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Generic sequences of polynomials

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ABSTRACT

Let I be an ideal in a polynomial ring S over an infinite field such that I is generated by a generic sequence of homogeneous polynomials of specified degrees. Fröberg has conjectured a formula for the Hilbert series of S/I. Moreno-Socías has conjectured a combinatorial property for the initial ideal of I with respect to degree reverse lexicographic order. I show that Moreno-Socías' Conjecture implies Fröberg's Conjecture. I also give a criterion for a Hilbert series to admit an ideal with the property proposed by Moreno-Socías and show that the Hilbert series proposed by Fröberg does have this property.

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1. Introduction

Let $S = k[x_1, \dots, x_n]$, where k is an infinite field, and let $f_1, \dots, f_r$ be homogeneous polynomials of degrees $d_1, \dots, d_r$ generating an ideal I. If the f_i are chosen "at random", then what properties should we expect of I? This paper is concerned with what we should expect of the Hilbert function of S/I and the reverse lexicographic initial ideal of I.

I use "at random" in a sense that is more evocative than rigorous. More precisely, this paper is concerned with generic properties of such sequences. To make sense of this, view $\prod_{i=1}^r S_{d_i}$, the set of all such sequences of the specified degrees, as an affine space for which the coordinates are the coefficients of the polynomials in the sequence. Say that a property P of such sequences is *generic* if it holds on a nonempty Zariski-open $U \subseteq \prod_{i=1}^r S_{d_i}$. Loosely, such a property holds "most of the time". The property ought to hold for a randomly chosen sequence. I will often say that a generic property holds "for a generic sequence". But, the precise meaning is that the property holds on a nonempty Zariski-open set, as above.

Fröberg gave a famous conjecture for the Hilbert function of an ideal generated by a generic sequence, Conjecture A in Section 3. Moreno-Socías gave a conjecture describing the initial ideal of such an ideal with respect to degree reverse lexicographic order, Conjecture D in Section 4. The main

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result of this paper is that Conjecture D implies Conjecture A, and that Conjecture A is equivalent to Conjecture E, a weak form of Conjecture D. I also formulate three more conjectures equivalent to Conjecture A to facilitate the proofs.

More specifically, the contents of the paper are as follows. In Section 2 I give some basic definitions and sketch a proof that there is some generically occurring Hilbert function and initial ideal. In Section 3 I introduce Fröberg's Conjecture and the notion of a semi-regular sequence. In Section 4, I discuss different versions of Moreno-Socías' Conjecture. In Section 5, I prove the main theorem of this paper, giving implications between the various conjectures. In the final section, I further explore the combinatorics of weakly reverse lexicographic ideals.

I am grateful to Karen Chandler, Tony Iarrobino, David Lieberman and David Wagner for helpful conversations related to this paper. David Wagner accumulated substantial computational evidence for Corollary 6 long before I discovered a proof. Tony Iarrobino caught an embarrassing error in an early version of this paper. I am also grateful to the referee for his or her helpful comments, which led me to clarify some issues with the notion of genericity.

Considerable time has elapsed since I submitted this paper, a situation for which I am responsible, and a great deal has been written on the subject since. To prevent further delays, I have not attempted to bring this paper up-to-date with the current literature, but have expanded the bibliography to include some more recent relevant papers, notably [1,4,5,7,16,18–22]. Most importantly, [4] and [21] include several results from this paper, independently discovered by Cho and Park.

Some of these papers cite "Generic Polynomials". That paper was an early draft of the present paper that did not include all of the results in the submitted version of this paper.

2. Hilbert functions, initial ideals and genericity

In this section, I give some basic definitions and then sketch a proof that Hilbert functions and initial ideals are constant in a nonempty open set in $\prod_{i=1}^r S_{d_i}$.

In this paper, k always represents an infinite field and $S = k[x_1, \ldots, x_n]$. We will view S as a graded k-algebra in the standard way, with all variables having degree 1.

If $I \subseteq S$ is a homogeneous ideal, then the *Hilbert function* of S/I is the function from $\mathbb{Z}$ to $\mathbb{N}$ given by

$$h_{S/I}(i) = \dim_k (S/I)_i,$$

and the *Hilbert series* of S/I is the generating function of the Hilbert function

$$H_{S/I}(t) = \sum_{i=0}^{\infty} h_{S/I}(i) t^i.$$

Degree reverse lexicographic order on the monomials of S is defined by $x^\mu > x^\nu$ if $\deg x^\mu > \deg x^\nu$, or $\deg x^\mu = \deg x^\nu$ and the last nonzero entry of $\mu - \nu$ is negative. This is the only monomial order that I use in this paper.

If f is the sum of αx^μ, where $\alpha \in k^*$, and a k-linear combination of monomials $x^\nu < x^\mu$, then $x^\mu = \mathbf{in}\, f$, the *initial term* of f. If I is an ideal, then $\mathbf{in}(I)$ is the ideal generated by the initial terms of elements of I. It is not hard to see that for a homogeneous ideal I, the Hilbert functions of S/I and $S/\mathbf{in}(I)$ are the same.

Let I be generated by an element of $\prod_{i=1}^r S_{d_i}$, which is to say a sequence $f_1, \ldots, f_r$ of homogeneous polynomials of degrees $d_1, \ldots, d_r$. We would like to see that there is a nonempty open set $U \subseteq \prod_{i=1}^r S_{d_i}$ such that the Hilbert function of S/I is the same for every sequence in that set. What is easy to see is that for each $d \in \mathbb{N}$ there is an open set U_d such that $h_{S/I}(d)$ is the same for every sequence in U_d. This is the set where I_d has as large dimension as possible. Failure to have that largest possible dimension is equivalent to the vanishing of certain polynomials in the coefficients, arising as determinants of certain minors of a coefficient matrix. The trick is to see that the intersection of all of the U_d is equal to the intersection of only finitely many, so that the intersection is open and nonempty.

First consider this problem in the case of $r \geqslant n$. Then for d sufficiently large, $d > D = d_1 + \cdots + d_r$ is more than sufficient, the largest that I_d can be is all of S_d. Indeed, this is the case if $f_1 = x_1^{d_1}, \ldots, f_n = x_n^{d_n}$. But, if $I_d = S_d$, then $I_{d+1} = S_{d+1}$ so that $U_d \subseteq U_{d+1}$ for every $d \geqslant D$. So, the intersection of all of the U_d is equal to the intersection of $U_0, \ldots, U_D$ and is thus open and nonempty.

Now, consider this problem in the case of $r < n$. We wish to show that there is an open set such that $f_1, \ldots, f_r$ is a regular sequence. But, $f_1, \ldots, f_r$ is a regular sequence if and only if there are independent linear homogeneous $\ell_{r+1}, \ldots, \ell_n$ such that $f_1, \ldots, f_r, \ell_{r+1}, \ldots, \ell_n$ is a regular sequence. So, let $d_i = 1$ for $r+1 \leqslant i \leqslant n$. As seen in the previous paragraph, there is a nonempty open set $U' \subseteq \prod_{i=1}^n S_{d_i}$ consisting of regular sequences $f_1, \ldots, f_r, \ell_{r+1}, \ldots, \ell_n$. Projecting $\prod_{i=1}^n S_{d_i}$ onto $\prod_{i=1}^r S_{d_i}$, the image of U' contains a nonempty open set U, since the projection is surjective.

The assertion that the image of U' contains an open set U follows from the theory of constructible sets. See Section 6 of Chapter 2 of [15], especially 6.C and 6.E. I will make further use of this method in the proof of Theorem 2.

Now, we will see that there is an open set $V \subseteq \prod_{i=1}^r S_{d_i}$ such that the initial ideal of I is the same for every sequence in V. Let U be the set on which the Hilbert function is constant, so that we at least know $a_d = \dim_k I_d$ for every d. It is easy to see that there is an open set V_d on which the space $(\mathbf{in}\, I)_d$ is constant. There is an order on monomial subspaces of dimension a_d in S_d such that the earliest space $W_d \subseteq S_d$ that ever occurs in this order for a sequence $f_1, \ldots, f_r$ from U is the one that must occur unless certain polynomial conditions on the coefficients of $f_1, \ldots, f_r$ are satisfied. $V_d \subseteq U$ is the open set on which these conditions are not satisfied so that for sequences in V_d, $(\mathbf{in}\, I)_d = W_d$. Again, the problem is to see that the intersection of all of the V_d is the same as the intersection of only finitely many.

Since V_d and V_{d+1} intersect nontrivially, we must have that $S_1 W_d \subseteq W_{d+1}$. Let J be the monomial ideal generated by all of the W_d. Then $J_d = W_d$. Since S is Noetherian, J is generated by only finitely many of the W_d, say $W_0, \ldots, W_t$. Then the intersection of all of the V_d is the intersection of $V_0, \ldots, V_t$. On this open set, $\mathbf{in}\, I = J$.

3. Semi-regular sequences

In this section, I develop the language of semi-regular sequences, which gives the most transparent version of Fröberg's Conjecture, and a convenient link to Moreno-Socías' Conjecture.

If $\sum_{d=0}^{\infty} a_d t^d$ is a power series with integer coefficients, then $|\sum_{d=0}^{\infty} a_d t^d| = \sum_{d=0}^{\infty} b_d t^d$ where $b_d = a_d$ if $a_i > 0$ for $0 \leqslant i \leqslant d$ and $b_d = 0$ otherwise.

Conjecture A. *(See Fröberg [10].) If k is an infinite field and I is generated by a generic sequence of polynomials of degrees $d_1, \ldots, d_r$, then*

$$H_{S/I}(t) = \left| \frac{\prod_{i=1}^r (1 - t^{d_i})}{(1-t)^n} \right|.$$

I will show that this conjecture is equivalent to several other conjectures. The most transparent conjecture equivalent to these is phrased in terms of semi-regular sequences.

Definition. If $A = S/I$, where I is a homogeneous ideal, and $f \in S_d$, then f is *semi-regular* on A if and only if for every e, the vector space map $A_{e-d} \to A_e$ given by multiplication by f is of maximal rank (either injective or surjective). A sequence of homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ is a *semi-regular sequence* if each f_i is semi-regular on $A/(f_1, \ldots, f_{i-1})$.

Notice that regular sequences are semi-regular. In fact, regular and semi-regular elements can be characterized by Hilbert series.

Proposition 1. *If $A = S/I$, and $f_1, \dots, f_r$ are homogeneous polynomials of degrees $d_1, \dots, d_r$, then $f_1, \dots, f_r$ is a semi-regular sequence if and only if*

$$H_{A/(f_1,\dots,f_s)} = \left| \left(\prod_{i=1}^{s} \left(1 - t^{d_i}\right) \right) H_A(t) \right|$$

for $1 \leqslant s \leqslant r$. $f_1, \dots, f_r$ is a regular sequence on A if and only if

$$H_{A/(f_1,\dots,f_r)} = \left(\prod_{i=1}^{r} \left(1 - t^{d_i}\right) \right) H_A(t).$$

Proof. If $r = 1$, then the dimension of $(A/f_1)_e$ is at least $\max\{\dim A_e - A_{e-d_1}, 0\}$, with equality if and only if the multiplication by f_1 map $A_{e-d_1} \to A_e$ has maximal rank. Also, should the dimension of $(A/f_1)_e$ be 0 for some e, then the dimension of $(A/f_1)_{e+1}$ is 0 as well. This proves the first statement in the case $r = 1$. That this statement holds in general now follows from the easy observation that if $H(t) = \sum_{i=0}^{\infty} a_i t^i$ is a power series, and $d \in \mathbb{N}$, then $|(1 - t^d)(|H(t)|)| = |(1 - t^d)H(t)|$.

The characterization of regular sequences is well known. □

The idea of a semi-regular sequence, although not the name, appears in Valla's discussion of Fröberg's Conjecture in [24]. A proof of one direction of Proposition 1 is also in that paper.

A corollary of this proposition is the well-known fact that permutations of homogeneous regular sequences are also regular. This does not hold for semi-regular sequences. For example, x^2, y^2, xy is a semi-regular sequence on $k[x, y]$, but x^2, xy, y^2 is not.

Notice that this proposition implies that Conjecture A is equivalent to the following natural conjecture.

Conjecture B. *If k is an infinite field and $S = k[x_1, \dots, x_n]$, and $d_1, \dots, d_r$ are non-negative integers, then a generic sequence of polynomials of degrees $d_1, \dots, d_r$ is semi-regular.*

Indeed, if Conjecture A is true, then there are nonempty Zariski-open sets $U_1, \dots, U_r \subseteq \prod_{i=1}^{r} S_{d_i}$ such that for a sequence $f_1, \dots, f_r \in U_s$, with $1 \leqslant s \leqslant r$, the ideal generated by $f_1, \dots, f_s$ gives the Hilbert series predicted by Conjecture A. If we let $U = U_1 \cap \cdots \cap U_r$, then Proposition 1 tells us that any sequence in U is a semi-regular sequence, proving Conjecture B. Conversely, if Conjecture B is true, then there is a nonempty open set $U \subseteq \prod_{i=1}^{r} S_{d_i}$ such that every sequence in U is semi-regular. But, Proposition 1 tells us that this sequence generates an ideal giving the Hilbert series predicted by Conjecture A.

These conjectures are known to be true in only a few cases. If $r \leqslant n$, then generic sequences of homogeneous polynomials are regular sequences. For $r = n + 1$, Stanley showed that these conjectures are true for characteristic 0 [13,23]. The conjectures are also true if $n \leqslant 3$ [10,2]. ([24] has a characteristic 0 proof of the case $n = 2$ that is very much in the spirit of this paper.) If $d_1 = \cdots = d_r = d$, Hochster and Laksov have shown that Conjecture A predicts the correct value of the Hilbert function at $i = d + 1$ [12]. Their work was extended to certain other values by Aubry [3]. Fröberg and Hollman have also established several cases through computer experimentation [11]. See [14] for a discussion of similar conjectures concerning powers of generic linear forms and of the Weak Fröberg Conjecture.

One other conjecture in the spirit of this section which is equivalent to those so far is the following. Note that in this conjecture, and some later conjectures, the number of polynomials and the number of variables is the same.

Conjecture C. *If k is an infinite field and $S = k[x_1, \dots, x_n]$ and $d_1, \dots, d_n \in \mathbb{N}$ and $f_i \in S_{d_i}$ are generic homogeneous polynomials, and $A = S/(f_1, \dots, f_n)$, then $x_n, x_{n-1}, \dots, x_1$ is a semi-regular sequence on A.*

4. Initial ideals in the generic case

In this section, I discuss Moreno-Socías' Conjecture and variants.

Conjecture D. *(See Moreno-Socías [17].) If k is an infinite field and $S = k[x_1, \dots, x_n]$ and $d_1, \dots, d_n \in \mathbb{N}$ and $f_i \in S_{d_i}$ are generic homogeneous polynomials generating an ideal I, and $J = \mathbf{in}(I)$, the initial ideal with respect to degree reverse lexicographic order, then J is a weakly reverse lexicographic ideal.*

A *reverse lexicographic ideal* is an ideal J generated by monomials such that if $x^\mu \in J$ then every monomial of the same degree which preceeds x^μ must be in J as well. A *weakly reverse lexicographic ideal* is an ideal J generated by monomials such that if $x^\mu \in J$ is *one of the minimal generators of J* then every monomial of the same degree which preceeds x^μ must be in J as well.

Deery studied reverse lexicographic ideals in his master's thesis [6]. In his thesis, he calls weakly reverse lexicographic ideals "almost reverse lexicographic ideals".

I have stated the Moreno-Socías Conjecture only in the case in which $r = n$, which is the only case treated in this paper, but this special case implies the cases in which the number of polynomials r and the number of variables n may be different. To see this for $n < r$, note that the computation of a reverse lexicographic initial ideal commutes with forming the quotient ring by the last variable, and the image of a weakly reverse lexicographic ideal in that quotient ring is also weakly reverse lexicographic. (See Proposition 15.12 in [8].) So, if Conjecture D holds for r polynomials in n variables, then it also holds for r polynomials in $n-1$ variables. This shows that Conjecture D as stated implies Conjecture D with n variables and $r \geqslant n$ polynomials.

On the other hand, if $r < n$, then for a generic sequence $f_1, \dots, f_r$ we have that $f_1, \dots, f_r, x_n, \dots, x_{r+1}$ is a regular sequence. Another property of reverse lexicographic order gives that the initial ideal of the ideal generated by $f_1, \dots, f_r$ is generated by monomials not divisible by any of $x_n, \dots, x_{r+1}$. (See Theorem 15.13 and Proposition 15.14 in [8].) This means that the generators for this ideal are the same as the generators in the case of $r = n$. This shows that Conjecture D as stated implies Conjecture D with more variables than polynomials.

In the case of the same number of variables as polynomials, we know that we have a regular sequence generically, so that we know the generic Hilbert function. For a given Hilbert function there is at most one weakly reverse lexicographic ideal, so Conjecture D allows us to write down generators for the predicted initial ideal. Whether or not there actually is a weakly reverse lexicographic ideal with the desired Hilbert function is a problem that I address in the last section of this paper.

I will show that Moreno-Socías' Conjecture implies the other conjectures in this paper, but it is not clear that the converse is true. The other conjectures in this paper are equivalent to a weak form of Moreno-Socías' Conjecture.

Conjecture E. *If k is an infinite field and $S = k[x_1, \dots, x_n]$ and $d_1, \dots, d_n \in \mathbb{N}$ and $f_i \in S_{d_i}$ are generic homogeneous polynomials generating an ideal I, and $J = \mathbf{in}(I)$, the initial ideal with respect to degree reverse lexicographic order, and x^μ is a minimal generator of J of degree d which is divisible by a variable x_m, then every monomial of degree d in the variables $x_1, \dots, x_{m-1}$ is in the ideal J.*

The last equivalent conjecture in this paper is the following.

Conjecture F. *If k is an infinite field and $S = k[x_1, \dots, x_n]$ and $d_1, \dots, d_n \in \mathbb{N}$ and $f_i \in S_{d_i}$ are generic homogeneous polynomials generating an ideal I, and $J = \mathbf{in}(I)$, the initial ideal with respect to degree reverse lexicographic order, then $x_n, \dots, x_1$ is a semi-regular sequence on S/J.*

5. Equivalence of the conjectures

Theorem 2. *Conjectures A, B, C, E and F are all equivalent, while Conjecture D implies the others.*

Proof. The equivalence of Conjectures A and B follows from Proposition 1 as noted in Section 3.

To see that Conjecture B implies Conjecture C, let $r = 2n$, and $d_i = 1$ for $n < i \leqslant 2n$. If Conjecture B holds, then there is a nonempty open set $U_0 \subseteq \prod_{i=1}^{n} S_{d_i} \times S_1^n$ consisting of semi-regular sequences.

We wish to show that there is a nonempty open set $U_1 \subseteq \prod_{i=1}^n S_{d_i}$ consisting of sequences $f_1, \ldots, f_n$ such that $f_1, \ldots, f_n, x_n, \ldots, x_1$ is a semi-regular sequence.

We may replace U_0 by a smaller nonempty open set consisting of sequences $f_1, \ldots, f_n, \ell_n, \ldots, \ell_1$ such that $\ell_n, \ldots, \ell_1$ are linearly independent. Choose a particular sequence $\ell_n, \ldots, \ell_1$ that occurs at the end of a sequence in U_0. Let

$$U_2 = U_0 \cap \prod_{i=1}^n S_{d_i} \times (\ell_n, \ldots, \ell_1).$$

Then U_2 is a nonempty open subset of $\prod_{i=1}^n S_{d_i} \times (\ell_n, \ldots, \ell_1)$, which is naturally homeomorphic to $\prod_{i=1}^n S_{d_i}$. Let $U_3 \subseteq \prod_{i=1}^n S_{d_i}$ be the image of U_2 under this homeomorphism. Then U_3 is a nonempty open set consisting of sequences $f_1, \ldots, f_n$ such that $f_1, \ldots, f_n, \ell_n, \ldots, \ell_1$ is semi-regular. Let $\gamma \in GL(n)$ be such that $\gamma(\ell_i) = x_i$. Then γ gives a homeomorphism of $\prod_{i=1}^n S_{d_i}$ to itself and we may take $U_1 = \gamma(U_3)$. Indeed, for every sequence $f_1, \ldots, f_n$ in U_3, $\gamma(f_1), \ldots, \gamma(f_n), x_n, \ldots, x_1$ is a semi-regular sequence. This proves that Conjecture B implies Conjecture C.

To see that Conjecture C implies Conjecture A, note that we only have to consider the case in which $r > n$, since Conjecture A is known to be true if $r \leqslant n$. So, assume that $r > n$ and that Conjecture C holds. Consider the degree sequence $d_1, \ldots, d_r$ for a polynomial ring $S' = k[x_1, \ldots, x_r]$ of r variables. By Conjecture C, there is a nonempty open set $U \subseteq \prod_{i=1}^r S'_{d_i}$ consisting of sequences $f'_1, \ldots, f'_r$ such that $f'_1, \ldots, f'_r, x_r, \ldots, x_{n+1}$ is a semi-regular sequence. Viewing S as $S = S'/(x_r, \ldots, x_{n+1})$, the image of U in $\prod_{i=1}^r S_{d_i}$ contains a nonempty open set U'.

For a sequence in U, the Hilbert series of $S'/(f'_1, \ldots, f'_r, x_r, \ldots, x_{n+1})$ is

$$\left|(1-t)^{r-n} \frac{\prod_{i=1}^r (1-t^{d_i})}{(1-t)^r}\right| = \left|\frac{\prod_{i=1}^r (1-t^{d_i})}{(1-t)^n}\right|.$$

Taking f_i to be the image of f'_i in $S = S'/(x_r, \ldots, x_{n+1})$, the f_i form a sequence of homogeneous polynomials of specified degrees which gives the Hilbert series predicted by Conjecture A. Thus, the Hilbert series predicted by Conjecture A holds on the nonempty open set U', proving that Conjecture C implies Conjecture A.

To see that Conjecture C is equivalent to Conjecture F, consider the following useful property of degree reverse lexicographic order: If I is a homogeneous ideal and I' is the image of I in $S' = S/(x_n, \ldots, x_m)$, then $\mathbf{in}(I')$ is the image of $\mathbf{in}(I)$ in S'. (See Proposition 15.12 in [8].) This implies that the Hilbert functions of $S/(I + (x_n, \ldots, x_m))$ and $S/(\mathbf{in}(I) + (x_n, \ldots, x_m))$ are the same. So, by Proposition 1, $x_n, \ldots, x_1$ is semi-regular on both S/I and $S/\mathbf{in}(I)$, or on neither of them.

To see that Conjecture F implies Conjecture E, let J be an ideal generated by monomials such that $x_n, \ldots, x_1$ is a semi-regular sequence on S/J. Let x^μ be a generator of J of degree d which is divisible by a variable x_m. Without loss of generality, we may assume that there is no lower degree generator of J divisible by x_m, and thus no monomial at all in J of lower degree and divisible by x_m. Furthermore, we may assume without loss of generality that x^μ is not divisible by any x_i with $i > m$. Consider the map $(S/(J + (x_n, \ldots, x_{m+1})))_{d-1} \to (S/(J + (x_n, \ldots, x_{m+1})))_d$ given by multiplication by x_m. Since $\frac{x^\mu}{x_m} \notin J$ (otherwise x^μ would not be a minimal generator), but $x^\mu \in J$, this multiplication map is not injective. But, x_m is semi-regular on $S/(J + (x_n, \ldots, x_{m+1}))$, so the map must be surjective. Thus, every monomial of degree d that is not divisible by x_m must be in $J + (x_n, \ldots, x_{m+1})$. Thus, every monomial of degree d in the variables $x_1, \ldots, x_{m-1}$ must be in J, so that J satisfies Conjecture E.

To see that Conjecture E implies Conjecture F, let J be an ideal satisfying the conclusion of Conjecture E: for every degree d and for every x_m, if there is a monomial generator of J of degree d which is divisible by x_m, then all monomials in the variables $x_1, \ldots, x_{m-1}$ of degree d are in J. We must show that for every m, multiplication by x_m is semi-regular on $S/(J + (x_n, \ldots, x_{m+1}))$. Let d be the smallest degree such that there is a generator of J of degree d which is divisible by x_m. Consider the map $(S/(J + (x_n, \ldots, x_{m+1})))_{i-1} \to (S/(J + (x_n, \ldots, x_{m+1})))_i$ given by multiplication by x_m. I claim that this map is injective if $i < d$ and surjective if $i \geqslant d$. Say that $i < d$ and $f \in S_{i-1}$ and

$x_m f \in J + (x_n, \ldots, x_{m+1})$. We may assume that f is a monomial. If f is divisible by any x_j, with $j > m$, then $f \in J + (x_n, \ldots, x_{m+1})$. Otherwise, $x_m f$ is divisible by a minimal generator x^ν of J. Since x^ν has degree less than d, it is not divisible by x_m. So, f is divisible by x^ν and $f \in J$. This proves injectivity. Now, say that $i \geqslant d$. Then every monomial of degree i in the variables $x_1, \ldots, x_{m-1}$ is in J. Thus, every monomial in the monomial basis of $(S/(J + (x_n, \ldots, x_{m+1})))_i$ is divisible by x_m. So, the map is surjective. Therefore, J satisfies the conclusion of Conjecture F.

We now have that Conjectures A, B, C, E and F are equivalent. To see that Conjecture D implies Conjecture E, and thus all of the others, note that a weakly reverse lexicographic ideal satisfies the condition of Conjecture E. □

I have stated and proven Theorem 2 with each conjecture taken in its entirety. The same arguments show that Conjecture E for a fixed number r of polynomials and variables is equivalent to Conjecture A for the same fixed number r of polynomials, but any number $n \leqslant r$ of variables.

6. Weakly reverse lexicographic ideals

One might guess that Conjecture D is too weak. One might expect instead that the initial ideal of a generic sequence of homogeneous polynomials should actually be a reverse lexicographic ideal. But, this is rarely true, thanks to Deery's characterization of which Hilbert series admit reverse lexicographic ideals. In this theorem, $\succcurlyeq$ is the coefficient-wise partial order on $\mathbb{Z}[[t]]$.

Theorem 3. *(See Deery [6].) Let $H(t) = \sum h(i)t^i \in \mathbb{Z}[[t]]$ be a series with non-negative coefficients. Then $H(t)$ is the Hilbert series of S/I for a reverse lexicographic ideal I if and only if*

(1) $\frac{1}{(1-t)^n} \succcurlyeq H(t)$ *and*
(2) *if the first degree in which $\frac{1}{(1-t)^n}$ differs from $H(t)$ is d, then $h(i+1) \leqslant h(i)$ for all $i \geqslant d$.*

For example, if I is generated by six generic quadratics in six variables, then the Hilbert series of S/I is $1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6$. This Hilbert series does not satisfy the criterion of Deery's Theorem: the first degree in which it differs from $(1-t)^{-6} = 1 + 6t + 21t^2 + \cdots$ is in degree 2, but the coefficients are not weakly decreasing thereafter. Thus, there is no reverse lexicographic ideal with this Hilbert series. In particular, the initial ideal of I is not reverse lexicographic.

What criterion is there for a Hilbert series to admit a weakly reverse lexicographic ideal, or even just an ideal modulo which $x_n, \ldots, x_1$ is a semi-regular sequence? If one can produce a Hilbert series of a regular sequence which does not admit an ideal such that $x_n, \ldots, x_1$ is a semi-regular sequence, then all of the conjectures in this paper are false. The following theorem gives the desired criterion.

Theorem 4. *Let $H(t) = \sum_{i=0}^{\infty} h(i)t^i$ be a series with non-negative coefficients such that $h(0) = 1$. Let $S = k[x_1, \ldots, x_n]$ where k is a field and $n \geqslant h(1)$. Then the following are equivalent:*

(1) *There is a weakly reverse lexicographic ideal J such that S/J has Hilbert series $H(t)$.*
(2) *There is an ideal J such that S/J has Hilbert series $H(t)$ and $x_n, \ldots, x_1$ is a semi-regular sequence on S/J.*
(3) *For every $r \in \mathbb{Z}$, let*

$$\sum_{i=0}^{\infty} h_r(i)t^i = \left|(1-t)^r H(t)\right|.$$

If $h_r(i) \leqslant h_r(i-1)$ then $h_r(i+1) \leqslant h_r(i)$.

Proof. $(1) \to (2)$: In the proof of Theorem 2, the argument that Conjecture F implies Conjecture E shows that $x_n, \ldots, x_1$ is a semi-regular sequence modulo a weakly reverse lexicographic ideal.

$(2) \to (3)$: First, I will prove that (3) is true for $r < 0$. Since $H(t)$ is a Hilbert series, should some coefficient be zero then all later coefficients are zero as well. Thus, if $H(t)$ is not a finite sum, then the coefficients of $(1-t)^r H(t)$ are positive and strictly increasing. If $H(t)$ is a polynomial of degree D, then the coefficients of $(1-t)^r H(t)$ are positive and, if $r < -1$, they are strictly increasing, while if $r = -1$, the coefficients are strictly increasing up to $i = D$ and constant thereafter.

(3) is also true for $r \geqslant h(1)$ irrespective of (2). In fact $(1-t)^r H(t) = 1 + (h(1) - r)t + \cdots$ so that $|(1-t)^r H(t)| = 1$ for $r \geqslant n$.

Now, if $0 \leqslant r \leqslant n-1$, let J be an ideal as in (2). Then $x_n, \dots, x_{n-r+1}$ is a semi-regular sequence on S/J, so that the Hilbert series of $S/(J + (x_n, \dots, x_{n-r+1}))$ is $|(1-t)^r H(t)|$. Say that $h_r(i) \leqslant h_r(i-1)$. Then, since x_{n-r} is semi-regular on $S/(J + (x_n, \dots, x_{n-r+1}))$, the multiplication by x_{n-r} map

$$S/\big(J + (x_n, \dots, x_{n-r+1})\big)_{i-1} \to S/\big(J + (x_n, \dots, x_{n-r+1})\big)_i$$

is surjective. But, this forces the next multiplication by x_{n-r} map

$$S/\big(J + (x_n, \dots, x_{n-r+1})\big)_i \to S/\big(J + (x_n, \dots, x_{n-r+1})\big)_{i+1}$$

to be surjective as well, giving the inequality $h_r(i+1) \leqslant h_r(i)$.

$(3) \to (1)$: If $h(1) = 0$, then $H(t) = 1$ and the weakly reverse lexicographic ideal giving $H(t)$ is generated by $x_1, \dots, x_n$.

I will now proceed by induction on n. If $n = 0$ then $h(1) = 0$ and we are in the case above. So, assume that $n > 0$ and $h(1) > 0$. Notice that the series $G(t) = |(1-t)H(t)|$ satisfies the hypotheses of the theorem with n replaced by $n-1$, since

$$G(t) = 1 + \big(h(1) - 1\big)t + \cdots,$$

and of condition (3), since $|(1-t)^r G(t)| = |(1-t)^{r+1} H(t)|$. So, there is a weakly reverse lexicographic ideal $I' \subseteq S' = k[x_1, \dots, x_{n-1}]$ such that S'/I' has Hilbert series $G(t)$. Let $I = I'S$. Then I is a weakly reverse lexicographic ideal and S/I has Hilbert series $\frac{1}{1-t}G(t)$. If $G(t) = (1-t)H(t)$, then I is the desired ideal. Otherwise,

$$G(t) = \sum_{i=0}^{D} \big(h(i) - h(i-1)\big)t^i$$

where $D + 1 = \min\{d\colon\ h(d) \leqslant h(d-1)\}$ and $h(-1) = 0$. Then, $\frac{1}{1-t}G(t) = \sum_{i=0}^{D-1} h(i)t^i + \sum_{i=D}^{\infty} h(D)t^i$. So, $\frac{1}{1-t}G(t) - H(t) = \sum_{i=D+1}^{\infty}(h(D) - h(i))t^i$, a series with positive, bounded, weakly increasing coefficients.

I will show how to add monomials to I so as to close the gap between the Hilbert series of S/I and $H(t)$, while preserving the weakly reverse lexicographic property. To do this, I will use a formula of Eliahou and Kervaire for the Hilbert series of a "stable" ideal [9]. I do not require the definition of a stable ideal, except that stable ideals are monomial ideals and weakly reverse lexicographic ideals are stable.

For a monomial x^μ, write $\deg \mu$ for the degree of x^μ, and $\max \mu$ for the highest index of a variable dividing x^μ. If $K \subseteq S$ is a stable ideal and $Gen(K)$ is its minimal set of monomial generators, then

$$H_{S/K} = \frac{1}{(1-t)^n} - \sum_{x^\mu \in Gen(K)} \frac{t^{\deg \mu}}{(1-t)^{n - \max \mu + 1}}.$$

In particular, if K is a weakly reverse lexicographic ideal and we add another monomial $x^\nu \notin K$ which does not divide any of the monomials of $Gen(K)$, and $K + x^\nu S$ is also weakly reverse lexicographic, then

$$H_{S/(K+x^\nu S)}(t) = H_{S/K}(t) - \frac{t^{\deg \nu}}{(1-t)^{n-\max \nu+1}}.$$

If $\max \nu = n$ then the Hilbert series decreases by exactly $t^{\deg \nu} + t^{\deg \nu+1} + \cdots$.

Notice that $(S'/I')_{D+1} = 0$, so that I' contains every monomial of degree $D+1$ in the variables $x_1, \ldots, x_{n-1}$, and is generated by monomials in degrees less than or equal to $D+1$. I will inductively construct an ascending chain of weakly reverse lexicographic ideals $I_{(s)}$ with $I_{(0)} = I$ and for $s > 0$, $I_{(s)} = I_{(s-1)} + m_{s-1}S$, where m_{s-1} is a monomial, such that the coefficients of $H_{S/I_{(s)}}(t) - H(t)$ are a non-negative, bounded, weakly increasing sequence and the degrees of m_s are at least $D+1$ and weakly increasing.

If $H_{S/I_{(s)}}(t) \neq H(t)$, then choose the first degree d in which the coefficients differ. Since the coefficient for t^d must then be positive, there is a monomial of degree d which is not in $I_{(s)}$. Choose the reverse lexicographically earliest such monomial, and call it m_s. Let $I_{(s+1)} = I_{(s)} + m_s S$.

Note that x_n divides m_s, since $d \geqslant D+1$ so that all monomials of degree d in the variables $x_1, \ldots, x_{n-1}$ are already in I. I also claim that d is at least as large as any of the degrees of the minimal generators of $I_{(s)}$. If $s = 0$, note that $d \geqslant D+1$ and $I_{(0)}$ is generated in degrees less than or equal to $D+1$. If $s > 0$, then d is at least equal to the degree of m_{s-1}. This is because m_{s-1} was chosen to be the first degree d' in which $H_{S/I_{(s-1)}}(t) - H(t)$ has a nonzero coefficient. Thus the Hilbert series $H_{S/I_{(s-1)}}(t)$ and $H_{S/I_{(s)}}(t)$ must be the same in degrees less than d', so $d \geqslant d'$. This proves the claim.

Thus, m_s does not divide any of the minimal generators of $I_{(s)}$. Since none of these minimal generators divide m_s either, $Gen(I_{(s+1)}) = Gen(I_{(s)}) \cup \{m_s\}$. By the choice of m_s, $I_{(s+1)}$ is a weakly reverse lexicographic ideal. By Eliahou and Kervaire's formula, $H_{S/I_{(s+1)}}(t) = H_{S/I_{(s)}}(t) - (t^d + t^{d+1} + \cdots)$. By the choice of d and induction, the coefficients of $H_{S/I_{(s+1)}} - H(t)$ are non-negative, bounded, and weakly increasing, and the lowest nonzero coefficient is in degree at least d.

Since the $I_{(s)}$ form an ascending chain of ideals in a polynomial ring, the chain must stabilize. But, we can extend the chain if the Hilbert series of $S/I_{(s)}$ is not equal to $H(t)$, so it must stabilize at a weakly reverse lexicographic ideal J such that S/J has Hilbert series $H(t)$. $\square$

The question still remains if there is a Hilbert series of a regular sequence which fails to satisfy condition (3) of Theorem 4. The next theorem shows that there is not. Thus, counterexamples to the conjectures in this paper are not apparent.

Theorem 5. *Let $d_1, \ldots, d_n$ be positive integers and for each $r \in \mathbb{Z}$, let*

$$\sum_i h_r(i) t^i = (1-t)^r \prod_{j=1}^{n} \bigl(1 + t + \cdots + t^{d_j - 1}\bigr).$$

Let $c_r = \max\{c\colon\ h_r(i) > 0$ for $0 \leqslant i \leqslant c\}$. If $0 \leqslant i \leqslant c_r$ and $h_r(i-1) \geqslant h_r(i)$, then $h_r(i) \geqslant h_r(i+1)$.

Proof. If $r \geqslant n$ then $h_r(1) = n - r \leqslant 0$ and $h_r(0) = 1$ so that $c_r = 0$. Since it is always true that $h_r(-1) = 0$, the conclusion is vacuous for $r \geqslant n$.

Now, consider the case $n = 0$. If $r \geqslant 0$, then this is included in the case above. Otherwise, $h_r(i) = \binom{i-r-1}{-r-1} > 0$ so that $c_r = \infty$. If $r = -1$, $h_{-1}(i) = 1$ is constant for $i \geqslant 0$, while if $r < -1$, $h_r(i)$ is strictly increasing for $i \geqslant 0$, both of which satisfy the conclusion of the theorem.

If $n > 0$, let

$$\sum h'_r(i)t^i = (1-t)^r \prod_{j=1}^{n-1} \left(1 + t + \cdots + t^{d_j - 1}\right)$$

and let $c'_r = \max\{c\colon\ h'_r(i) > 0 \text{ for } 0 \leqslant i \leqslant c\}$. By induction on n, the theorem is true for all h'_r. Let $d = d_n$. Since

$$\begin{aligned}\sum h_r(i)t^i &= (1-t)^r \prod_{j=1}^{n} \left(1 + t + \cdots + t^{d_j - 1}\right)\\ &= \left(1-t^d\right)(1-t)^{r-1} \prod_{j=1}^{n-1} \left(1 + t + \cdots + t^{d_j - 1}\right) = \left(1-t^d\right) \sum h'_{r-1}(i)t^i,\end{aligned}$$

comparing coefficients gives that $h_r(i) = h'_{r-1}(i) - h'_{r-1}(i-d)$. This turns out to be a key equation in the argument. First, I use it to establish that $c_r \leqslant c'_{r-1}$. Examining the equation at $i = c'_{r-1} + 1$, and noting that $h'_{r-1}(c'_{r-1} + 1) \leqslant 0$ while $h'_{r-1}(c'_{r-1} + 1 - d) \geqslant 0$ gives that $h_r(c'_{r-1} + 1) \leqslant 0$. This shows that $c_r \leqslant c'_{r-1}$.

Now, assume that the conclusion of the theorem is false for some h_r. Then there is some i such that $0 \leqslant i \leqslant c_r$ and $h_r(i) \leqslant h_r(i-1)$, but $h_r(i) < h_r(i+1)$. Rewriting $h_r(i) \leqslant h_r(i-1)$ in terms of h'_{r-1}, gives

$$\begin{aligned}h'_{r-1}(i) - h'_{r-1}(i-d) &\leqslant h'_{r-1}(i-1) - h'_{r-1}(i-d-1),\\ h'_{r-1}(i) - h'_{r-1}(i-1) &\leqslant h'_{r-1}(i-d) - h'_{r-1}(i-d-1),\\ h'_r(i) &\leqslant h'_r(i-d),\end{aligned}$$

where the last inequality follows from the identity $h_r(j) = h_{r-1}(j) - h_{r-1}(j-1)$. Rewriting $h_r(i) < h_r(i+1)$ gives

$$\begin{aligned}h'_{r-1}(i) - h'_{r-1}(i-d) &< h'_{r-1}(i+1) - h'_{r-1}(i-d+1),\\ h'_{r-1}(i-d+1) - h'_{r-1}(i-d) &< h'_{r-1}(i+1) - h'_{r-1}(i),\\ h'_r(i+1-d) &< h'_r(i+1).\end{aligned}$$

First assume that $i \leqslant c'_r$. Since $h'_r(i) \leqslant h'_r(i-d)$, the induction hypothesis says that there is some $j \leqslant i$ such that h'_r is weakly decreasing on the interval $[j, c'_r + 1]$. In particular, $h'_r(i+1) \leqslant h'_r(i)$. This, together with the inequalities above, gives that $h'_r(i+1-d) < h'_r(i-d)$. This implies that $h'_r(i-d) > 0$ so that $i \geqslant d$ and the induction hypothesis again says that h'_r is weakly decreasing on the interval $[i-d, c'_r + 1]$. But, this contradicts that $h'_r(i+1-d) < h'_r(i+1)$.

So, it must be that $c'_r < i \leqslant c_r \leqslant c'_{r-1}$. By the definition of c'_r, $h'_r(c'_r + 1) \leqslant 0$, so that $h'_{r-1}(c'_r + 1) \leqslant h'_{r-1}(c'_r)$. The induction hypothesis then says that h'_{r-1} is weakly decreasing on $[c'_r, c'_{r-1} + 1]$, which in turn implies that $h'_r(j) \leqslant 0$ on $[c'_r + 1, c'_{r-1} + 1]$. Since $i+1$ is in this interval, it follows from one of the inequalities derived above that $h'_r(i+1-d) < h'_r(i+1) \leqslant 0$. Rewriting $h'_r(i+1-d)$ in terms of h'_{r-1}, gives that $h'_{r-1}(i+1-d) < h'_{r-1}(i-d)$. Since the left-hand side is not negative, the right-hand side is positive, implying that $i \geqslant d$. Applying the induction hypothesis again, h'_{r-1} is weakly decreasing on $[i-d, c'_{r-1}]$ so that $h'_{r-1}(i-d) \geqslant h'_{r-1}(i)$. Then $h_r(i) = h'_{r-1}(i) - h'_{r-1}(i-d) \leqslant 0$, contradicting that $0 \leqslant i \leqslant c_r$. $\square$

Corollary 6. *Let $S = k[x_1, \dots, x_n]$, $r \in \mathbb{N}$, $d_1, \dots, d_r$ be positive integers and*

$$H(t) = \left| \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^n} \right|.$$

Then there is a weakly reverse lexicographic ideal J in S, such that S/J has Hilbert series $H(t)$.

Proof. First consider the case in which $r \leqslant n$. Then

$$H(t) = \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^n} = \frac{\prod_{i=1}^{r}(1 + t + \dots + t^{d_i - 1})}{(1-t)^{n-r}}.$$

Let D be the number of i with $1 \leqslant i \leqslant r$ such that $d_i \geqslant 2$. Then the leading coefficient of $H(t)$ is 1 and the coefficient of t is $n - r + D \leqslant n$. So, the hypotheses of Theorem 4 apply to $H(t)$. By Theorem 5, condition (3) of Theorem 4 applies to $H(t)$, so that condition (1) gives the desired weakly reverse lexicographic ideal.

Now, consider the case in which $r > n$. Let

$$H'(t) = \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^r}.$$

Then by the case considered above, there is a weakly reverse lexicographic ideal J' in $S' = k[x_1, \dots, x_r]$ such that S'/J' has Hilbert series $H'(t)$. Let J be the image of J' in $S = S'/(x_r, \dots, x_{n+1})$. Then J is a weakly reverse lexicographic ideal in S.

Since J' is weakly reverse lexicographic, $x_r, \dots, x_{n+1}$ is a semi-regular sequence on S'/J'. So, the Hilbert series of

$$S'/\bigl(J' + (x_r, \dots, x_{n+1})\bigr) \simeq S/J$$

is

$$\left| (1-t)^{r-n} \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^r} \right| = \left| \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^n} \right|.$$

So, J is the desired ideal. □

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