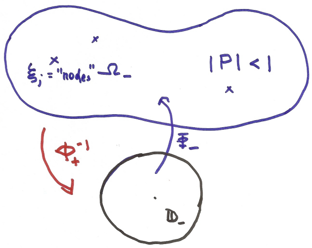

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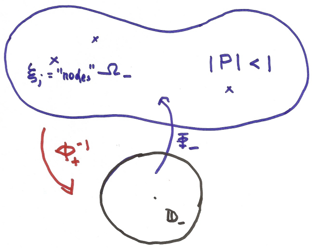

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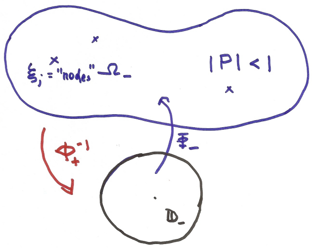

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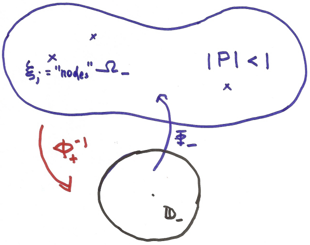

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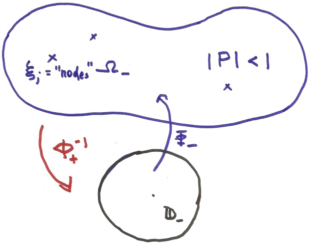

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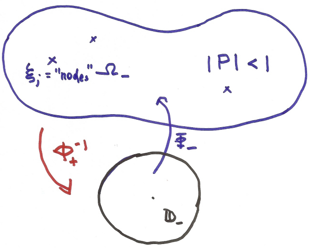

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Theorem

The fingerprint of the lemniscate $\Gamma := \partial\Omega$ equals

$$k := \mathbb{T} \to \mathbb{T},\ k = \Phi_+^{-1} \circ \Phi_- = \sqrt[n]{B_1(z)}.$$

Evolution of Bernoulli's Lemniscates

Bernoulli's Lemniscate

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$r < 1$

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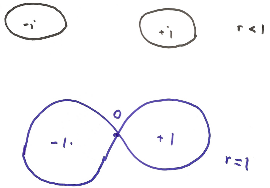

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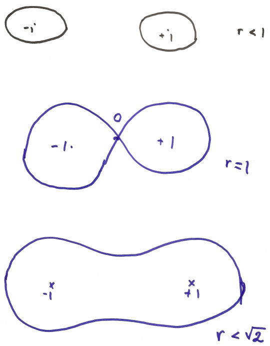

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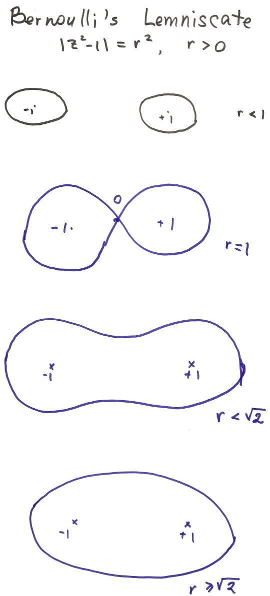
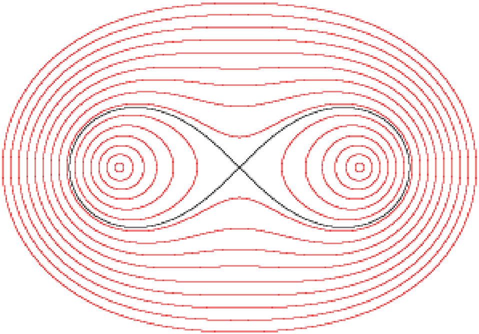
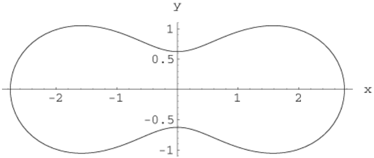




Fingerprinting Bernoulli's lemniscate



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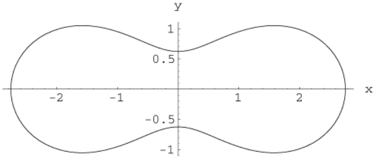

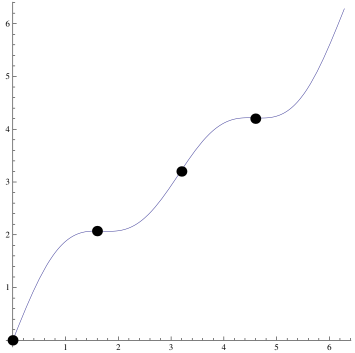

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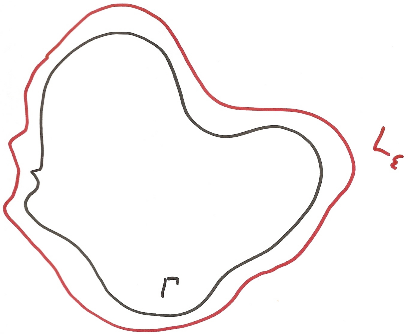

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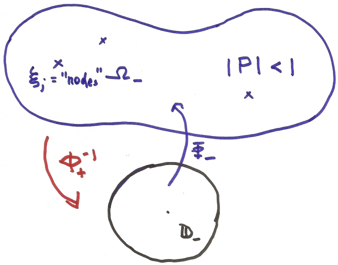

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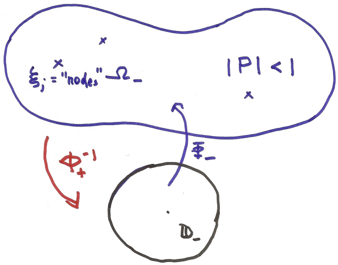

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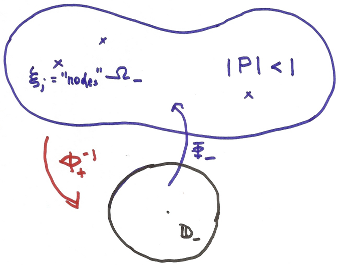

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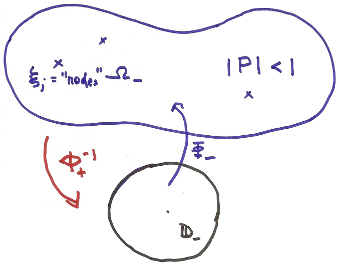

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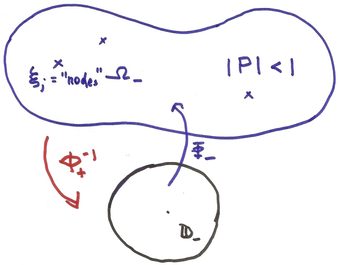

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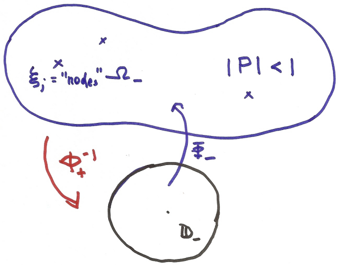
Questions: (i) Are such k dense in $\mathrm{Diff}_+(\mathbb{T})$?

Main Questions



Recall: Fingerprints k of n- lemniscates are n-th roots of Blaschke products B, i.e.

$$k \in \mathrm{Diff}_+, \;\; k : \mathbb{T} \to \mathbb{T}, \;\; k = \sqrt[n]{B(z)}.$$

Questions: (i) Are such k dense in $\mathrm{Diff}_+(\mathbb{T})$?

(ii) Does each such k "fingerprint" a polynomial lemniscate?

Results: Ebenfelt - DK - Shapiro, 2011

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Theorem (I)

Algebraic diffeomorphisms of the unit circle

$$k = \sqrt[n]{B(z)}, \; B = e^{i\theta} \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \; |a_j| < 1,$$

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A simpler proof is recently given by M. Younsi, June 2014.

Theorem I

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The key is the injectivity of $\mathfrak{F}$.

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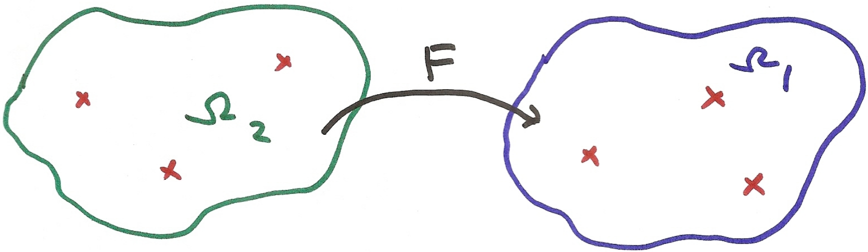

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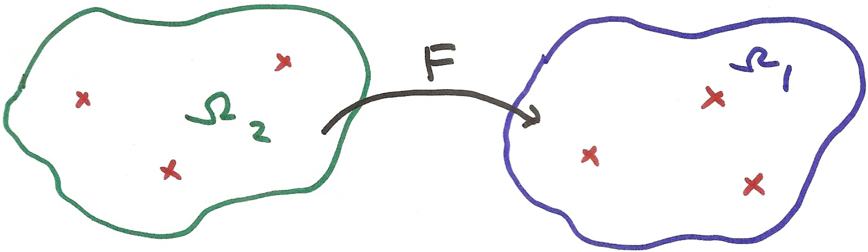

Theorem (III)

Let Ω_1, Ω_2 be (connected) n-lemniscates $\{|P| < 1\}$, $\{|Q| < 1\}$. If $F : \Omega_2 \to \Omega_1$ is a conformal mapping that maps nodes into nodes, then F is an affine mapping, i.e., $F = Aw + B$.

High Ground: Critical Values Problem

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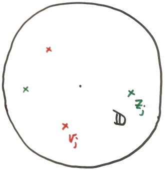

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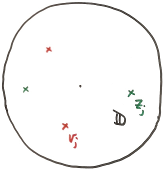

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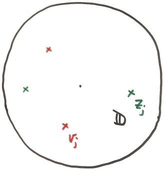

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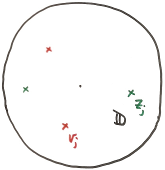

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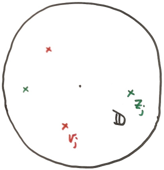

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Given the set $V := \{v_1, \ldots, v_{n-1}, |v_j| < 1\}$, let $CV_{\mathcal{B}}[V]$ denote the set of equivalence classes in $\mathcal{B}$ with the same set of critical values V.

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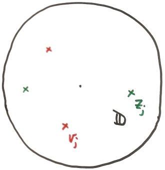

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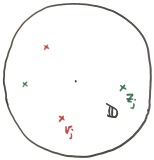

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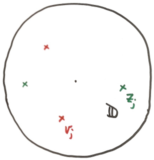

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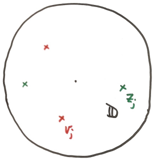

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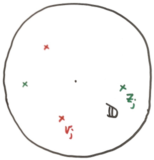

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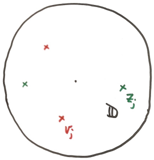

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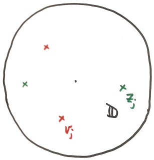

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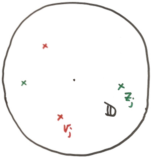

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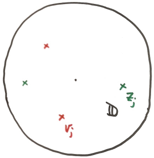

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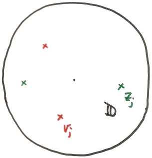

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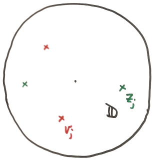

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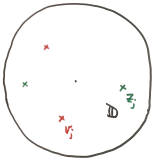

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Rational Lemniscates

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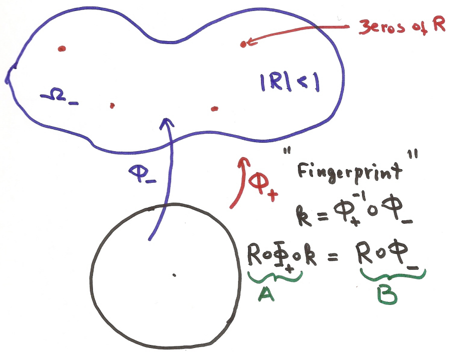

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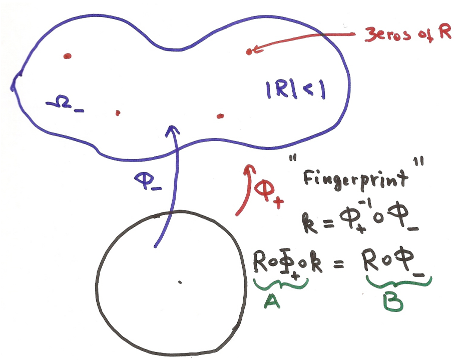

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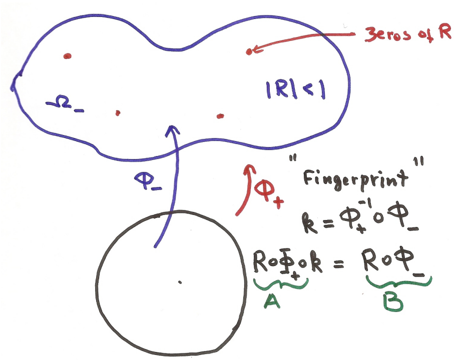

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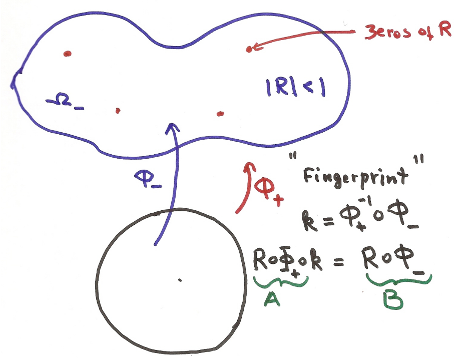

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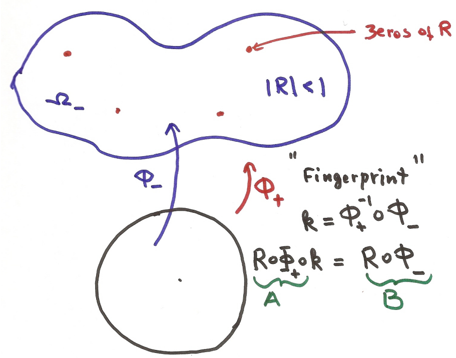

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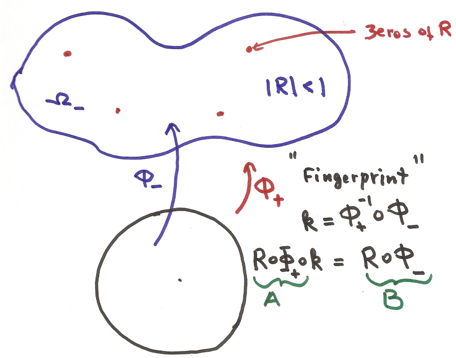
Rational Lemniscates



R is a rational function of degree n. A, B are Blaschke products of degree n. Yet, $A \neq z^n$ as for polynomial lemniscates. Thus, $k = A^{-1} \circ B$. Kirillov's theorem $\Rightarrow$ the converse,

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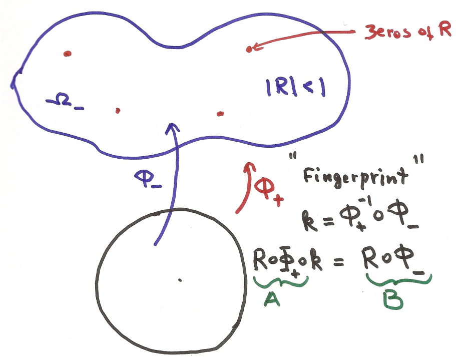

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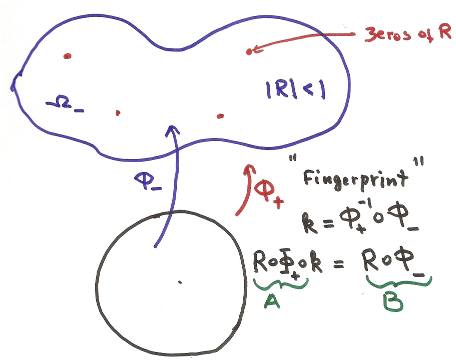

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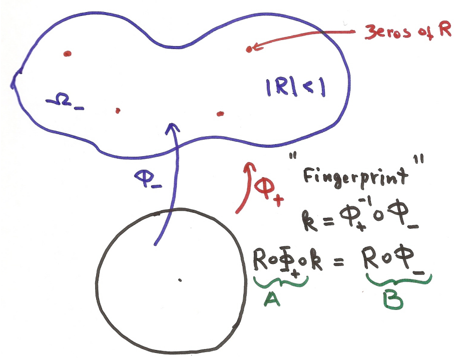

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Further Questions

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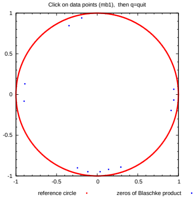

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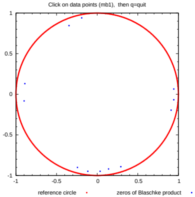

Second Blaschke product B_2

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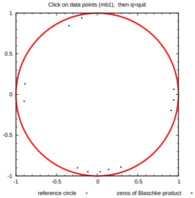

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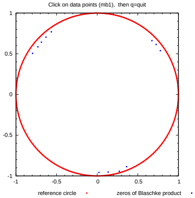

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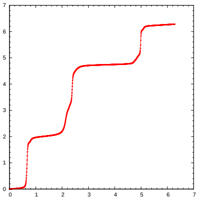

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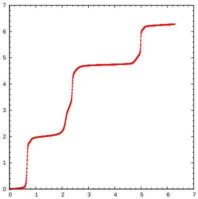

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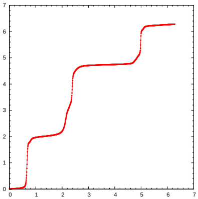

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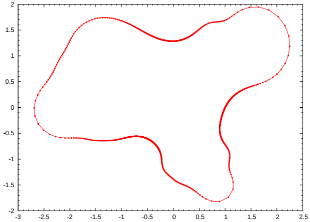

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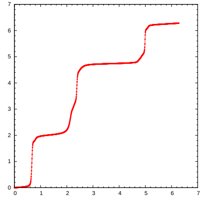

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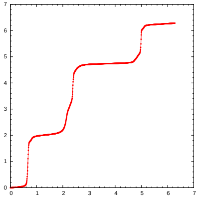

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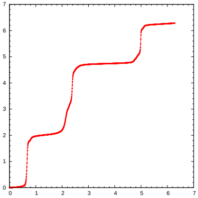

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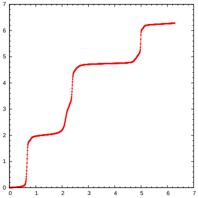

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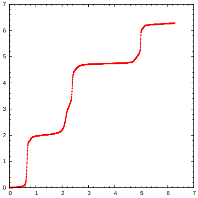

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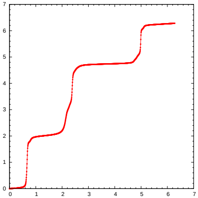

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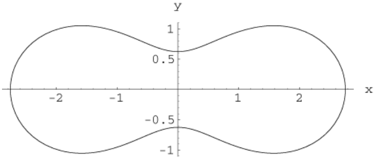
- In reality, probably a smaller number, perhaps $2n$?

Fingerprinting Bernoulli's lemniscate again



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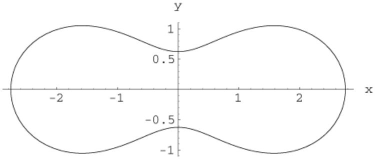

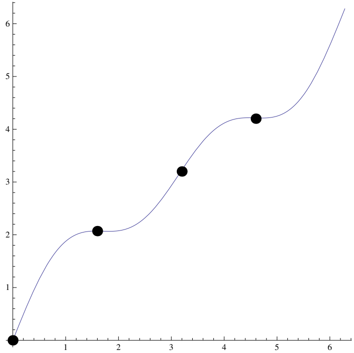

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- If the rate of convergence is $O(n^{-k})$, does this indicate the degree of smoothness of the shape?
- How does the motion of zeros in the Blaschke product fingerprint $k = \sqrt[n]{B(z)}$ reflect the changes in the shape?

THANK YOU!