

Varieties of sums of powers

Dedicated to the memory of Alf B. Aure and Michael Schneider

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Abstract. The variety of sums of powers of a homogeneous polynomial of degree d in n variables is defined and investigated in some examples, old and new. These varieties are studied via apolarity and syzygies. Classical results (cf. [Sylvester 1851], [Hilbert 1888], [Dixon, Stuart 1906]) and some more recent results of Mukai (cf. [Mukai 1992]) are presented together with new results for the cases $(n, d) = (3, 8), (4, 2), (5, 3)$. In the last case the variety of sums of 8 powers of a general cubic form is a Fano 5-fold of index 1 and degree 660.

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0. Introduction

0.1 Let $f \in \mathbf{C}[x_0, \dots, x_n]$ be a homogeneous form of degree d . f can be written as a sum of powers of linear forms

$$f = l_1^d + \dots + l_s^d$$

for s sufficiently large. Indeed, if we identify the map $l \mapsto l^d$ with the d^{th} Veronese embedding $\mathbf{P}^n \hookrightarrow \mathbf{P}^{N_d}$, where $N_d = \binom{n+d}{n} - 1$, this amounts to say that the image spans \mathbf{P}^{N_d} . What is the minimal possible value of s ?

A dimension count shows that

$$s \geq \lceil \frac{1}{n+1} \binom{n+d}{n} \rceil$$

for a **general** f . With a few exceptions for (d, n) equality holds by a result of Alexander and Hirschowitz [-1995] and Terracini's Lemma [-1911]:

0.2 Theorem. (Alexander-Hirschowitz) *A general form f of degree d in $n+1$ variables is a sum of $\lceil \frac{1}{n+1} \binom{n+d}{n} \rceil$ powers of linear forms, unless*

$d = 2$, where $s = n+1$ instead of $\lceil \frac{n+2}{2} \rceil$, or

$d = 4$ and $n = 2, 3, 4$, where $s = 6, 10, 15$ instead of $5, 9, 14$ respectively, or

$d = 3$ and $n = 4$, where $s = 8$ instead of 7 .

The exceptions were classically known cf. [Clebsch 1861], [Reye 1874], [Sylvester 1886], [Richmond 1902], [Palatini 1903], [Dixon 1906]. See also 1.4.

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0.3 For **special** forms f the minimal value s can both be smaller or bigger (cf. 1.6). As far as we know it is an open problem to determine the function $\tilde{s} = \tilde{s}(d, n)$ such that **every** form of degree d in $n+1$ variables can be written as a sum of less than $\tilde{s} + 1$ powers.

0.4 Given the answer to the first question, one might ask whether the presentation as a sum of powers is unique. Of course a **general** form can have a unique expression only if $\frac{1}{n+1} \binom{n+d}{n}$ is an integer. The following is known:

A general form f of degree d in $n + 1$ variables has a unique presentation as a sum of $s = \frac{1}{n+1} \binom{n+d}{n}$ powers of linear forms in the following cases:

$n = 1, d = 2k - 1$ and $s = k$, [Sylvester 1851], or

$n = 3, d = 3$ and $s = 5$ Sylvester's Pentahedral Theorem [Sylvester 1851] [Reye 1873], or

$n = 2, d = 5$ and $s = 7$ [Hilbert 1888], [Richmond 1902], [Palatini 1903].

Question: Are there further examples?

Iarrobino and Kanev treat this question for polynomials with fewer summands (cf. [Iarrobino, Kanev 1996]).

0.5 We put this question in a somewhat more general frame work: Let $F = V(f) \subset \mathbf{P}^n$ be a hypersurface of degree d . If the equation $f = l_1^d + \dots + l_s^d$ is a sum of s powers of linear forms, then the linear forms give hyperplanes in $L_i = V(l_i) \subset \mathbf{P}^n$, hence points $L_i \in \check{\mathbf{P}}^n$ in the dual space. The union of these points is a point in the Hilbert scheme $Hilb_s(\check{\mathbf{P}}^n)$. We define the variety of sums of powers presenting F as the closure

$$VSP(F, s) = \overline{\{ \{L_1, \dots, L_s\} \in Hilb_s(\check{\mathbf{P}}^n) \mid \exists \lambda_i \in \mathbf{C} : f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d \}}$$

of power sums presenting f in the Hilbert scheme. Note that taking d^{th} roots of the λ_i one could put them into the equations l_i .

We study the set of power sum presentations as subsets of natural Grassmannians. There may be several Grassmannians to choose from, and it is not clear to us whether the compactification in these Grassmannians always coincide with the Hilbert scheme compactification for a given polynomial. In the cases treated here they do. The natural Grassmannians where we embed $VSP(F, s)$ are presented in section 2.

Both for special and for general hypersurfaces $F = V(f)$ the most basic questions about the schemes $VSP(F, s)$ are:

Questions: What is the degree $\deg VSP(F, s)$ in case $\dim VSP(F, s) = 0$?

Is $VSP(F, s)$ irreducible and smooth, in case $\dim VSP(F, s) > 0$?

The schemes $VSP(F, s)$ are contravariants of the hypersurfaces $F \subset \mathbf{P}^n$ under the action of $PGL(n+1)$. They attracted an enormous amount of work during the last decades of the 19th century, most notably in the work started by [Sylvester 1851], [Rosanes 1873], [Reye 1874] and [Scorza 1899]. Nevertheless little is known on e.g. the degrees resp. the global structure of these schemes.

Our interest in these schemes arose from work of Iarrobino [- 1994] and work of Mukai, who proved:

0.6 Theorem [Mukai 1992]. *Let $F \subset \mathbf{P}^2$ be a general plane quartic. Then $VSP(F, 6)$ is a smooth Fano 3-fold V_{22} , i. e. of index 1 and genus 12 with anti-canonical embedding of degree 22. Moreover, every V_{22} arises this way.*

The fact that plane quartics F are exceptions of Theorem 0.2 lead to interesting geometric properties of a $V_{22} \cong VSP(F, 6)$: There are 6 conics passing through a general point of a V_{22} , each corresponding to one of the 6 summands of the power sum. This was observed classically, e.g. by [Rosanes 1873] and [Scorza 1899]. However the Fano property was not known classically. Fano overlooked the existence of the V_{22} 's in his famous paper [Fano 1937].

0.7 For the other exceptional cases in Theorem 0.2 there are interesting subvarieties of $VSP(F, s)$ by similar arguments. In case of general quartics $F \subset \mathbf{P}^3$ or \mathbf{P}^4 or general cubics $F \subset \mathbf{P}^4$ the varieties $VSP(F, s)$ with $s = 10, 15$ or 8 respectively, are all 5-folds. The main emphasis of this paper lies on the study of $VSP(F, 8)$ for cubics in \mathbf{P}^4 , perhaps the easiest case among those three. Our main result is this:

0.8 Theorem. *Let $F \subset \mathbf{P}^4$ be a general cubic. Then $VSP(F, 8)$ is a smooth Fano 5-fold of index 1 and degree 660. More precisely: Let $\mathbf{S}^{10} \subset \mathbf{P}^{15}$ denote the spinor variety of isotropic \mathbf{P}^4 's in the 8-dimensional smooth quadric $Q^8 \subset \mathbf{P}^9$. There exists a linear subspace $\mathbf{P}^{10} \subset \mathbf{P}^{15}$, which depends on F , such that $VSP(F, 8)$ is isomorphic to the variety of lines in the 5-fold $Y = Y(F) := \mathbf{P}^{10} \cap \mathbf{S}^{10} \subset \mathbf{P}^{15}$.*

On first sight the occurrence of the spinor variety might look like a surprise. However it is less so in view of apolarity and the following result of Mukai:

0.9 Theorem [Mukai 1995]. *Let $C \subset \mathbf{P}^6$ be a smooth canonical curve of genus 7 and Clifford index 3, (e.g. a general canonical curve of genus 7). Then $C \cong \mathbf{P}^6 \cap \mathbf{S}^{10}$ for a suitable linear subspace $\mathbf{P}^6 \subset \mathbf{P}^{15}$.*

Our second result generalizes Mukai's theorem to the case of "the general empty Gorenstein subscheme of degree 12 in \mathbf{P}^{4n} ": Let R_S denote the homogeneous coordinate ring of $\mathbf{S}^{10} \subset \mathbf{P}^{15}$. Let $A = \mathbf{C}[x_0, \dots, x_4]/I$ be a graded Artinian Gorenstein ring with Hilbert function $(1, 5, 5, 1)$. Then we have:

0.10 Theorem. *Let R_S and A be as above. If A is general, then there exist a regular sequence of linear forms $h_0, \dots, h_{10} \in R_S$ such that $A \cong R_S/(h_0, \dots, h_{10})$.*

The restriction general here means that the result hold for the general cubic dual polynomial. The connection to Mukai's theorem is the following: Let R_C be the homogeneous coordinate ring of a smooth canonical curve $C \subset \mathbf{P}^6$. Then for a regular sequence $h_0, h_1 \in R_C$ of linear forms the Artinian ring $\hat{A} = R_C/(h_0, h_1)$ has Hilbert function $(1, 5, 5, 1)$. By [Schreyer 1986] R_C hence \hat{A} is syzygy general, iff C has Clifford index 3.

The connection with Theorem 0.8 is projective duality: The quadric $Q^8 \subset \mathbf{P}^9$ has 2 families of isotropic \mathbf{P}^4 's, hence two (isomorphic) spinor varieties $\mathbf{S}_{ev}^{10} \subset \mathbf{P}_{ev}^{15}$ and $\mathbf{S}_{odd}^{10} \subset \mathbf{P}_{odd}^{15}$, where $\mathbf{P}_{ev}^{15} = \check{\mathbf{P}}_{odd}^{15}$ naturally. With this identification \mathbf{S}_{ev}^{10} is the dual variety $\check{\mathbf{S}}_{odd}^{10}$, cf. [Ein 1986]. So, if A^F denotes the apolar Artinian Gorenstein ring of a general cubic $F \subset \mathbf{P}^4$, cf. section 1, then A^F has Hilbert function $(1, 5, 5, 1)$ and is syzygy general. If $\mathbf{P}^4 \subset \mathbf{P}_{ev}^{15}$ denotes the linear space defined by $h_0 = \dots = h_{10} = 0$, where $A^F \cong R_{\mathbf{S}_{ev}^{10}}/(h_0, \dots, h_{10})$, then $\mathbf{P}^{10} = \mathbf{P}^{4\perp} \subset \mathbf{P}_{odd}^{15}$ is the linear space such that $Y(F) \cong \mathbf{P}^{10} \cap \mathbf{S}_{odd}^{10}$.

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0.12 Notation. \mathbf{C} denotes an algebraically closed field of characteristic 0. However with minor modifications all results in this paper hold for arbitrary fields of characteristic zero or fields of sufficiently large characteristic p , e.g. $char(k) > d$ is necessary even to define the variety of power sums. All computer experiments in [MACAULAY], which lead us to discover our results, were done over a finite field. $\mathbf{G}(d, n)$ denotes the Grassmannian of d -dimensional subspaces of k^n , while $\mathbf{G}(n, d)$ denotes the Grassmannian of d -dimensional quotient spaces of k^n .

We give the numerical information of the minimal free resolution of a graded $S = \mathbf{C}[x_0, \dots, x_r]$ -module

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_n \leftarrow 0$$

with $F_i = \bigoplus_{j \in \mathbf{Z}} \beta_{ij} S(-j)$ in MACAULAY notation, i. e. in the form

$$\begin{array}{ccccccc} \beta_{00} & \beta_{11} & \beta_{22} & \dots & \beta_{n,n} \\ \beta_{01} & \beta_{12} & \beta_{23} & \dots & \beta_{n,n+1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \beta_{0m} & \beta_{1,m+1} & \beta_{2,m+2} & \dots & \beta_{n,n+m}. \end{array}$$

Note that the integer m is the Castelnuovo-Mumford regularity of M . We indicate β_{ij} 's which are zero by a $-$. For example the syzygies of the twisted cubic in \mathbf{P}^3 look like:

$$\begin{array}{ccc} 1 & - & - \\ - & 3 & 2 \end{array},$$

there are 3 quadric generators of the ideal which have 2 linear syzygies.

1. Apolarity and syzygies

1.1 Consider $S = \mathbf{C}[x_0, \dots, x_n]$ and $T = \mathbf{C}[\partial_0, \dots, \partial_n]$. T acts on S by differentiation:

$$\partial^\alpha(x^\beta) = \alpha! \binom{\beta}{\alpha} x^{\beta-\alpha}$$

if $\beta \geq \alpha$ and 0 otherwise. Here α and β are multi-indices, $\binom{\beta}{\alpha} = \prod \binom{\beta_i}{\alpha_i}$ and so on. In particular we have a perfect pairing between forms of degree d and homogeneous differential operators of order d . Note that the polar of a form $f \in S$ in a point $a \in \mathbf{P}^n$ is given by $P_a(f)$ for $a = (a_0, \dots, a_n)$ and $P_a = \sum a_i \partial_i \in T$.

One can interchange the role of S and T by defining

$$x^\beta(\partial^\alpha) = \beta! \binom{\alpha}{\beta} \partial^{\alpha-\beta}.$$

With this notation we have for forms of degree n

$$P_a^d(f) = f(P_a^d) = d!f(a).$$

Moreover

$$f(P_a^m) = 0 \iff f(a) = 0 \tag{1.1.1}$$

if $m \geq d$. More generally we say that homogeneous forms $f \in S$ and $D \in T$ are **apolar** if $f(D) = D(f) = 0$ (According to [Salmon 1885] the term was coined by Reye).

Apolarity allows to define Artinian Gorenstein graded quotient rings of T via forms: For f a homogeneous form of degree d and $F = V(f) \subset \mathbf{P}^n$ define

$$F^\perp = f^\perp = \{D \in T \mid D(f) = 0\}$$

and

$$A^F = T/F^\perp.$$

The socle of A^F is in degree d . Indeed $P_a(D(f)) = 0 \forall P_a \in T_1 \iff D(f) = 0$ or $D \in T_d$. In particular the socle of A^F is 1-dimensional, and A^F is indeed Gorenstein and is called the **apolar Artinian Gorenstein ring** of $F \subset \mathbf{P}^n$.

Conversely for a graded Gorenstein ring $A = T/I$ with socle degree d multiplication in A induces a linear form $f: S_d(T_1) \rightarrow \mathbf{C}$ which can be identified with a homogeneous polynomial $f \in S$ of degree d . This proves:

1.2 Lemma [Macaulay 1916]. *The map $F \mapsto A^F$ is a bijection between hypersurfaces $F = V(f) \subset \mathbf{P}^n$ of degree d and graded Artinian Gorenstein quotient rings $A = T/I$ of T with socle degree d .*

Macaulay's result in terms of "inverse systems" is explained more generally in [Eisenbud 1995] Theorem 21.6 and exercise 21.7. The polynomial f is called the **dual socle generator** or the **dual polynomial** of A^F . Note that the dual polynomial is defined only up to nonzero scalar.

1.3 In the following we identify S with homogeneous coordinate ring of \mathbf{P}^n and T with the homogeneous coordinate ring of the dual space $\check{\mathbf{P}}^n$. $F = V(f) \subset \mathbf{P}^n$ denotes always a hypersurface of degree d . We call a subscheme $\Gamma \subset \check{\mathbf{P}}^n$ **apolar** to F , if the homogeneous ideal $I_\Gamma \subset F^\perp \subset T$.

For example, if $f = l_1^d + \dots + l_s^d$ and $\Gamma = \{L_1, \dots, L_s\} \subset \mathbf{P}^n$ the collection of hyperplanes $L_i = V(l_i) \subset \mathbf{P}^n$, then Γ is apolar to $F = V(f)$. Indeed $g(l_i^d) = 0$ for $g \in I_\Gamma$ by (1.1.1).

Conversely, if $\Gamma = \{L_1, \dots, L_s\} \subset \check{\mathbf{P}}^n$ is apolar to F , then the inclusions $(0) \subset I_\Gamma \subset F^\perp$ induces inclusions

$$\text{Hom}(A_d^F, k) \subset \text{Hom}(R_d, k) \subset \text{Hom}(T_d, k)$$

where $R = R_\Gamma = T/I_\Gamma$ is the homogeneous coordinate ring of Γ . The linear forms

$$\{D \mapsto D(l_i^d)\}$$

span $\text{Hom}(R_d, k)$ so $\{D \mapsto D(f)\} \in \text{Hom}(A_d^F, k)$ takes the form $D(f) = D(\lambda_1 l_1^d + \dots + \lambda_s l_s^d)$ for suitable $\lambda_1, \dots, \lambda_s \in k$. Hence $f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d$ and $\Gamma \in \text{VSP}(F, s)$.

Let $\mathcal{H} \subset \text{Hilb}(\check{\mathbf{P}}^n)$ be a component or a union of components of the Hilbert scheme. Then we call the scheme

$$\text{VPS}(F, \mathcal{H}) = \{\Gamma \in \mathcal{H} \mid \Gamma \text{ is apolar to } F\}$$

the variety of apolar schemes to F in \mathcal{H} . For $p \in \mathbf{Q}[t]$ we abbreviate

$$\text{VPS}(F, p) = \text{VPS}(F, \text{Hilb}_p(\check{\mathbf{P}}^n)).$$

For $s \in \mathbf{N} \subset \mathbf{Q}[t]$ we have

$$\text{VSP}(F, s) \subset \text{VPS}(F, s),$$

This can be a proper inclusion even after taking closures in the Hilbert scheme. $\text{VPS}(F, p)$ are contravariants of F .

1.4 The consideration of A^F gives a quick explanation, why quartics in $\mathbf{P}^2, \mathbf{P}^3$ and \mathbf{P}^4 are exceptions of Theorem 0.2: If a quartic is a sum of 5, 9 or 14 powers respectively, then F^\perp contains a quadric. However for a general F the ideal F^\perp is generated by cubics in all three cases.

In case of cubics $F \subset \mathbf{P}^4$ the argument uses syzygies: As \mathbb{T} -module an A^F has syzygies

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array}$$

for a general F . On the other hand for cubics, which are the sum of 7 general powers, F^\perp contains the ideal I_C of the rational normal curve $C \subset \mathbf{P}^4$ which passes through the 7 points. Since the homogeneous coordinate ring of C has syzygies

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 6 & 8 & 3 & - & - \end{array},$$

the corresponding Gorenstein ring has syzygies at least

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & 3 & - & - \\ - & - & 3 & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array}.$$

So these F are not general.

1.5 Syzygies of A^F also give a quick uniform proof of the assertions in 0.4: For a binary form f of odd degree $d = 2k - 1$ the apolar Artinian Gorenstein ring is a complete intersection

$$A^F \cong \mathbf{C}[\partial_0, \partial_1]/(a, b)$$

with $\deg a + \deg b = 2k + 1$, say $\deg a < \deg b$. For a general f we have $\deg a = k$ and the k roots give the unique set $l_1, \dots, l_k \in \mathbf{C}[x_0, x_1]$ such that $f = \lambda_1 l_1^d + \dots + \lambda_k l_k^d$. Similarly, of course, for a general form of even degree $d = 2k$, the apolar Artinian Gorenstein ring is a complete intersection

$$A^F \cong \mathbf{C}[\partial_0, \partial_1]/(a, b)$$

with $\deg a = \deg b = k + 1$. Thus $VSP(F, k + 1) \cong \mathbf{P}^1$.

In Hilbert's case A^F is Gorenstein of codimension 3 and the structure theorem of Buchsbaum-Eisenbud applies. In particular the number of generators of F^\perp is odd. For a general quintic A^F has syzygies

$$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 4 & 1 & - \\ - & 1 & 4 & - \\ - & - & - & - \\ - & - & - & 1 \end{array},$$

with the middle matrix ϕ is skew symmetric, its Pfaffians generate the ideal. The 4 linear entries of the first column are dependent, since $\mathbf{C}[\partial_0, \partial_1, \partial_2]$ has only 3 linearly independent linear forms. Thus after column and row operations we may assume

$$\phi = \begin{pmatrix} 0 & 0 & \partial_0 & \partial_1 & \partial_2 \\ 0 & 0 & q_{23} & q_{24} & q_{25} \\ -\partial_0 & -q_{23} & 0 & q_{34} & q_{35} \\ -\partial_1 & -q_{24} & -q_{34} & 0 & q_{45} \\ -\partial_2 & -q_{25} & -q_{35} & -q_{45} & 0 \end{pmatrix},$$

for a general F . The 2×2 minors of the block

$$\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ q_{23} & q_{24} & q_{25} \end{pmatrix}$$

are among the Pfaffians of ϕ and generate the ideal of the unique 7 points in $\check{\mathbf{P}}^2$.

Finally for the pentahedral theorem we note that A^F for a general cubic $F \subset \mathbf{P}^3$ has syzygies

$$\begin{array}{cccc} 1 & - & - & - \\ - & 6 & 5 & - \\ - & - & 5 & 6 \\ - & - & - & 1 \end{array} \cong \begin{array}{cccc} 1 & - & - & - \\ - & 5 & 5 & - \\ - & - & - & 1 \end{array} \otimes \begin{array}{cc} 1 & - \\ - & 1 \end{array},$$

cf. [Kustin, Miller 1985] for the decomposition. The 5 Pfaffians among the 6 quadrics define the ideal of 5 distinguished points. (This decomposition of syzygies is well-known in case of canonical curves $C \subset \mathbf{P}^5$ of genus 6: A general genus 6 canonical curve C is a complete intersection of an unique Del Pezzo surface of degree 5 and a non unique hyperquadric, cf. [ACGH 1985], [Schreyer 1986], [Mukai 1988].)

1.6 In the cases above one can give examples of forms which need more summands than the general form for a power sum presentation easily: Just take the unique finite length subscheme non-reduced.

For binary forms the syzygy approach solves the question posed in 0.3 and 0.5. E.g. every binary form of degree d is sum of $\tilde{s} = d$ or less d^{th} powers. An example of a form which achieves this bound is $f = xy^{d-1}$. Here

$$A^F = \mathbf{C}[\partial_x, \partial_y]/(\partial_x^2, \partial_y^d).$$

Since every element of F^\perp of degree $\leq d - 1$ has a double root $\tilde{s} = d$. For $\partial_x^d - \partial_y^d \in F^\perp$ the corresponding power sum is

$$d^2 xy^{d-1} = \sum_{j=1}^d (\zeta_{d^2}^j x + \zeta_{d^2}^{j+d} y)^d$$

where ζ_{d^2} is a primitive d^2 -th root of unity.

Mukai's result $VSP(F, 6) \cong V_{22}$ for a general plane quartic $F \subset \mathbf{P}^2$, is obtained with syzygies. The approach is similar for general planes curves of even degree up to 8.

1.7 Theorem. Let F be a general plane curve of degree $d = 2n - 2$, $2 \leq n \leq 5$, then

$$VSP(F, \binom{n+1}{2}) \cong \mathbf{G}(n, V, \eta) = \{E \in \mathbf{G}(n, V) \mid \Lambda^2 E \subset \ker(\eta)\}$$

where V is a $2n + 1$ -dimensional vector space and η is a net of alternating forms $\eta: \Lambda^2 V \rightarrow \mathbf{C}^3$ on V .

i) [Mukai 1992] When F is a smooth plane conic section, then $VSP(F, 3)$ is a Fano 3-fold of index 2 and degree 5 in \mathbf{P}^6 .

ii) [Mukai 1992] When F is a general plane quartic curve, then $VSP(F, 6) \cong V_{22}$ (cf. 0.6).

iii) [Mukai 1992] When F is a general plane sextic curve, then $VSP(F, 10)$ is isomorphic to polarized K3-surface of genus 20.

iv) When F is a general plane octic curve, then $VSP(F, 15)$ is finite of degree 16, i. e. consists of 16 points.

Proof. A ternary form f of even degree $d = 2n - 2$ is called non-degenerate, if F^\perp contains no elements of degree $n - 1$. Here, as always, $F = V(f)$ denotes the corresponding plane curve of degree d . For non-degenerated f the Artinian Gorenstein ring A^F has syzygies

$$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ \vdots & \vdots & \vdots & \vdots \\ - & 2n+1 & 2n+1 & - \\ \vdots & \vdots & \vdots & \vdots \\ - & - & - & - \\ - & - & - & 1 \end{array},$$

because by the symmetry of the Hilbert function of A^F there are $2n + 1 = \binom{n+2}{2} - \binom{n}{2}$ generators of degree $2n$ in F^\perp and by the symmetry of the resolution there are no generators in higher degree. Conversely a general skew symmetric $2n + 1 \times 2n + 1$ matrix ϕ of linear forms defines via its Pfaffians an Artinian ring with socle in degree $d = 2n - 2$, whose form is non-degenerate.

Interpreting ϕ as a net of alternating forms $\eta: \Lambda^2 V \rightarrow \mathbf{C}^3$ on a $2n+1$ -dimensional vector space $V = (F^\perp)_n^*$ one obtains:

1.8 Lemma. Let f be a non-degenerate ternary form of degree $d = 2n - 2$, $n = 2, 3, 4, 5$ and $\eta: V \rightarrow \mathbf{C}^3$ the corresponding alternating net. Then

$$VPS(F, \binom{n+1}{2}) = \mathbf{G}(n, V, \eta)$$

Proof. Let $X \in \mathbf{G}(n, V, \eta)$ be a point, and let

$$\phi = \begin{pmatrix} B & A \\ -A^t & 0 \end{pmatrix}$$

be the corresponding decomposition of the syzygy matrix with a $n + 1 \times n + 1$ skew-symmetric matrix B , a $n + 1 \times n$ matrix A and a $n \times n$ block of zeros. Among the Pfaffians of the syzygy matrix are the $n \times n$ minors of A . We claim that the minors define a scheme of length $\binom{n+1}{2}$ in \mathbf{P}^2 . By Hilbert-Burch [Eisenbud, Thm 20.15], this is the case unless the $n + 1$ minors f_0, \dots, f_n have a common factor h of degree $k > 0$. We will derive a contradiction by studying the syzygies of the ideal $J = (g_0, \dots, g_n)$ with $f_i = hg_i$. g_0, \dots, g_n are linearly independent, since f_0, \dots, f_n have this properties. Their free resolution has shape:

$$0 \leftarrow T/J \leftarrow T \leftarrow (n+1)T(-n+k) \leftarrow nT(-n+k-1) \oplus T(-2n+2k-1) \leftarrow T(-2n+k-1) \leftarrow 0.$$

Indeed syzygies among the g 's are syzygies among the f 's, hence linear combinations of the columns of ϕ with n zero components. Since $\ker -A^t = (g_0, \dots, g_n)^t$ the syzygies are generated by the columns of A

and the vector $B(g_0, \dots, g_n)^t$. The second syzygy is the vector $(f_{n+1}, \dots, f_{2n}, h)^t$. Here f_{n+1}, \dots denote the remaining Pfaffians. It follows

$$\deg T/J = \binom{n+1}{2} - (2n+1)k + k^2$$

which gives a negative number in the possible range of (n, k) , unless $k = 1$ and $n = 3, 4, 5$. In these remaining cases we consider the resolution of the last matrix transposed. The n forms of degree n modulo the linear form h , generate an ideal of codimension 2 in T/h , hence have a determinantal resolution

$$0 \leftarrow T/J_1 \leftarrow T/h \leftarrow nT/h(-n) \leftarrow (n-2)T/h(-n-1) \oplus T/h(-n-2) \leftarrow 0,$$

where $J_1 = (h, f_{n+1}, \dots, f_{2n})$, and over T the resolution is

$$0 \leftarrow T/J_1 \leftarrow T \leftarrow T(-1) \oplus nT(-n) \leftarrow (2n-2)T(-n-1) \oplus T(-n-2) \leftarrow (n-2)T \oplus T(-n-3) \leftarrow 0.$$

Consider the sheafification of this complex and the middle syzygy sheaf \mathcal{F} . (g_0, \dots, g_n) generate the kernel of

$$\mathcal{F}^{\leftarrow}(n+1)\mathcal{O}(-n-1) \leftarrow \mathcal{O}(-2n) \leftarrow 0.$$

However from the presentation of \mathcal{F}

$$0 \leftarrow \mathcal{F} \leftarrow (2n-2)\mathcal{O}(-n-1) \oplus \mathcal{O}(-n-2) \leftarrow (n-2)\mathcal{O}(-n-2) \oplus \mathcal{O}(-n-3) \leftarrow 0$$

we obtain a short exact sequence

$$0 \leftarrow \mathcal{O}_H(-n-2) \leftarrow \mathcal{F} \leftarrow \mathcal{G} \leftarrow 0.$$

So $\text{Im} \alpha \subset \mathcal{G}$ and $c_1(\ker \alpha) \geq -(n+1)^2 - c_1(\mathcal{G}) = -(n+1)^2 - c_1(\mathcal{F}) + 1 = -2n+1$ a contradiction to the degrees of the g 's.

Thus

$$VPS(F, \binom{n+1}{2}) \supseteq \mathbf{G}(n, V, \eta).$$

For the other inclusion, consider a finite subscheme $\Gamma \in VPS(F, \binom{n+1}{2})$. Since F is nondegenerate, Γ imposes independent conditions on forms of degree $n-1$. By Hilbert-Burch Γ has syzygies:

$$0 \leftarrow R_\Gamma \leftarrow T \leftarrow (n+1)T(-n) \xleftarrow{\psi} nT(-n-1) \leftarrow 0. \quad (1.8.1)$$

By apolarity the ideal $J_\Gamma \subset F^\perp$. Hence we have a sequence

$$0 \leftarrow A^F \leftarrow R_\Gamma \leftarrow F^\perp/J_\Gamma \leftarrow 0. \quad (1.8.2)$$

Since A^F and $R = R_\Gamma$ have Hilbert functions

$$(1, 3, \dots, \binom{n+1}{2}, \binom{n}{2}, \dots, 3, 1)$$

and

$$(1, 3, \dots, \binom{n+1}{2}, \binom{n+1}{2}, \binom{n+1}{2}, \dots)$$

respectively, F^\perp/J_Γ has n generators of degree n with $n+1$ linear relations:

$$0 \leftarrow F^\perp/J_\Gamma \leftarrow nT(-n) \leftarrow (n+1)T(-n-1).$$

The minors of the presentation matrix are contained in the annihilator, which is J_Γ . Hence, this matrix is ψ^t again, and $F^\perp/J_\Gamma \cong \omega_R(-2n+2)$. A mapping cone between the complex (1.8.1) and its dual over the sequence (1.8.2), gives syzygies of A^F with the desired block structure. \square

End of the proof of (1.7). General curves F correspond via dual socle generators to general alternating matrices. Hence for a general F the variety $\mathbf{G}(n, V, \eta)$ is smooth of expected dimension as zero loci of a general section of a globally generated bundle. In fact, let Q be the universal rank n quotient bundle on $\mathbf{G} = \mathbf{G}(n, V)$ with chern classes $c_i = c_i(Q)$. Then $X = \mathbf{G}(n, V, \eta)$ is the zero locus of a section of $3 \wedge^2 Q$, so

$$\deg X = c_1^m \cdot c_{3\binom{n}{2}}(3 \wedge^2 Q),$$

where

$$\begin{aligned} m = \dim X &= n(n+1) - 3\binom{n}{2} = \frac{1}{2}n(5-n) \\ &= 3, 3, 2, 0 \quad \text{for } n = 2, 3, 4, 5. \end{aligned}$$

A calculation in the Chow ring of the respective \mathbf{G} 's give the degrees 5, 22, 38 and 16, respectively, for X . Consider now the incidence correspondence

$$\{(\Gamma, F) \in \text{Hilb}^o\left(\binom{n+1}{2}, \mathbf{P}^2\right) \times \mid (2n-2)L \mid^o \mid \Gamma \text{ is apolar to } F\}$$

where o indicates that we consider only the open parts of Γ 's, respectively F 's, which impose independent conditions on forms of degree $n-1$. The incidence correspondence is irreducible, since its fibers over Hilb^o are irreducible of constant dimension. The fibers of the other projection are the $\mathbf{G}(n, V, \eta)$'s by the lemma. Since smooth Γ are dense in the Hilbert scheme, smooth Γ are dense in every component of $\mathbf{G}(n, V, \eta)$ for a general F . This proves (iv).

In the other cases all components have the same degree and have induced canonical bundle by the irreducibility of the correspondence.

The tangent bundle $T_{\mathbf{G}}$ has first chern class $c_1(T_{\mathbf{G}}) = (2n+1)c_1$, while the normal bundle of X in \mathbf{G} , the bundle $N_X = 3 \wedge^2 Q$, has first chern class $3(n-1)c_1 \cap X$. Therefore the chern class of the canonical bundle on X is $(n-4)c_1 \cap X$. Hence the components are Fano 3-folds of index 2 in case $\deg F = 2$, Fano 3-folds of index 1 in case $\deg F = 4$, and K3- or abelian surfaces in case $\deg F = 6$. The sums of the degrees of the components are 5, $22 = 2 \cdot 11$ and $38 = 2 \cdot 19$ respectively. Since 5, 11 and 19 are prime numbers the fiber is irreducible for a general F . This proves (i) and (ii).

In case (iii) it remains to show that X is K3. The total chern classes of $T_{\mathbf{G}}$ and N_X are

$$c(T_{\mathbf{G}}) = 1 + 9c_1 + 39c_1^2 + c_2 + \dots \quad \text{and} \quad c(N_X) = (1 + 9c_1 + 36c_1^2 + 6c_2) \cap X,$$

respectively. Thus T_X has total chern class $c(T_X) = c(T_{\mathbf{G}})/c(N_X) = 1 + 3c_1^2 - 5c_2$. Since the degree of X is $c_1^2 \cap X = 38$, we get $c_2(T_X) = 114 - 5c_2 \cap X$, i.e. X can only be K3 (with $c_2 \cap X = 18$). \square

This proves the easy part of Mukai's theorem for plane quartics. For the difficult part, that every V_{22} arises this way, cf. [Mukai 1992].

1.9 To describe $VSP(F, 8)$ for a general cubic $F \subset \mathbf{P}^4$ we look for subcomplexes

$$\begin{array}{cccccc} 1 & - & - & - & - & \\ - & 7 & 8 & - & - & \\ - & - & 3 & 8 & 3 & , \end{array}$$

corresponding to 8 points $\Gamma \subset \check{\mathbf{P}}^4$, of the syzygies of A^F

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array} .$$

Thus we have embeddings $VSP(F, 8) \hookrightarrow \mathbf{G}(7, 10)$ or $VSP(F, 8) \hookrightarrow \mathbf{G}(8, 16)$. However unlike Mukai's case, we could not find a description of the compactifications of those subschemes, which would give e.g. a correct dimension estimate. Our method is more involved.

2. Duality, projections and secants

2.1 The syzygy approach of the previous section has a geometric counterpart in terms of duality and projections which will clarify our solution of the case of a cubic in \mathbf{P}^4 . In the next section we shall see how this more geometric approach and the syzygy approach both apply to the cases of plane curves of degree 7 and quadric surfaces.

With notation as above let $f \in S_d$ be a homogeneous form of degree d , and assume $f = l_1^d + \dots + l_s^d$ for some s and some linear forms l_i i. e. $\Gamma = \{L_1, \dots, L_s\} \subset \check{\mathbf{P}}^n$ is apolar to $F = V(f)$. The first simple but crucial observation is that for any homogeneous $D \in T$ of degree $0 \leq d' \leq d$, we get

$$D(f) = \sum_{i=1}^s \lambda_i^D l_i^{d-d'},$$

with $\lambda_i^D \in \mathbf{C}$. Thus Γ is apolar to $F_D = V(D(f))$, i. e. to all multiple partials of f .

2.2 Now fix an $e \leq d$ and let

$$F_e^\perp = \{D \in T_e \mid D(f) = 0\} \subset F^\perp.$$

Then

$$F_e^\perp = \{D \in T_e \mid D(D'(f)) = 0 \text{ for all } D' \in T_{d-e}\} \subset T_e,$$

so F_e^\perp is the dual space to the space

$$F_e = \{D'(f) \mid D' \in T_{d-e}\} \subset S_e$$

of multiple partials of f . We consider the map

$$\pi_e^F : \check{\mathbf{P}}^n \rightarrow \mathbf{P}_*(F_e^\perp) = \mathbf{P}^{n_e}$$

defined by F_e^\perp , with $n_e = \dim F_e^\perp - 1$. By construction π_e^F is the composition of the e -uple embedding of $\check{\mathbf{P}}^n$ and the **projection from the partials** of f of order $d - e$ considered as points in $\check{\mathbf{P}}^{\binom{n+d}{d}-1}$.

2.3 If $\Gamma = \{L_1, \dots, L_s\} \subset \check{\mathbf{P}}^n$ is apolar to F_e , then $I_\Gamma(e) \subset F_e^\perp$ and the span of $\pi_e^F(\Gamma)$ has dimension

$$d_\Gamma(e) = \dim F_e^\perp - \dim I_\Gamma(e) - 1.$$

Therefore, when F_e^\perp has no basepoints, $\pi_e^F(\Gamma)$ is contained in a fiber of π_e^F over a linear space of dimension $d_\Gamma(e)$. When π_e^F is an embedding the apolar set Γ determine a s -secant $\mathbf{P}^{d_\Gamma(e)}$ to $\pi_e^F(\check{\mathbf{P}}^n)$ this way.

2.4 If $d_\Gamma(e) = d_e(s)$ is independent of $\Gamma \in VSP(F, s)$, then

$$\Gamma \mapsto \langle \pi_e^F(\Gamma) \rangle$$

defines a map

$$\rho : VSP(F, s) \rightarrow \mathbf{G}(d_e(s) + 1, n_e + 1).$$

This occurs naturally when the $(d - e)$ -uple partials of f are linearly independent and the points of Γ impose independent conditions on forms of degree e for any $\Gamma \in VSP(F, s)$. In this case $n_e = \binom{n+d}{d} - \binom{n+d-e}{d-e}$ and

$$d_\Gamma(e) = d_e(s) = s - \binom{n+d-e}{d-e} - 1.$$

In general ρ is a rational map, when it is birational it gives a birational model of $VSP(F, s)$. Of course, this model may not coincide with the Hilbert scheme compactification of (0.5), and in general we see no reason why it should.

In the cases treated here we choose the minimal e such that $d_e(s) \geq 0$. This number coincides with the highest degree of the generators of F^\perp and the homogeneous ideal I_Γ of $\Gamma \in VSP(F, s)$. Since A^F is Artinian, F_e^\perp cannot have base points, and the map $\rho : VSP(F, s) \rightarrow \mathbf{G}(d_e(s) + 1, n_e + 1)$ is a morphism. In each case treated here we check in fact that it is an embedding of $VSP(F, s)$.

By construction the image of ρ is a subvariety of the variety of s -secant spaces to $\pi_e^F(\check{\mathbf{P}}^n)$ of dimension $d_e(s)$, and in general it is proper.

In fact, the image $\pi_e^F(L)$ of any line $L \subset \check{\mathbf{P}}^n$ have a span of dimension at most e . So as soon as $e \leq d_e(s)$ any scheme of length s on the line will contribute to the variety of s -secant spaces to $\pi_e^F(\check{\mathbf{P}}^n)$ of dimension $d_e(s)$, while such schemes are not apolar to F .

The general criterion for a subscheme Z to be apolar to F can be weakened to a useful sufficient criterion: Consider a scheme $Z \subset \check{\mathbf{P}}^n$ of length s such that the span of $\pi_e^F(Z)$ has dimension $d_e(s)$. Let I_Z be the homogeneous ideal of Z . If Z impose s conditions on forms of degree e , then $I_Z(e) \subset F_e^\perp$. If furthermore I_Z is generated by forms of degree e , then $I_Z \subset F^\perp$ and Z is apolar to F .

2.5 The projection from partials is well illustrated by the case of the cubic consisting of three lines in the plane. Let $f = x_0x_1x_2$, $F = V(f)$. Then

$$F_2^\perp = \{D \in T_e | D(f) = 0\} = \langle \partial_0^2, \partial_1^2, \partial_2^2 \rangle,$$

and

$$\pi_2^F : \check{\mathbf{P}}^2 \rightarrow \mathbf{P}^2$$

is a 4 : 1-map. If $\Gamma \in VSP(F, 4)$, then I_Γ is generated by two quadrics, so Γ is a fiber of π_2^F . Conversely any general fiber Z of π_2^F is a collection of 4 distinct points, the complete intersection of two quadrics. Therefore Z is apolar to F . We conclude that $VSP(F, 4) \cong \mathbf{P}^2$.

Note that this approach works for the general plane cubic so $VSP(F, 4) \cong \mathbf{P}^2$ for any general cubic $F \subset \mathbf{P}^2$.

Remark. We get the Hilbert scheme compactification in this case since the map π_2^F is finite and every fiber have length 4.

3. Another example

In this section we illustrate the syzygy approach and the duality and projection approach applied to the case of plane curves of degree 7.

3.1 Theorem. [Dixon, Stuart 1908]. *Let $F \subset \mathbf{P}^2$ be a general curve of degree 7. Then $VSP(F, 12)$ consists of 5 distinct points, i. e. the form of degree 7 defining F have precisely 5 distinct presentations as the sum of 12 powers of linear forms.*

We consider first the apolar graded Artinian Gorenstein ring $A^F = \mathbf{C}[\partial_0, \partial_1, \partial_2]/F^\perp$. It is a certain "Veronese" quotient of the homogeneous coordinate ring of a smooth Del Pezzo surface D of degree 5 in \mathbf{P}^5 . More precisely let $I_D \subset \mathbf{C}[y_0, \dots, y_5]$ be the homogeneous ideal of a smooth Del Pezzo surface in \mathbf{P}^5 :

3.2 Proposition. *Let A^F be as above. Then there is a Veronese embedding of $\check{\mathbf{P}}^2 \rightarrow \mathbf{P}^5$ with corresponding ring homomorphism $\varphi : \mathbf{C}[y_0, \dots, y_5] \rightarrow \mathbf{C}[\partial_0, \partial_1, \partial_2]$ such that $A^F = \mathbf{C}[\partial_0, \partial_1, \partial_2]/\varphi(I_D)$ where $\varphi(I_D)$ denotes the ideal generated by the image of I_D under φ .*

Proof. For a general F the Artinian Gorenstein ring A^F has the maximal possible Hilbert function (1,3,6,10,10,6,3,1). So there are 5 quartic generators in F^\perp . The syzygies are

$$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & 5 & - & - \\ - & - & 5 & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & 1 \end{array} .$$

because those without relations in degree 6 are the most general. The syzygy matrix has 10 off diagonal quadratic entries, and it is an open condition on F that these generate all 6 linearly independent quadrics in \mathbf{P}^2 . Thus we obtain a Veronese embedding $\mathbf{P}^2 \hookrightarrow \mathbf{P}^5 \subset \mathbf{P}^9$ in which we have the Del Pezzo surface $\mathbf{P}^5 \cap \mathbf{G}(2, 5)$. The two surfaces do not intersect because A^F is Artinian. Conversely given a Veronese embedding $\varphi: \mathbf{C}[y_0, \dots, y_5] \rightarrow \mathbf{C}[\partial_0, \partial_1, \partial_2]$ and a Del Pezzo surface D such that $\mathbf{P}^2 \cap D = \emptyset$, then $\phi(I_D)$ defines an Artinian Gorenstein ring with desired syzygies \square

Proof of (3.1): Recall from the proof of the proposition that F^\perp is generated by 5 quartics, they are restrictions of the quadrics defining a Del Pezzo surface D to the Veronese embedding V of $\check{\mathbf{P}}^2$. The map $\pi_4^F: \check{\mathbf{P}}^2 \rightarrow \mathbf{P}^4$ defined by these 5 quartics is therefore a morphism. A set of 12 points $\Gamma \subset \check{\mathbf{P}}^2$ is apolar to F when they are collinear under this map, and the ideal I_Γ is generated by three quartics, cf. 2.4. Now, the Del Pezzo is a linear section of the Grassmannian $\mathbf{G}(2, 5) \subset \mathbf{P}^9$. The 5 quadrics generating the ideal of $\mathbf{G}(2, 5)$ define a rational map $\pi: \mathbf{P}^9 \rightarrow \mathbf{P}^4$. The preimage of a line is a twisted cubic 6-fold scroll with vertex a plane, it lies in a tangent hyperplane to $\mathbf{G}(2, 5)$. Similarly the preimage of a line for the map defined by the quadrics in I_D are restrictions of these twisted cubic scrolls to \mathbf{P}^5 . The general one meets \mathbf{P}^5 in a twisted cubic surface, which does not meet V . The tangent hyperplanes to $\mathbf{G}(2, 5)$ are naturally parameterized by $\mathbf{G}(5, 3)$ in the dual space $\check{\mathbf{P}}^9$. Dual to $\mathbf{P}^5 \subset \mathbf{P}^9$ is a $\mathbf{P}^3 \subset \check{\mathbf{P}}^9$. This \mathbf{P}^3 meets $\mathbf{G}(5, 3)$ in 5 points. These 5 points correspond to the 5 twisted cubic threefold scrolls that contain D . They each intersect V in a scheme of length 12 which become collinear under the map to \mathbf{P}^4 . The ideal of these points is generated by the three minors of a 2×3 matrix with quadratic entries, so these 5 schemes of length 12 are precisely the elements of $VPS(F, 12)$. When the Veronese embedding is general with respect to the Del Pezzo surface D , these apolar schemes are all smooth such that $VPS(F, 12) = VSP(F, 12)$, and the theorem follows. \square

Together with (1.5), (1.7) and (2.5) this result treats general plane curves of degree ≤ 8 . For higher degrees the problems seems more difficult.

4. Cubic threefolds, first properties

4.0 Let $f \in S = \mathbf{C}[x_0, \dots, x_4]$ be a general cubic polynomial, and $F = V(f) \subset \mathbf{P}^4$, and let $T = \mathbf{C}[\partial_0, \dots, \partial_4]$. In the remaining sections we investigate $VSP(F, 8)$. Chronologically we started to study its local geometry. We show in this section that there are 8 conic sections through each point in $VSP(F, 8)$. In the next section we study the syzygies of 8 general points in \mathbf{P}^4 , and in particular describe the pencil of elliptic quintic curves through 8 general points. For a general cubic, we then started to look for the variety of apolar elliptic quintic curves. With MACAULAY we computed examples of apolar Artinian Gorenstein rings for cubics f with general syzygies, and found a rank 10 quadratic relation among the apolar quadrics to f . The quadrics defining an apolar elliptic quintic curve through an apolar subscheme of length 8, form an isotropic \mathbf{P}^4 for the corresponding quadratic form. Thus we look for a certain subvariety Y of the set of isotropic \mathbf{P}^4 's of a smooth quadric Q in \mathbf{P}^9 parametrizing apolar elliptic quintic curves. The isotropic \mathbf{P}^4 's are parameterized by a 10-dimensional spinor variety \mathbf{S} , which plays a crucial role in our investigation. We introduce \mathbf{S} in section 6 with its natural half spin embedding in \mathbf{P}^{15} , and show that the apolar Artinian Gorenstein ring of a general cubic F is an Artinian quotient of the homogeneous coordinate ring of this spinor variety in section 7. In section 8 we then use the facts that the spinor variety is isomorphic to its dual, and that the quadrics generating its homogeneous ideal define a map of \mathbf{P}^{15} onto the smooth quadric Q in \mathbf{P}^9 , to find the variety of apolar quintic elliptic curves, and prove the main result on $VSP(F, 8)$. In the last section we show that $VSP(F, 8)$ is a Fano variety of index 1 and degree 660.

4.1 First we construct a rational map of $VSP(F, 8)$ into the Grassmannian of $\mathbf{G}(3, 10)$ of planes in \mathbf{P}^9 . The apolar quadrics

$$F_2^\perp = \{D \in T_2 \mid D(f) = 0\},$$

define in the notation of section 2, a map

$$\pi_2^F: \check{\mathbf{P}}^4 \rightarrow \mathbf{P}^9 (= \mathbf{P}_*(F_2^\perp)),$$

the composition of the Veronese embedding $\check{\mathbf{P}}^4 \rightarrow \check{\mathbf{P}}^{14}$ with the projection from the partials of f considered as points in $\check{\mathbf{P}}^{14}$. For general F this map is an embedding as we shall see later (cf. 8.3). We denote the image by V_F .

As in (2.4), if $\{L_1, \dots, L_8\} \in VSP(F, 8)$ then

$$\langle \pi_2^F(L_1, \dots, L_8) \rangle = \text{span}\{\pi_2^F(L_1), \dots, \pi_2^F(L_8)\}$$

is only a plane in \mathbf{P}^9 . It is an 8-secant plane to $\pi_2^F(\check{\mathbf{P}}^4) = V_F \subset \mathbf{P}^9$. Thus we have defined a map of $VSP(F, 8)$ into $\mathbf{G}(3, 10)$ as promised. In section 8 we show that this map is in fact an embedding. For the purpose of this section we identify $VSP(F, 8)$ with its image in $\mathbf{G}(3, 10)$.

Notice that all 8-secant planes to V_F need not come from points on $VSP(F, 8)$. In fact any line in $\check{\mathbf{P}}^4$ is embedded as a conic on V_F , but 8 points on it does not belong to $VSP(F, 8)$.

4.2 Recall from (1.4) that for cubics $F = V(f) \subset \mathbf{P}^4$ where f is a sum of 7 general powers, called a degenerate cubic, the unique rational normal quartic curve $C \subset \check{\mathbf{P}}^4$ passing through the 7 points is apolar to F . By the 3-uple embedding of $\check{\mathbf{P}}^4$ into \mathbf{P}^{34} , the curve C has degree 12 and spans a \mathbf{P}^{12} which contain the cubic f . This \mathbf{P}^{12} can be interpreted as the space of binary forms of degree 12, and the fact, from (1.5), that $VSP(f_{bin}, 7) \cong \mathbf{P}^1$ when one considers f as such a binary form f_{bin} , means that C has a \mathbf{P}^1 of 7-secant \mathbf{P}^6 's passing through f . In the interpretation of f as a cubic the conclusion is that $VSP(F, 7)$ contains a \mathbf{P}^1 . On the other hand the quadrics through C are apolar to F . The apolar quadrics

$$F_2^\perp = \{D \in T_2 \mid D(f) = 0\},$$

define a map

$$\pi_2^F : \check{\mathbf{P}}^4 \rightarrow \mathbf{P}^9 (= \mathbf{P}_*(F_2^\perp)).$$

Recall that a rational normal quartic curve is defined by the 2×2 minors of a 2×4 matrix with linear entries. Among these minors there is the Plücker quadric relation, a rank 6 quadric. Therefore $\pi_2^F(\check{\mathbf{P}}^4) \subset \mathbf{P}^9$ is contained in a rank 6 quadric, and $C_F = \pi_2^F(C)$ spans the vertex \mathbf{P}^3 of this quadric. Any 7 points apolar to F become collinear on C_F . Thus C_F is a curve of degree 8 in \mathbf{P}^3 with at least a \mathbf{P}^1 of 7-secant lines. This, of course, occurs precisely when C_F is a curve of type (1, 7) on a smooth quadric surface. Thus for general degenerate cubic F , the variety $VSP(F, 7) \cong \mathbf{P}^1$, identified by one of the rulings of a quadric in a \mathbf{P}^3 .

4.3 Proposition. *There are 8 conic sections through the general point on $VSP(F, 8) \subset \mathbf{G}(3, 10)$ corresponding to the 8 summands of the power sum presentation.*

Proof. Let $\{L_1, \dots, L_8\} \in VSP(F, 8)$ be a general point with $L_i = V(l_i)$. Then $f = l_1^3 + \dots + l_8^3$. Thus $f_i = f - l_i^3$ is a degenerate cubic, and the union of any element of $VSP(F_i, 7)$ with L_i belongs to $VSP(F, 8)$. From (4.2) we get altogether 8 \mathbf{P}^1 's through $\{L_1, \dots, L_8\}$. Now to show that these \mathbf{P}^1 's are conic sections in the image of $VSP(F, 8)$ in $\mathbf{G}(3, 10)$ we consider again the rational normal curve apolar to F_i . The image of this curve on V_F has degree 8 and has a pencil of 7-secant planes all passing through the point L_i . The projection from L_i therefore have degree 8 and have a pencil of 7-secant lines, which must form a quadric surface like above. Therefore the 7-secant planes form a pencil of planes of a rank 4 quadric in a \mathbf{P}^4 . But such a pencil clearly form a conic section in $\mathbf{G}(3, 10)$, so the proposition follows. \square

5. Syzygies of 8 general points in \mathbf{P}^4

In this section we describe the family of elliptic quintic normal curves through 8 general points in \mathbf{P}^4 .

5.1 Let $\Gamma = \{p_1, \dots, p_8\} = D_1 \cap D_2 \subset S \subset \mathbf{P}^4$ be the 8 intersection points of two elliptic normal curves $D_1, D_2 \in |2H - R|$ on a cubic scroll $S \subset \mathbf{P}^4$, where $H, R \in \text{Pic}S$ is the hyperplane class respectively ruling on S . The resolution

$$0 \leftarrow \mathcal{O}_\Gamma \leftarrow \mathcal{O}_S \leftarrow 2\mathcal{O}_S(-2H + R) \leftarrow \mathcal{O}_S(-4H + 2R) \leftarrow 0$$

of Γ on S induces a filtration

$$\begin{array}{cccccc} 1 & - & - & - & - & \\ - & 3 & 2 & - & - & \\ - & - & - & - & - & \end{array} + 2 \begin{pmatrix} - & - & - & - & - \\ - & 2 & 3 & - & - \\ - & - & - & 1 & - \end{pmatrix} + \begin{array}{cccccc} - & - & - & - & - & \\ - & - & - & - & - & \\ - & - & 3 & 6 & 3 & \end{array}$$

for the syzygies of $\Gamma \subset \mathbf{P}^4$

$$\begin{array}{cccccc} 1 & - & - & - & - & \\ - & 7 & 8 & - & - & \\ - & - & 3 & 8 & 3 & . \end{array}$$

Hence the piece

$$\begin{array}{cccccc} - & - & - & - & - & \\ - & - & - & - & - & \\ - & - & 3 & 8 & 3 & . \end{array}$$

of the syzygies of Γ recovers S as the support of the cokernel of the map $3\mathcal{O}(-4) \leftarrow 8\mathcal{O}(-5)$. The pencil can be recovered from $2\mathcal{O}(-5)$ in the kernel of this map.

In the rest of this section we show that any general set of 8 points in \mathbf{P}^4 has syzygies as above.

5.2 Proposition. *There is a pencil of elliptic quintic curves through 8 points in general position in \mathbf{P}^4 . This pencil sweep out a rational cubic scroll.*

Proof. Recall the classical result (Castelnuovo) that there is a unique rational normal quartic curve through 7 general points in \mathbf{P}^4 . So given 8 points consider the rational normal curve passing through all but one of them, say the point p . The ideal of the rational quartic curve is generated by the 2×2 minors of a 2×4 matrix with linear entries. This matrix has rank 2 at p , but clearly one can find a 2×3 submatrix which have rank 1 at p . This submatrix has rank 1 on a rational cubic scroll. Thus we find rational cubic scrolls through the 8 points.

Given a rational cubic scroll through the 8 points there is either one or a pencil of anticanonical curves on the scroll passing through the 8 points. On the other hand, an anticanonical curve on the scroll is an elliptic quintic normal curve. Any complete linear series of degree 2 on this curve define a scroll, the union of the secant lines to the curve meeting the curve in pairs of points which belong to the linear series. These scrolls are rational cubic scrolls, and their union is precisely the secant variety of the elliptic curve. Since the complete linear series of degree 2 on the elliptic curve E is parameterized by $Jac_2 E$ which is again an elliptic curve, we have found for each elliptic curve through the 8 points an elliptic pencil of rational cubic scrolls through the 8 points.

Given 8 points on an elliptic quintic curve, the quadrics through the 8 points will define a linear series of degree 2 on the curve. Consider the rational cubic scroll of the curve determined by this linear series. On it the elliptic curve will move in a pencil through the 8 points. Thus there is a linear pencil of elliptic curves through the 8 points and they sweep out a rational cubic scroll. To show that there is only one such pencil, consider two elliptic quintic curves through the 8 points. The 5 quadrics through one of the curves define a linear series of degree 2 residual to the 8 points on the other, so there are 3 quadrics containing both curves. Since their union has degree 10, the three quadrics cannot define a complete intersection, so in fact they define a cubic scroll. Clearly it is the one described above, so it is unique. \square

With (5.2) it is natural to see $VSP(F, 8)$ for a cubic threefold F as a variety of lines in the variety of apolar elliptic quintic curves to F . We come back to this in section 8 after introducing the spinor variety and the apolar Artinian Gorenstein ring A^F .

6. Equations and geometry of the spinor varieties S_{ev} and S_{odd}

6.0 Let W be a 10-dimensional vector space and let Q^8 be a rank 10 quadric in $\mathbf{P}^9 = \mathbf{P}_*(W)$, i. e. defined by a nondegenerate quadratic form $q : W \rightarrow k$. The quadric Q contains two families of isotropic \mathbf{P}^4 's, parameterized by the (isomorphic) spinor varieties S_{ev} and S_{odd} . We recall the description of these varieties cf. [Room 1952], [Chevalley 1954], [Lazarsfeld, Van de Ven 1984],[Zak 1984], [Ein 1986], [Mukai 1995].

6.1 Let Q be defined by the equation

$$q = \sum_{i=1}^5 y_i y_{-i} = 0$$

Then $U_0 = \{y_1 = \dots y_5 = 0\}$ and $U_\infty = \{y_{-1} = \dots y_{-5} = 0\} \subset W$ represent a pair of disjoint isotropic \mathbf{P}^4 's. Let $e_{\pm 1}, \dots, e_{\pm 5} \in W$ be the dual basis of the $y_{\pm i}$'s. Then

$$U_0 = \langle e_{-1}, \dots, e_{-5} \rangle \text{ and } U_\infty = \langle e_1, \dots, e_5 \rangle .$$

These space are dual to each other via q . Consider the Clifford-operators

$$\partial_+ = \sum_{i=1}^5 y_i e_i \text{ and } \partial_- = \sum_{i=1}^5 y_{-i} e_{-i} .$$

The operation $\partial_\pm: \wedge^\bullet U_\infty \rightarrow W^* \otimes \wedge^{\bullet \pm 1} U_\infty$ as wedge-product respectively contraction induce a map of bundles $\Lambda_{\mathbf{P}^9} \rightarrow \Lambda_{\mathbf{P}^9}(1)$, where $\Lambda = \wedge^\bullet U_\infty$. For $\partial = \partial_+ + \partial_-$ one has $\partial \circ \partial = q \cdot id_\Lambda$. Hence decomposing in even and odd parts one obtains two rank 8 vector bundles on Q :

$$E^{ev} = \text{coker}(\partial^{odd}: \Lambda_Q^{odd}(-1) \rightarrow \Lambda_Q^{ev}) \text{ and } E^{odd} = \text{coker}(\partial^{ev}: \Lambda_Q^{ev}(-1) \rightarrow \Lambda_Q^{odd}) .$$

The bundles are related to each other via the short exact sequences

$$0 \rightarrow E^{odd}(-1) \rightarrow \Lambda_Q^{ev} \rightarrow E^{ev} \rightarrow 0 \text{ and } 0 \rightarrow E^{ev} \rightarrow \Lambda_Q^{odd}(1) \rightarrow E^{odd}(1) \rightarrow 0 .$$

$E^{ev/odd}$ are homogeneous bundles under the action of $SO(q)$. They are the unique indecomposable vector bundles E on Q with $H^i(Q, E(n)) = 0$ for all n and $0 < i < \dim Q$, cf. [Knörrer 1987]. The induced action on $\Lambda^{ev/odd} = H^0(Q, E^{ev/odd})$ are the half spin representations of $SO(q)$.

6.2 Every isotropic \mathbf{P}^4 arises as zero loci of a section in E^{ev} respectively E^{odd} . We treat E^{ev} :

Let $1, e_1 \wedge e_2, \dots, e_4 \wedge e_5, e_2 \wedge e_3 \wedge e_4 \wedge e_5, \dots, e_1 \wedge e_2 \wedge e_3 \wedge e_4$ be the basis of monomials in e_1, \dots, e_5 of Λ^{ev} , and let $x_0, x_{12}, \dots, x_{45}, x_{2345}, \dots, x_{1234}$ be corresponding dual coordinate functions on Λ^{ev} . In these coordinates the section $(1, 0, \dots, 0)$ vanishes along $\mathbf{P}_*(U_0)$:

$$\partial^{ev}: (1, 0, \dots, 0) \mapsto y_1 e_1 + \dots + y_5 e_5 .$$

Let $(id_5, A) = (\delta_{ij}, a_{ij})$ be point in the affine neighborhood of $0 \in Hom(U_0, U_\infty) \subset \mathbf{G}(5, W)$. $(id, A) \in Hilb_{\mathbf{P}^4}(Q)$ iff

$$(id, A) \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix} \begin{pmatrix} id \\ A^t \end{pmatrix} = A + A^t = 0$$

i. e. if $A = (a_{ij})$ is skew symmetric. More intrinsically, an affine peace of the spinor variety \mathbf{S}_{ev} is given by $\Lambda^2 U_\infty \subset Hom(U_\infty^*, U_\infty) = Hom(U_0, U_\infty)$. Concerning the spinor embedding we consider the exponential map

$$\text{exp}: (a_{ij}) \mapsto (1, a_{ij}, \text{Pfaff}_{ijkl}(A)) \in \Lambda^{ev} .$$

Here $\text{Pfaff}_{ijkl}(A) = a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}$. In more intrinsic terms the exponential map is given by

$$\text{exp}: \Lambda^2 U_\infty \rightarrow \Lambda^{ev}, A \mapsto 1 + A + \frac{1}{2} A \wedge A$$

A computation shows, that the corresponding section has zero loci defined by

$$y_i + \sum_j a_{ij} y_{-j} = 0 \text{ for } i = 1, \dots, 5$$

as desired.

6.3 The image of the exponential map satisfies the homogeneous equations

$$\begin{aligned}
q_1^+ &= x_0x_{2345} + x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25} \\
q_2^+ &= x_0x_{1345} - x_{13}x_{45} + x_{14}x_{35} - x_{34}x_{15} \\
q_3^+ &= x_0x_{1245} + x_{12}x_{45} - x_{14}x_{25} + x_{24}x_{15} \\
q_4^+ &= x_0x_{1235} - x_{12}x_{35} + x_{13}x_{25} - x_{23}x_{15} \\
q_5^+ &= x_0x_{1234} + x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} \\
q_1^- &= x_{12}x_{1345} + x_{13}x_{1245} + x_{14}x_{1235} + x_{15}x_{1234} \\
q_2^- &= -x_{12}x_{2345} + x_{23}x_{1245} + x_{24}x_{1235} + x_{1234}x_{25} \\
q_3^- &= -x_{13}x_{2345} - x_{23}x_{1345} + x_{34}x_{1235} + x_{1234}x_{35} \\
q_4^- &= -x_{14}x_{2345} - x_{24}x_{1345} - x_{34}x_{1245} + x_{1234}x_{45} \\
q_5^- &= -x_{15}x_{2345} - x_{25}x_{1345} - x_{35}x_{1245} - x_{1235}x_{45}
\end{aligned}$$

Indeed in the open set $x_0 = 1$ the first five equations express x_{ijkl} as the Pfaffians of $X = (x_{ij})$, and the last 5 equations become the well-known syzygies among the Pfaffians.

6.4 Proposition.(e.g. [Mukai 1995]). $\mathbf{S}_{ev} \subset \mathbf{P}^{15}$ is defined by the equation above. It has degree 12, its homogeneous coordinate ring is Gorenstein, with syzygies

$$\begin{array}{cccccc}
1 & - & - & - & - & - \\
- & 10 & 16 & - & - & - \\
- & - & - & 16 & 10 & - \\
- & - & - & - & - & 1
\end{array}$$

Proof. We already saw that the spinor variety $\mathbf{S}_{ev} \subset \mathbf{P}^{15}$ is defined by these equations in the affine part $x_0 = 1$. Moreover the equations have syzygies as claimed. This follows from e.g. a MACAULAY computation. By the length of the resolution we conclude that this ideal is projectively Cohen-Macaulay. Furthermore by a similar computation x_0 is a nonzero divisor. Since projective Cohen-Macaulay ideals are unmixed, this is the homogeneous ideal of $\mathbf{S}_{ev} \subset \mathbf{P}^{15}$. \square

6.5 Consider the point $o = (1 : 0 : \dots : 0) \in \mathbf{S}_{ev}$ corresponding to U_0 . The tangent space at o is $T_{\mathbf{S},o} = \{x_{1234} = x_{1235} = x_{1245} = x_{1345} = x_{2345} = 0\}$. The equations

$$\begin{pmatrix} q_1^- \\ q_2^- \\ q_3^- \\ q_4^- \\ q_5^- \end{pmatrix} = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{2345} \\ x_{1345} \\ x_{1245} \\ x_{1235} \\ x_{1234} \end{pmatrix} = 0$$

define the image $\tilde{\mathbf{S}} \subset \mathbf{P}^{14}$ of the birational projection of \mathbf{S}_{ev} from o . Notice that $\tilde{\mathbf{S}}$ is a generic syzygy variety, cf. [Eusei, Schreyer 1995]. $\tilde{\mathbf{S}}$ has syzygies

$$\begin{array}{cccccc}
1 & - & - & - & - & \\
- & 5 & 1 & - & - & \\
- & - & 11 & 10 & 1 & \\
- & - & - & - & - & 1
\end{array}$$

The equation of $\mathbf{S}_{ev} \subset \mathbf{P}^{15}$ can be recovered from $\tilde{\mathbf{S}}$ as

$$\varphi_4 \cdot \begin{pmatrix} x_0 \\ 1 \end{pmatrix} = 0$$

where φ_4 is the last syzygy matrix of $\tilde{\mathbf{S}}$.

The exceptional set of the projection is inside $T_{\mathbf{S},o}$ and there given by

$$\begin{pmatrix} x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25} \\ -x_{13}x_{45} + x_{14}x_{35} - x_{34}x_{15} \\ x_{12}x_{45} - x_{14}x_{25} + x_{24}x_{15} \\ -x_{12}x_{35} + x_{13}x_{25} - x_{23}x_{15} \\ x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} \end{pmatrix} = 0$$

These are just Plücker quadrics. So the exceptional set is the seven dimensional cone

$$\mathbf{G}_o^7 = \{\mathbf{P}_*(U) \mid \dim U \cap U_0 \geq 3\}$$

consisting of even isotropic \mathbf{P}^4 's whose intersection with $\mathbf{P}_*(U_0)$ is at least 2-dimensional. Thus \mathbf{G}_o^7 has vertex o and is isomorphic to the cone over $\mathbf{G}(3, U_0)$.

6.6 The rational map

$$v^+ : \mathbf{P}_*(\Lambda^{ev}) \dashrightarrow \mathbf{P}^9 = \mathbf{P}_*(W), a \mapsto [q_1^-(a) : \dots : q_5^+(a)]$$

defined by the 10 quadrics maps a section of $a \in \mathbf{P}_*(\Lambda^{ev}) = \mathbf{P}_*(H^0(Q, E^{ev}))$ to its zero loci, which is a point on Q for general sections since $c_8(E^{ev}) = 1$. Indeed

$$q_1^- \cdot q_1^+ + \dots + q_5^- \cdot q_5^+ = 0,$$

so the image is $Q \subset \mathbf{P}^9$. To see the statement about the zero loci we notice that from a different point of view the map

$$\partial^{ev} : \Lambda^{ev} \rightarrow W^* \otimes \Lambda^{odd}$$

coincides with the first syzygy matrix of \mathbf{S}_{ev}

$$\varphi_2 : \Lambda^{ev} \rightarrow W^* \otimes (\Lambda^{ev})^*$$

if we identify $\Lambda^{odd} \cong (\Lambda^{ev})^*$ via the wedge-product pairing into $\Lambda^5 U_\infty$. For a point $[a] \in \mathbf{P}^{15} \setminus \mathbf{S}_{ev}$ the matrix $\varphi_2(a)$ has rank 9 precisely and $[q_1^+(a) : \dots : q_5^+(a)]$ represents the cokernel. This means that the section $[a] \in \mathbf{P}^{15}$ vanishes at $[q_1^+(a) : \dots : q_5^-(a)] \in Q$.

6.7 Consider the inverse image by the map v^+ of an isotropic \mathbf{P}^4 . In both families this can be computed from the equations. For the isotropic subspace $P_\infty = \mathbf{P}_*(U_\infty) \subset Q$ the inverse image by v^+ is defined by

$$q_1^+ = q_2^+ = q_3^+ = q_4^+ = q_5^+ = 0$$

Let H be the hyperplane defined by $x_0 = 0$. The quadrics, restricted to H , are the same Plücker quadrics as above. This time they define an 11-dimensional cone $G_{P_\infty}^{11}$ isomorphic to a cone over $\mathbf{G}(2, U_\infty)$. Note that the hyperplane H belongs to the dual variety. It is tangent to \mathbf{S}_{ev} along the contact locus C_H , the \mathbf{P}^4 which is the vertex of the 11-dimensional cone $G_{P_\infty}^{11}$, it is defined by $x_{ij} = 0$, all $i < j$ and coincides naturally with the dual space to $\mathbf{P}_*(U_\infty)$. On the complement of H , where $x_0 = 1$, we get as noted in (6.3) precisely the image of the exponential map. From the equations we get that the quadrics q_i^+ define the graph in $\text{span} \langle T_{\mathbf{S},o}, C_H \rangle = \mathbf{P}^{15}$ of the rational map of $T_{\mathbf{S},o}$ into C_H defined by the Plücker quadrics. This graph is 10 dimensional and its closure in \mathbf{P}^{15} is of course again just \mathbf{S}_{ev} (cf. also [Ein 1986]). Thus the inverse image of P_∞ is the 11-dimensional cone $G_{P_\infty}^{11}$ which intersect \mathbf{S}_{ev} along a tangent hyperplane section H defined by $x_0 = 0$. Therefore the isotropic space P_∞ canonically correspond to the hyperplane which contains its inverse image by v^+ . The point $H \in \mathbf{S}_{odd}$ is the point associated to P_∞ .

6.8 For the isotropic space $P_0 = \mathbf{P}_*(U_0)$, the inverse image by v^+ is given by

$$q_1^- = q_2^- = q_3^- = q_4^- = q_5^- = 0.$$

As noted in (6.5) this is the cone with vertex $o = (1 : 0 : \dots : 0) \in \mathbf{S}_{ev}$ over a generic syzygy variety. Two isotropic subspaces of Q which intersect in a \mathbf{P}^3 form the intersection of Q with a \mathbf{P}^5 . Therefore the union of the preimages of the two subspaces by v^+ is the complete intersection of 4 quadrics. Since the preimage of P_∞ has degree 5 and dimension 11 the preimage of P_0 must therefore have degree 11 and dimension 11.

Note that any hyperplane containing $T_{\mathbf{S},o}$ will intersect the preimage of P_0 in $T_{\mathbf{S},o}$ and a residual variety which intersect $T_{\mathbf{S},o}$ along a quadric.

7. The apolar Artinian Gorenstein ring of a general cubic threefold

Proof of (0.10): Recall from (6.4) that the homogeneous coordinate ring R_S of the spinor variety $\mathbf{S} \subset \mathbf{P}^{15}$ is Gorenstein with syzygies

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array} .$$

The spinor variety has dimension 10, so eleven general linear forms h_0, \dots, h_{10} define a \mathbf{P}^4 which does not intersect \mathbf{S} . Therefore the quotient $A = R_S/(h_0, \dots, h_{10})$ is an Artinian Gorenstein ring. Its Hilbert function is $(1, 5, 5, 1)$ and it has socle degree 3, so A is the apolar Artinian Gorenstein ring A^F for some cubic hypersurface $F \subset \mathbf{P}^4$. Now, let X be the open set in the Grassmannian $\mathbf{G}(5, \Lambda^{ev})$ of subspaces which does not intersect the spinor variety $\mathbf{S}+$. The action of $SO(W, q)$ on Λ^{ev} induces an action on X . Clearly, any isomorphism between two quotients A and A' is induced by an automorphism of \mathbf{P}^4 . Therefore, the above construction defines a map $m : X/SO(W, q) \rightarrow H_3$, to the space of cubics in \mathbf{P}^4 modulo projective equivalence. That this map is injective follows from the

7.1 Lemma. *For any cubic F in the image of m there is precisely one quadratic relation between the quadrics in F_2^\perp*

The proof of this lemma is postponed until (8.7).

The dimension of $X/SO(W, q)$ is $\dim \mathbf{G}(5, 16) - \dim SO(W, q) = 55 - 45 = 10$ while $\dim H_3 = h^0(\mathcal{O}_{\mathbf{P}^4}(3)) - \dim GL(\mathbf{C}^5) = 35 - 25 = 10$ so the map m is dominating onto its image components. But both source and target are irreducible, so the image of m must be a dense subvariety, and (0.10) follows. \square

We conjecture that the generality condition on A can be made precise via syzygies:

7.2 Conjecture. *An Artinian Gorenstein ring A with Hilbert function $(1, 5, 5, 1)$ is isomorphic to a quotient $R_S/(h_0, \dots, h_{10})$ for some linear forms $h_0, \dots, h_{10} \in R_S$ iff A has syzygies*

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array} .$$

as $\mathbf{C}[x_0, \dots, x_4]$ -module.

This conjecture implies Mukai's Theorem (Theorem 0.9) by applying the concept of 'very rigidity' cf. [Buchweitz 1981]. For a different proof of Mukai's Theorem see also [Euseen, Schreyer 1995].

8. Proof of the main results

8.0 Let A^F be the homogeneous coordinate ring of the *empty* intersection of the spinor variety \mathbf{S} with a general $P = \mathbf{P}^4$, i. e. the apolar Artinian Gorenstein ring of a general cubic $F \in \mathbf{P}^4$.

To describe $VSP(F, 8)$ we follow the procedure of section 3 and consider duality and the projection from partials π_2^F . The projection from partials $\pi_2^F : P \rightarrow \mathbf{P}^9$ is nothing but the restriction of the map v^+

to P , and the points in $VSP(F, 8)$ representing smooth apolar subschemes to F of degree 8 correspond to proper 8-secant planes to the image $V_P = \pi_2^F(P) \subset \mathbf{P}^9$. We show in lemma 8.4 below that any such 8-secant plane is contained in the quadric $Q \subset \mathbf{P}^9$ which is the image of v^+ . Since any plane in Q is the intersection of a pencil of isotropic \mathbf{P}^4 's, the preimage of a plane is contained in a pencil of tangent hyperplanes to \mathbf{S} , cf. 6.7. Thus any plane in Q correspond to a line in the dual variety \mathbf{S}_{odd} to $\mathbf{S} = \mathbf{S}_{ev}$.

Recall that the preimage of an isotropic \mathbf{P}^4 corresponding to a point on \mathbf{S}_{odd} is a 11-dimensional cone over the Grassmannian variety $\mathbf{G}(2, 5)$ contained in a tangent hyperplane. Clearly P intersects this inverse image properly in a curve precisely when P is contained in the tangent hyperplane. In this case the curve of intersection is an elliptic quintic curve. Therefore $Y = \mathbf{P}^{10} \cap \mathbf{S}_{odd} \subset \mathbf{P}_*(\Lambda^{odd})$ where \mathbf{P}^{10} is dual to $P \subset \mathbf{P}_*(\Lambda^{ev})$, parameterizes apolar elliptic quintic curves to F , i. e. defines a subvariety of $VPS(F, 5t)$ in the notation of 1.3.

On the other hand, it follows from (5.2) that apolar subschemes of degree 8 are contained in a pencil of apolar elliptic quintic curves. Let M_Y denote the Fano variety of lines in Y . We show that for general cubic 3-fold F , the variety M_Y is smooth of dimension 5 and that every point on M_Y correspond to a finite apolar subscheme to F of length 8, i. e. that F^5 coincide with $VSP(F, 8)$.

8.1 Lemma. *Let $Y = \mathbf{P}^{10} \cap \mathbf{S}_{odd} \subset \mathbf{P}_*(\Lambda^{odd})$ for a general \mathbf{P}^{10} , and let M_Y be the Fano variety of lines in Y . Then M_Y is smooth of dimension 5.*

Proof. Recall for each point $H \in \mathbf{S}_{odd}$ the tangent cone $G_H^7 = \mathbf{S}_{odd} \cap T_H$ with vertex H inside the tangent space T_H is isomorphic to a 7-dimensional cone over the Grassmannian of planes in the isotropic \mathbf{P}^4 corresponding to H . For each point H in Y let $g_H^2 = G_H^7 \cap \mathbf{P}^{10} \subset Y$. Then g_H^2 is the cone over an elliptic normal quintic curve. In particular there is an elliptic curve of lines through a general point H in Y . Since a line is a pencil of points, Y contains a 5-dimensional family of lines, i. e. M_Y has dimension 5.

For smoothness we consider the normal bundle N_L of a line L in Y . M_Y is smooth as soon as $h^1(N_L) = 0$ for every line L in Y . This is verified by a dimension count. For each line L in \mathbf{S} the normal bundle is $3\mathcal{O}_L + 6\mathcal{O}_L(1)$ (cf. [Ein 1986]). Since Y is a section of \mathbf{S}_{odd} with a \mathbf{P}^{10} , the normal bundle sequence becomes

$$0 \longrightarrow N_L \longrightarrow 3\mathcal{O}_L \oplus 6\mathcal{O}_L(1) \longrightarrow 5\mathcal{O}_L(1) \longrightarrow 0$$

This is exact so the rank of the map $\alpha_L : 6\mathcal{O}_L(1) \longrightarrow 5\mathcal{O}_L(1)$ is at least 3; otherwise the map would have a summand $3\mathcal{O}_L \longrightarrow 3\mathcal{O}_L(1)$ which cannot be surjective at every point of L . On the other hand $h^1(N_L) \neq 0$ only if N_L has a summand $\mathcal{O}_L(-d)$ with $d \geq 2$. Therefore L is a singular point on M_Y only if the map α_L has rank 3. But the set of flags (L, \mathbf{P}^{10}) of lines L in $\mathbf{S}_{odd} \cap \mathbf{P}^{10}$ where α_L has rank 3 is a determinantal variety of codimension 6 in the variety of all flags so for general Y , M_Y is smooth. \square

For the correspondence between 8-secant planes to V_P and lines in Y , we first study the fibers of the map v^+ more carefully.

Let $Bl_{\mathbf{S}} : \tilde{\mathbf{P}} \rightarrow \mathbf{P}_*(\Lambda^{ev})$ be the the blowup map along \mathbf{S} , then v^+ lifts to a morphism $\tilde{v}^+ : \tilde{\mathbf{P}} \rightarrow Q \subset \mathbf{P}^9$. For a subscheme $Z \subset Q$ let $F_Z = Bl_{\mathbf{S}}((\tilde{v}^+)^{-1}(Z))$. By abuse of notation we often call F_Z the preimage of Z by v^+ . We consider first the preimage of points, lines and planes.

8.2 Lemma. *The fiber of v^+ , i. e. F_p for a point $p \in Q$, is a linear \mathbf{P}^7 which intersect \mathbf{S} in a smooth quadric hypersurface. The preimage of a line, i. e. F_L for a line $L \subset Q$ is a rational normal quartic scroll in a \mathbf{P}^{11} with vertex a \mathbf{P}^3 . The preimage of a conic section, which is not contained in Q , is a rational normal scroll of dimension 8 and degree 8.*

Proof. By homogeneity we may choose any point and any line and compute the preimage from the equations.

Consider the point on Q with $y_1 = q_1^+$ as the only nonzero coordinate. The other quadrics all vanish on the $\mathbf{P}^7 = \{x_{12} = x_{1345} = x_{13} = x_{1245} = x_{14} = x_{1235} = x_{15} = x_{1234} = 0\}$ which is the singular locus of q_1^- , and q_1^+ defines in it a smooth quadric. For any point outside the \mathbf{P}^7 one of the variables defining it is nonzero. But on the complement of any coordinate hyperplane \mathbf{S} is defined by 5 quadrics as explained in (6.3), and q_1^+ is not among these 5 quadrics for any of the coordinates defining the \mathbf{P}^7 . Therefore any point outside this \mathbf{P}^7 on all but the first quadric also lie on the first quadric. Thus we have recovered F_p as a \mathbf{P}^7 which intersect \mathbf{S}_{ev} in a quadric. In particular the fiber over a point intersect \mathbf{S}_{ev} in codimension one.

Next look at the line in Q with all coordinates but $y_1 = q_1^-$ and $y_2 = q_2^-$ equal zero. In this case the remaining quadrics

$$\begin{aligned}
q_1^+ &= x_0x_{2345} + x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25} \\
q_2^+ &= x_0x_{1345} - x_{13}x_{45} + x_{14}x_{35} - x_{34}x_{15} \\
q_3^+ &= x_0x_{1245} + x_{12}x_{45} - x_{14}x_{25} + x_{24}x_{15} \\
q_3^- &= -x_{13}x_{2345} - x_{23}x_{1345} + x_{34}x_{1235} + x_{1234}x_{35} \\
q_4^+ &= x_0x_{1235} - x_{12}x_{35} + x_{13}x_{25} - x_{23}x_{15} \\
q_4^- &= -x_{14}x_{2345} - x_{24}x_{1345} - x_{34}x_{1245} + x_{1234}x_{45} \\
q_5^+ &= x_0x_{1234} + x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} \\
q_5^- &= -x_{15}x_{2345} - x_{25}x_{1345} - x_{35}x_{1245} - x_{1235}x_{45}
\end{aligned}$$

cut the union of \mathbf{S} and the rational quartic scroll in $\mathbf{P}^{11} = \{x_0 = x_{34} = x_{45} = x_{35} = 0\}$ defined by the 2×2 minors of:

$$\begin{pmatrix} x_{13} & x_{14} & x_{15} & x_{1345} \\ x_{23} & x_{24} & x_{25} & -x_{2345} \end{pmatrix}$$

In fact the 5 quadrics defining \mathbf{S} on the complement of any of the coordinate hyperplanes which contain \mathbf{P}^{11} does not involve q_1^- or q_2^- , so any point outside the \mathbf{P}^{11} on the 8 quadrics lie on \mathbf{S} .

Thus the preimage F_L of the line is a rational normal quartic scroll in \mathbf{P}^{11} with vertex a \mathbf{P}^3 . One can check that the intersection of \mathbf{P}^{11} with \mathbf{S} is not proper, in fact this intersection has codimension one in the fiber over the line.

For the preimage of a conic section, the computation is similar. \square

A plane $\pi \subset Q$ is contained in a pencil of isotropic \mathbf{P}^4 's of each family on Q . These pencils of isotropic \mathbf{P}^4 's correspond to a line $L_\pi \subset \mathbf{S}$ and by duality a pencil of tangent hyperplanes defining a $\mathbf{P}_\pi^{13} \subset \mathbf{P}_*(\Lambda^{ev})$. The following lemma should be compared with (5.2).

8.3 Lemma. *The preimage by v^+ of a plane, i. e. F_π for a plane $\pi \subset Q$ is a cone C_π over a codimension 4 subvariety of degree 8 and sectional genus 3 with vertex line $L_\pi \subset \mathbf{S}$ inside \mathbf{P}_π^{13} . More precisely, the tangent spaces to \mathbf{S} along L_π form the rulings of a rational cubic scroll and F_π is the baselocus of a pencil of divisors on this scroll determined by the tangent hyperplanes $H \supset \mathbf{P}_\pi^{13}$. These divisors are all isomorphic to the 10-dimensional cone over a tangent hyperplane section of $\mathbf{G}(2, 5)$.*

Proof. We consider the plane π in Q with all coordinates but $y_1 = q_1^-$, $y_2 = q_2^-$ and $y_3 = q_3^-$ equal zero. The remaining quadrics

$$\begin{aligned}
q_1^+ &= x_0x_{2345} + x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25} \\
q_2^+ &= x_0x_{1345} - x_{13}x_{45} + x_{14}x_{35} - x_{34}x_{15} \\
q_3^+ &= x_0x_{1245} + x_{12}x_{45} - x_{14}x_{25} + x_{24}x_{15} \\
q_4^+ &= x_0x_{1235} - x_{12}x_{35} + x_{13}x_{25} - x_{23}x_{15} \\
q_4^- &= -x_{14}x_{2345} - x_{24}x_{1345} - x_{34}x_{1245} + x_{1234}x_{45} \\
q_5^+ &= x_0x_{1234} + x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} \\
q_5^- &= -x_{15}x_{2345} - x_{25}x_{1345} - x_{35}x_{1245} - x_{1235}x_{45}
\end{aligned}$$

cut the union of \mathbf{S} and a cone C_π over a codimension 4 subvariety of degree 8 and sectional genus 3 with a vertex line

$$L_\pi : \quad x_{ij} = 0 \quad ij \neq 45 \quad x_{2345} = x_{1345} = x_{1245} = 0$$

in $\mathbf{P}_\pi^{13} = \{x_0 = x_{45} = 0\}$. In fact any point outside \mathbf{P}_π^{13} on the 7 quadrics lie on \mathbf{S} like above.

In each tangent hyperplane $H \supset \mathbf{P}_\pi^{13}$ the preimage G_H^{11} of the isotropic \mathbf{P}^4 corresponding to H is defined by 5 quadrics. For the pencil of tangent hyperplanes there is a net of common quadrics, the 2×2 minors of the matrix

$$\begin{pmatrix} x_{14} & x_{24} & x_{34} \\ x_{15} & x_{25} & x_{35} \end{pmatrix}$$

which define a rational cubic scroll of dimension 11 with vertex a \mathbf{P}^7 inside \mathbf{P}^{13} . For each H , the intersection $G_H^{11} \cap \mathbf{P}^{13}$ has dimension 10 and is a divisor on this cubic scroll. Now G_H^{11} is naturally the cone over the Grassmannian $\mathbf{G}(2, 5)$ of lines in the isotropic \mathbf{P}^4 corresponding to H , while the pencil of isotropic \mathbf{P}^4 's corresponding to the pencil of hyperplanes through \mathbf{P}^{13} meet in a plane. Therefore $G_H^{11} \cap \mathbf{P}^{13}$ is isomorphic to the cone over a tangent hyperplane section of $\mathbf{G}(2, 5)$. The preimage F_π is the intersection on cubic scroll of this pencil of divisors.

Explicitly F_π is defined by the above net of quadrics and the 4 quadric entries of the product matrix :

$$\begin{pmatrix} x_{14} & x_{24} & x_{34} \\ x_{15} & x_{25} & x_{35} \end{pmatrix} \cdot \begin{pmatrix} x_{23} & x_{2345} \\ -x_{13} & x_{1345} \\ x_{12} & x_{1245} \end{pmatrix}.$$

Each of the divisors on the cubic scroll is defined by an element in the column space of the second factor above. The vertex of this divisor as a cone is the \mathbf{P}^4 defined by the corresponding column vector in the vertex \mathbf{P}^7 of the cubic scroll.

A point on the line L_π correspond to a vector in the row space of the first matrix factor above. In fact the entries in the row vector vanish in \mathbf{P}^{13} exactly along the tangent \mathbf{P}^{10} to \mathbf{S} at the corresponding point. The inverse image of the isotropic subspace with vertex at a point on the line L_π restricts to the \mathbf{P}^{13} as the union of this tangent space and the cone F_π . In fact the tangent spaces to \mathbf{S} along the vertex line form the ruling of the distinguished cubic scroll that contains F_π , and the \mathbf{P}^7 inside F_π is precisely the intersection of these tangent spaces. \square

Thus v^+ identifies the line in both \mathbf{S}_{ev} and \mathbf{S}_{odd} which correspond to a plane in Q . One can check that the intersection of \mathbf{S}_{ev} with the inverse image F_π has codimension 2.

Next we restrict v^+ to the $P = \mathbf{P}^4 \subset \mathbf{P}_*(\Lambda^{ev})$ which correspond to the apolar Artinian Gorenstein ring A^F . Note that P does not intersect \mathbf{S} , and v^+ have only linear fibers which intersect \mathbf{S} in codimension 1, so this restriction is an embedding. Like above we set $V_P = v^+(P) \subset Q \subset \mathbf{P}^9$.

8.4 Lemma. *Any proper 8-secant plane to V_P is contained in the quadric Q .*

Proof. Since V_P is contained in Q , the intersection of V_P with any plane which is not contained in Q is contained in a conic section. But the preimage of a conic section is a rational normal scroll of degree 8 and codimension 7 by (8.2). This scroll cannot meet any \mathbf{P}^4 in 8 points, so the lemma follows. \square

8.5 Lemma. *Assume that P does not intersect the spinor variety, and that the intersection Y of the dual $\mathbf{P}^{10} = P^\perp \subset \mathbf{P}_*(\Lambda^{odd})$ with the dual spinor variety is smooth and has a smooth Fano variety of lines M_Y . Then every line in Y correspond to a plane in $Q \subset \mathbf{P}^9$ which intersect V_P along a finite scheme of length 8.*

Proof. Any line $l_\pi \subset Y$ correspond to a dual $\mathbf{P}^{13} \subset \mathbf{P}_*(\Lambda^{ev})$ which contain P . The line l_π correspond to a plane $\pi \subset Q \subset \mathbf{P}^9$ whose preimage F_π by v^+ span this \mathbf{P}^{13} . The preimage of $\pi \cap V_P$ is therefore the intersection $P \cap F_\pi$. But F_π has degree 8 and codimension 4, by (8.3), so as soon as the intersection $P \cap F_\pi$ is proper the lemma follows.

On the other hand since v^+ restricted to P is an embedding, the intersection $\pi \cap V_P$ contains at most a curve. And, since the map is defined by quadrics, any plane curve on V_P is a conic section, the image of a line on P . Therefore we need only to rule out the existence of a line L in the intersection $P \cap F_\pi$.

Assume that $P \cap F_\pi$ contains a line L . For a case by case argument we use the coordinates of the proof of (8.3). Thus \mathbf{P}^{13} is defined by $\mathbf{P}^{13} = \{x_0 = x_{45} = 0\}$ and F_π is defined by the by the 2×2 minors of the matrix

$$\mathcal{M} = \begin{pmatrix} x_{14} & x_{24} & x_{34} \\ x_{15} & x_{25} & x_{35} \end{pmatrix}$$

and the quadratic entries of the product matrix:

$$\mathcal{M} \cdot \mathcal{N} = \begin{pmatrix} x_{14} & x_{24} & x_{34} \\ x_{15} & x_{25} & x_{35} \end{pmatrix} \cdot \begin{pmatrix} x_{23} & x_{2345} \\ -x_{13} & x_{1345} \\ x_{12} & x_{1245} \end{pmatrix}.$$

The intersection of F_π with \mathbf{S} is defined with the additional three quadrics

$$\begin{aligned} q_1^- &= x_{12}x_{1345} + x_{13}x_{1245} + x_{14}x_{1235} + x_{15}x_{1234} \\ q_2^- &= -x_{12}x_{2345} + x_{23}x_{1245} + x_{24}x_{1235} + x_{1234}x_{25} \\ q_3^- &= -x_{13}x_{2345} - x_{23}x_{1345} + x_{34}x_{1235} + x_{1234}x_{35} \end{aligned}$$

The 2×2 -minors of \mathcal{M} define the distinguished cubic scroll S_π which contain F_π .

The row vectors of \mathcal{M} define the ruling of the cubic scroll and it is the intersection with tangent spaces to \mathbf{S}_{ev} with the \mathbf{P}^{13} . Since any tangent space intersect \mathbf{S} in codimension 3, the intersection of P with a tangent space is at most a plane, i. e. the rank of any row vector of \mathcal{M} restricted to P is at least 2. If every row vector has rank 2 restricted to P , then $S = S_\pi \cap P$ is an irreducible quadric hypersurface in P , otherwise S is a possibly reducible cubic surface scroll.

Now, the line L is clearly contained in S . Consider the rank of a general row vector of \mathcal{M} restricted to L . This rank may be 0, 1 or 2. If the rank is 0 then S is a cone with L inside the vertex. If the rank is 1 then L intersect the general ruling of S in a point, and if the rank is 2 then L is contained in one ruling and do not intersect any other ruling of S . We go case by case.

When \mathcal{M} have rank 0 on L , then the three additional quadrics reduce to the 2×2 minors of \mathcal{N} . Therefore L need not intersect the spinor variety. But we will show that the line $l_\pi \subset Y$ in this case correspond to a singular point on the Fano variety M_Y of lines on Y . Now, the rulings in S all pass through L ; if S is a surface it is three plans through L , if it is a quadric it is a cone with vertex L , in either case it contains three planes which together span P . Each plane is the intersection of P with a tangent space, so dually each of these planes correspond to planes in Y through the line l_π . Moreover, these planes in Y must span a \mathbf{P}^4 ; otherwise Y contains a \mathbf{P}^3 . Any $\mathbf{P}^3 \subset Y$ correspond by duality to a \mathbf{P}^{11} containing P and the tangent space T_p to \mathbf{S} at a point p . Thus P would intersect this tangent space in a \mathbf{P}^3 . But the tangent cone to \mathbf{S} has codimension 3 inside T_p , cf. (6.5), so P would in fact intersect \mathbf{S} , which is absurd. Since the planes in Y which contain l_π span a \mathbf{P}^4 , the normal bundle of l_π in Y has at least 6 sections, 2 from each of three independent planes. But the Fano variety M_Y has dimension 5 so the line l_π must be a singular point on M_Y , contrary to our assumption.

When the general row of \mathcal{M} have rank 1 on L , then we may assume that x_{14} and x_{15} are the only nonzero terms of \mathcal{M} on L . The remaining quadrics defining F_π restrict to L as the quadratic forms $x_{14}x_{23} = x_{14}x_{2345} = x_{15}x_{23} = x_{15}x_{2345} = 0$. The last three quadrics defining the intersection with the spinor variety reduce to the quadric $x_{12}x_{1345} + x_{13}x_{1245} + x_{14}x_{1235} + x_{15}x_{1234}$, so the line must intersect the spinor variety, contrary to our assumption that P does not.

When the general row of \mathcal{M} have rank 2 on L , then we may assume that x_{14} and x_{24} are the only nonzero terms on L . The remaining quadrics defining F_π restrict to L as the quadric forms $x_{14}x_{23} - x_{24}x_{13} = x_{14}x_{2345} + x_{24}x_{1345} = 0$. The last three quadrics defining the intersection with the spinor variety reduce to the quadrics

$$x_{12}x_{1345} + x_{13}x_{1245} + x_{14}x_{1235} = -x_{12}x_{2345} + x_{23}x_{1245} + x_{24}x_{1235} = -x_{13}x_{2345} - x_{23}x_{1345}.$$

These 5 quadrics are the principal Pfaffians of the following 5×5 -dimensional skew symmetric matrix:

$$\begin{pmatrix} 0 & 0 & -x_{24} & -x_{2345} & x_{23} \\ 0 & 0 & x_{14} & -x_{1345} & -x_{13} \\ x_{24} & -x_{14} & 0 & x_{1245} & -x_{12} \\ x_{2345} & x_{1345} & -x_{1245} & 0 & -x_{1235} \\ x_{23} & x_{13} & x_{12} & x_{1235} & 0 \end{pmatrix}$$

The two quadrics defining F_π are the last two principal Pfaffians of this matrix. When these vanish on the line L , then all the Pfaffians vanish on the point defined by x_{1235} on L , so again L must intersect the spinor variety.

We have shown that the intersection $P \cap F_\pi \subset \mathbf{P}^{13}$ is proper for any plane π corresponding to a line in Y , so the lemma follows. \square

From (0.10), (8.2), (8.4) and (8.5) we get

8.6 Theorem. *Let $F \subset \mathbf{P}^4$ be a general cubic. There exists a linear subspace $\mathbf{P}^{10} \subset \mathbf{P}^{15}$, which depends on F , such that $VSP(F, 8)$ is isomorphic to the variety of lines in the 5-fold $Y = Y(F) := \mathbf{P}^{10} \cap \mathbf{S}^{10} \subset \mathbf{P}^{15}$. Moreover $VSP(F, 8)$ is smooth of dimension 5.*

8.7 Proof of (7.1). For this proof we investigate the geometry of M_Y a bit further. Recall from the proof of (8.1) that every point H in Y is the vertex of a cone g_H over an elliptic quintic curve. But each line in the cone g_H correspond to a plane in P_H , the isotropic \mathbf{P}^4 corresponding to H . So for each $H \in Y$ there is an elliptic quintic scroll V_H of planes in P_H . This threefold scroll is singular along an elliptic quintic surface scroll S_H of lines in P_H . Turning to the dual space, note that for a general \mathbf{P}^{10} the dual $P = \mathbf{P}^4$ do not intersect \mathbf{S}_{ev} at all. The space P meets each cone G_H^{11} for a general $H \in \mathbf{S}_{odd}$ in 5 points, but for a point H in Y , the span of G_H^{11} contains P and the intersection is an elliptic curve E_H , the intersection of a Grassmannian with a general linear space. For two points H and H' on a line, G_H^{11} and $G_{H'}^{11}$ have a common line in the vertex, it is the dual of the plane of intersection of P_H and $P_{H'}$. Since G_H^{11} is the cone over the Grassmannian of lines in P_H each point on E_H determines a line in $P_H \subset Q$. But P lies in the span of both G_H^{11} and $G_{H'}^{11}$, so P intersects G_H^{11} only in fibers of lines in P_H which meet $P_{H'}$. The elliptic curve E_H altogether determines an elliptic scroll of lines. Since each line in the scroll meet the plane $P_H \cap P_{H'}$ for any $H' \in g_H$, the scroll must coincide with the scroll S_H in P_H described above.

Now, any line in Y correspond to an 8-secant plane to the image $v^+(P) = V_P \subset \mathbf{P}^9$. On the scroll S_H these planes are the planes of plane cubic curves. The image of E_H is an elliptic curve of degree 10 on the scroll S_H , which must therefore intersect the plane cubic curves on S_H in 8 points. More precisely the image of E_H is a curve of type $C_0 + 7R$ on S_H , where C_0 is a section with selfintersection 1 and R is a member of the ruling.

Since the scroll S_H is not contained in any quadric, it follows that the image of E_H is not contained in any quadrics either. Therefore the isotropic \mathbf{P}^4 of any apolar elliptic curve E_H is contained in any quadric which contains V_P . But for a general point on the quadric Q there is a 6-dimensional quadric of isotropic \mathbf{P}^4 's through the point, they form a 6-dimensional quadric in the spinor variety, and necessarily intersects the 5-dimensional subvariety Y . Therefore the isotropic subspaces of apolar elliptic curves fill all of Q , and this is the unique quadric containing V_P . \square

9. Invariants of $VSP(F, 8)$

9.1 To find the invariants of M_Y we describe the Fano variety M of lines in $\mathbf{S}(= \mathbf{S}_{ev})$ via the correspondence with the Fano variety of planes in the quadric Q . These varieties are isomorphic. Each line in \mathbf{S} correspond to a pencil of isotropic \mathbf{P}^4 's of the same family through a plane in the quadric Q , and this correspondence is made precise by the map v^+ (cf. 8.3). In fact v^+ induces a map of $\mathbf{G}(3, \mathcal{N}(-2h) = \mathbf{G}(3, B)$ into $\mathbf{G}(3, W)$, so that the universal subbundle on $\mathbf{G}(3, B)$ is induced from $M \subset \mathbf{G}(3, 10)$. The Fano variety F_q of planes in Q is easily described in the Grassmannian $G = \mathbf{G}(3, 10)$ of planes in \mathbf{P}^9 . But we need to identify the subvariety M_Y so we compute $M \subset \mathbf{G}(2, 16)$.

9.2 Proposition. *M_Y is a Fano 5-fold of index 1 and degree 660.*

Proof. As above, let M denote the Fano variety of lines in the spinor variety \mathbf{S} and let $G \subset \mathbf{S} \times M$ denote the universal family. $pr_1: G \rightarrow \mathbf{S}$ is a $\mathbf{G}(3, 5)$ -bundle. We compute the cohomology rings of \mathbf{S} , G and M , starting with \mathbf{S} .

Let B denote the tautological rank 5 subbundle on the spinor variety \mathbf{S} with tautological sequence

$$0 \rightarrow B \rightarrow 10\mathcal{O}_{\mathbf{S}} \rightarrow B^* \rightarrow 0.$$

Then $\Omega_{\mathbf{S}}^1 \cong \Lambda^2 B \subset \text{Hom}(B, B^*)^*$. The cohomology ring $H^*(\mathbf{S}, \mathbf{Q})$ is generated by the Chern classes of B and the hyperplane class h of $\mathbf{S} \subset \mathbf{P}^{15}$: Relations among these classes arise from the tautological sequence, the conormal bundle sequence

$$0 \rightarrow \mathcal{N}_{\mathbf{S} \subset \mathbf{P}^{15}}^* \rightarrow \Omega_{\mathbf{P}^{15}}^1|_{\mathbf{S}} \rightarrow \Omega_{\mathbf{S}}^1 \rightarrow 0$$

and $\mathcal{N}_{\mathbf{S} \subset \mathbf{P}^{15}}^* \cong B^*(-2h)$.

With $b_i = c_i(B)$ these relations give $b_1 = -2h, b_2 = 2h^2, b_4 = -2h^4 - 2hb_3, b_5 = 0$ and

$$b_3^2 + 8b_3h^3 + 8h^6 = 0 = 6h^5b_3 + 7h^8.$$

The last 2 equations define a complete intersection in $\mathbf{Q}[h, b_3]$. Since

$$h^{10} = 12 \in H^{20}(\mathbf{S}, \mathbf{Q}) \cong \mathbf{Q},$$

there are no further relations between h and b_3 . To deduce

$$H^*(\mathbf{S}, \mathbf{Q}) \cong \mathbf{Q}[h, b_3]/(b_3^2 + 8b_3h^3 + 8h^6, 6h^5b_3 + 7h^8)$$

we note that the odd cohomology of \mathbf{S} vanishes and compute the Euler number of \mathbf{S} :

$$e(\mathbf{S}) = c_{10}(\Omega_{\mathbf{S}}^1) = \frac{4}{3}h^{10} = 16,$$

which equals the length of the right hand side ring above.

$G \cong \mathbf{G}(3, B)$. So $H^*(G, \mathbf{Q})$ is generated as $H^*(\mathbf{S}, \mathbf{Q})$ -algebra by the Chern classes u_1, u_2, u_3 of the universal subbundle U and relation follow from the exact sequence

$$0 \rightarrow U \rightarrow B_G \rightarrow Q \rightarrow 0.$$

This gives

$$H^*(G, \mathbf{Q}) \cong H^*(\mathbf{S}, \mathbf{Q})[u_1, u_2]/(f, g)$$

with

$$f = h^4 - h^2u_2 - \frac{1}{2}u_2^2 - \frac{1}{2}b_3u_1 + 2h^3u_1 - 2hu_2u_1 + 3h^2u_1^2 - \frac{1}{2}u_2u_1^2 + 2hu_1^3 + \frac{1}{2}u_1^4$$

and

$$g = b_3h^2 - \frac{1}{2}b_3u_2 + 2h^3u_2 - hu_2^2 + b_3hu_1 + 3h^2u_2u_1 - 2u_2^2u_1 \\ - \frac{1}{2}b_3u_1^2 + 2h^2u_1^3 + \frac{1}{2}u_2u_1^3 + 2hu_1^4 + \frac{1}{2}u_1^5.$$

$pr_2: G \rightarrow M$ is a \mathbf{P}^1 -bundle: $G \cong \mathbf{P}(E)$ with $E = pr_{2*}pr_1^*\mathcal{O}(h)$. Hence

$$H^*(G, \mathbf{Q}) \cong H^*(M, \mathbf{Q})[h]/(h^2 - c_1h + c_2)$$

with $c_1 = c_1(E), c_2 = c_2(E)$. To compute these classes we note that u_1, u_2, u_3 are classes on M , since U is induced from $M \subset \mathbf{G}(3, 10)$ and compute

$$Ann(u_1^{14}) = (u_1^2, u_2, h^2 + u_1h) \subset H^*(G, \mathbf{Q}).$$

Since $H^4(M, \mathbf{Q}) \subset H^4(G, \mathbf{Q})$ is 3- respectively 4-dimensional $u_1^2, u_2, h^2 + u_1h \in H^4(M, \mathbf{Q})$. So

$$c_2 = -h^2 - u_1h, \quad c_1 = -u_1.$$

For canonical classes we obtain

$$K_{\mathbf{S}} = c_1(\Omega_{\mathbf{S}}^1) = -8h, K_{G/\mathbf{S}} = c_1(\mathcal{H}om(U, Q)^*) = 6h + 5u_1,$$

hence $K_G = -2h + 5u_1, K_{G/M} = -2h + c_1(E) = -2h - u_1$, so

$$K_M = 6u_1.$$

$M_Y \subset M$, the set of lines in $Y = \mathbf{S} \cap \mathbf{P}^{10} \subset \mathbf{P}^{15}$, is defined by the vanishing of 5 sections of $E = pr_{2*}pr_1^*\mathcal{O}(h)$. Hence $[M_Y] = (-h^2 - u_1h)^5$ and $K_{M_Y} = K_M + 5c_1(E) = u_1$. So M_Y is Fano with anti-canonical class $-u_1$ induced by the Plücker embedding of $M_Y \subset M \subset \mathbf{G}(3, 10) \subset \mathbf{P}^{119}$. Since

$$(-h^2 - u_1h)^5(-u_1)^5h = 11u_1^6h^{10} \in H^*(G, \mathbf{Q}),$$

the degree of the anti-canonical embedding of M_Y is $11 \cdot 5 \cdot 12 = 660$. In particular $-K_{M_Y}$ is not divisible in $\text{Pic}(M_Y)$, i. e. M_Y is a Fano variety of index 1. \square

Theorem (0.8) now follows from (8.6) and (9.2).

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