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Waring's Problem for the Ring of Polynomials

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We investigate the question of which polynomials are not representable as the sum of "few" powers of polynomials. In particular for Waring's problem over the ring of polynomials we show that there exist polynomials which are not the sum of fewer than $n^{1/2}$ nth powers of polynomials.

INTRODUCTION

We investigate the question: Is it possible to represent any polynomial as a sum of nth powers of polynomials? If we attempt a comparison of coefficients the problem quickly gets out of hand. Even if we dealt only with linear polynomials, an nth power of a general linear polynomial $ax + b$ has $n + 1$ related coefficients. If we form the sum of k nth powers of linear polynomials, we have on comparing coefficients $n + 1$ equations in $2k$ unknowns. The equations, however, are not linear in the unknowns so it is not clear that there will be a solution for $2k \geqslant n + 1$. Indeed it is not obvious that the equations are solvable for any value of k.

In Section 1 we apply finite differences to show that x can always be represented as a sum of n nth powers. Thus we abandon the seemingly hopeless complications of coefficient comparisons. Incidentally, the finite difference argument is due to Professor S. Hurwitz of CCNY. A similar argument would show that constants are representable as a sum of $(n + 1)$ nth powers. In these arguments it is necessary to allow our polynomials to have complex coefficients. In general we will allow our polynomials to have complex coefficients. With complex coefficients, Molluzzo [4] has shown that constants are expressible as a sum of $[(4n + 1)^{1/2}]$ non-constant nth powers of polynomials in a non-trivial manner.

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If we allow sums and differences of nth powers it would be of great interest to achieve solutions allowing only integral coefficients. A solution in this form could be used to give an upper bound for $v(k)$, where $v(k)$ is the smallest value of r such that every integer

$$n = \pm x_1^k \pm x_2^k \pm \cdots \pm x_r^k$$

where the $\{x_i\}$ are integers [1].

The problem of representing any polynomial as a sum of nth powers of polynomials is known as the Waring problem for polynomials. We show in Section II that a lower bound for this problem is $n^{1/2}$ nth powers. This is the first proof that the number of nth powers necessary to represent x even goes to infinity with n. When the method of section II is applied to the representation of constants by nth powers of polynomials, we show, together with the construction of Molluzzo, that $n^{1/2}$ is the correct order of magnitude for representing constants.

The method of Section II is actually quite general. We go on to show that a large class of Polynomial Diophantine equations is insoluble.

The problem of replacing polynomials with rational functions is also investigated. It is shown that x is not representable as the sum of less than $(n/8)^{1/2}$ nth powers of rational functions.

The results on rational functions are used to investigate the Fermat problem for polynomials. Namely, are there polynomial solutions to

$$P_1^n + P_k^n = Q^n$$

in non-constant polynomials with no common zeros?

We show that an nth power polynomial can be the sum of no fewer than $(n/8)^{1/2}$ nth powers of polynomials. On the other hand we show that every nth power polynomial is the sum of no more than $2(n)^{1/2}$ nth powers.

1. Crude Bounds for the Waring Problem for Polynomials

The question "For what value of k, is every polynomial the sum of k nth powers of polynomials" is known as Waring's problem for polynomials. For any polynomial $Q(x)$ do there exist k polynomials such that $P_1^n(x) + P_2^n(x) + \cdots + P_k^n(x) = Q(x)$, where the polynomials may have complex coefficients.

We note that if the polynomial "x" is the sum of k nth powers

$$P_1^n(x) + P_2^n(x) + \cdots + P_k^n(x) = x,$$

then the substitution of $Q(x)$ for x yields

$$P_1^n(Q(x)) + P_2^n(Q(x)) + \cdots + P_k^n(Q(x)) = Q(x),$$

which shows that any polynomial is representables a the sum of k nth powers. Thus the representation of "x" is pivotal.

By considering $(n-1)$th differences of x^n it can be shown [1] that "x" is representable as the sum of n nth powers. Thus $k \leqslant n$. The construction is as follows:

$$\begin{aligned} \Delta^{n-1}(x^n) &= (x+(n-1))^n + c_1(x+(n-2))^n + \cdots + c_{n-1}x^n \\ &= (n-1)!\, x + a \end{aligned}$$

Setting $x = x_1/(n-1)! - a/(n-1)!$, we have the desired representation. S. Hurwitz has conjectured that n nth powers is the minimum needed. In fact Heilbronn has conjectured that n is minimal even if entire functions are allowed. [2]

On the other hand for $n > 2$

$$P_1{}^n + P_2{}^n = x$$

is impossible. The expression on the left can be factorized as

$$\prod_{\omega_i \text{ an } n\text{th root of } -1} (P_1 + \omega_i P_2).$$

Since factors on the left must match factors on the right

$$P_1 + \omega_i P_2 = ax$$

for only one value of i. Thus if $n > 2$ there are certainly two values of i for which

$$\begin{aligned} P_1 + \omega_e P_2 &= b, \\ P_1 + \omega_f P_2 &= c, \end{aligned}$$

b and c constants. Solution of this system yields P_2 and P_1 to be constants and this is of course impossible. This gives the lower bound $k \geqslant 3$ for $n > 2$.

For the case $n = 2$ we have

$$(x + \tfrac{1}{4})^2 - (x - \tfrac{1}{4})^2 = x.$$

For cubes, the above results tell us that x is the sum of 3 cubes but no fewer than 3.

Since $\Delta^2 x^3 = 6x + 6$, setting $x_1 = x + 1$, we have $1/6(x+1)^3 - 1/3x^3 + 1/6(x-1)^3 = x$. In the next section we will prove that if k is the minimum number of nth powers necessary to represent x then $k^2 - k > n$. Setting $k = 3$ we find that if we hope to represent x as a sum of 3 nth powers then $n \leqslant 5$.

The same lower bounds hold for representing 1 as a sum of nth powers.

In fact for $n = 5$ a construction of Molluzo [4] shows that 1 is the sum of 4 fifth powers of Polynomials.

2. Proof that $k(n) \to \infty$ as $n \to \infty$

THEOREM. *Let $\sum_{i=1}^{k(n)} P_i^n(x) = x$ and suppose that $k(n)$ is the minimum number of* n*th powers which give x. Then $k > n^{1/2}$. In fact $k^2 - k > n$.*

Proof. Suppose that $\sum_{i=1}^{k} P_i^n(x) = x$ and that k is minimal.

Consider the following two Wronskians:

$$W(P_1^n, P_2^n, \dots, P_k^n),$$
$$W(x, P_2^n, \dots, P_k^n).$$

Since $x = P_1^n + \cdots + P_k^n$ these two Wronskians are equal identically.

The P_i^n form a linearly independent set. If they were not linearly independent, one of the P_i^n could be replaced by a sum of the others and x would then be represented as a sum of $k - 1$ nth powers contradicting the minimality of k.

$$W(P_1^n, \dots, P_k^n) \neq 0.$$

Since P_i^{n-r} is a factor of the rth derivative of P_i^n

$$W(P_1^n, \dots, P_k^n) = Q \prod_{i=1}^{k} P_i^{(n-k+1)}, \qquad \text{where} \quad Q \not\equiv 0 \text{ is a polynomial.}$$

If D_i is the degree of P_i it follows that

$$\deg W(P_1^n, \dots, P_k^n) \geqslant (n - k + 1) \sum_{i=1}^{k} D_i .$$

On the other hand, evaluating $W(x, P_2^n, \dots, P_k^n)$ by the definition of a determinant we find that

$$\deg W(x, P_2^n, \dots, P_k^n) \leqslant 1 + nD_2 - 1 + nD_3 - 2 + \cdots + nD_k - (k - 1)$$
$$= n \sum_{i=2}^{k} D_i - \frac{k(k-1)}{2} + 1.$$

Since $\deg W(P_1^n, \dots, P_k^n) = \deg W(x, P_2^n, \dots, P_k^n)$

$$(n - k + 1) \sum_{i=1}^{k} D_i \leqslant n \sum_{i=2}^{k} D_i - \frac{k(k-1)}{2} + 1,$$
$$nD_1 \leqslant (k - 1) \sum_{i=1}^{k} D_i - \frac{k(k-1)}{2} + 1,$$

and letting D_1 be the largest of the D_i's we have

$$nD_1 \leqslant (k-1)\,kD_1 - \frac{k(k-1)}{2} + 1,$$

and dividing by D_1 the result is

$$n < k^2 - k \qquad \text{for} \quad k > 2.$$

It is interesting to note that this method gives a sharp result in that it makes use only of the low degree of the polynomial of x.

3. Representation of Other Polynomials

We can use the results of Section 2 to decide whether or not a wide class of Polynomial Diophantine equations has solutions with coefficients in the complex field. In fact we will show that $P_1^n + P_2^n + \cdots + P_k^n = R$ is insoluble for

$$\deg R < n - k^2 + k.$$

This result generalizes a Theoıem of Newman-Klamkin [3]. Their result states that $P^a + Q^b = R$ is insoluble if a or $b > 2c$ where c is the degree of R. In the proof of the Newman–Klamkin result use is made of the multiplicity of zeroes of P^a and Q^b. It is in this sense that our result is a generalization. The nth powers of course have high multiplicity zeros. It is precisely this fact which allows the extraction of a high degree polynomial from the Wronskian and produces the theorem.

In fact we prove the following

THEOREM. *Let $P_1^n + \cdots + P_k^n = R$ where the $P_1, \ldots, P_k$ are non constant. Then*

$$\deg R \geqslant n - \frac{(k^2 - k)}{2}.$$

Further if all the P_i's are not linear we have

$$\deg R \geqslant D(n - k^2 + k) + \frac{k(k-1)}{2}$$

where D is the largest of the degrees of any of the P_i's.

Proof. If $P_1^n + \cdots + P_k^n = R$ we can write, as in the previous section,

$$W(P_1^n, \ldots, P_k^n) = W(R, P_2^n, \ldots, P_k^n)$$

and this produces the following inequality,

$$(n - k + 1) \sum_{i=1}^{k} D_i \leqslant n \sum_{i=2}^{k} D_i - \frac{k(k-1)}{2} + \deg R$$

Which, assuming $D_1 = D$ is the largest, yields

$$\deg R \geqslant nD - k(k-1)\, D + \frac{k(k-1)}{2}.$$

An application of the above result is that x^a is representable as a sum of no fewer than $(n - a)^{1/2}$ nth powers. So in fact no monomial comes cheaply. Representing even $x^{[n/2]}$ would cost of the order of magnitude of $n^{1/2}$ polynomials. In a later section using rational functions we will show that all powers of x up to x^n require more than $c(n)^{1/2}$ nth powers of polynomials.

Our proof that sums of nth powers represent only high degree polynomials really makes use only of the fact that an nth power has high multiplicity zeroes.

The same would be true of expressions such as xP^n, $(x^2 - 7)\, Q^n$ etc. Thus we can also be sure that an equation such as

$$x\, P^{100} + (x^2 - 7)\, Q^{100} = x^{80}$$

is not solvable for P & Q polynomials and further the equation

$$x\, P^{100} + (x^2 - 7)\, Q^{99} = x^{80}$$

would also have no solutions.

Furthermore, by the same method we have an exact generalization of Newman–Klamkin [3]:

$$\sum_{i=1}^{k} P_i^{n_i} = R$$

is unsolvable for

$$\deg R < (\min_i n_i) - k^2.$$

4. Extension to Rational Functions

THEOREM. *Let $R_1^n(x) + R_2^n(x) + \cdots + R_k^n(x) = x$ where the R_i's are rational functions, then $k > 2^{1/2} n^{1/2}/4$.*

Proof. A proof similar to the proof for nth powers of polynomials could be used. However the proof will be clearer in a somewhat modified form.

Consider the equation

$$R_1^n + R_2^n + \cdots + R_k^n = x$$

and differentiate $(k - 1)$ times

$$(R_1^n)' + (R_2^n)' + \cdots + (R_k^n)' = 1$$
$$\vdots$$
$$(R_1^n)^{(k-1)} + (R_2^n)^{(k-1)} + \cdots + (R_k^n)^{(k-1)} = 0.$$

We note that the ith term in each of the k equations on the left side has the factor $(R_i)^{n-k+1}$. We assume each of the R_i's to be in lowest terms.

Now consider the k equations to be a linear system in the unknowns $(R_i)^{n-k+1}$ and proceed to solve by use of Cramer's Rule. In fact solve for $(R_1)^{n-k+1}$ where we rearrange so that R_1 has the polynomial numerator or denominator of largest degree. Call this degree A.

Cramer's rule gives

$$R_1^{n-k+1} = \left| \begin{matrix} x & R_2^{k-1} & \cdots & R_n^{k-1} \\ 1 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ 0 & \dfrac{(R_2^n)^{(k-1)}}{R_2^{n-k+1}} & \cdots & \dfrac{(R_n^n)^{(k-1)}}{R_n^{n-k-1}} \end{matrix} \right| \Big/ \Delta,$$

where

$$\Delta = \left| \begin{matrix} R_1^{k-1} & \cdots & R_n^{k-1} \\ \vdots & & \vdots \\ \dfrac{(R_1^n)^{(k-1)}}{R_1^{n-k+1}} & \cdots & \dfrac{(R_n^n)^{(k-1)}}{R_n^{n-k-1}} \end{matrix} \right|.$$

We want to derive an inequality based on the sums of the degrees of the numerator and the denominator on both sides.

An obvious lower bound for the left side is $2(n - k + 1)A$ and R_1 is assumed in lowest terms.

To get an upper bound on the right hand side, we note that the entries in the first row are powers of rational functions. More important the denominators are powers. When such a function is differentiated

$$\left(\frac{F}{g^r}\right)' = \frac{gF' - Frg'}{g^{r+1}},$$

we note that the degree of the numerator and denominator each increases by at most the degree of g and also that the resulting derivative has as its denominator a power of g.

Therefore by mathematical induction the largest degree numerator or denominator in the determinant is $(k-1)A + (k-1)A = 2(k-1)A$ and common denominators can be formed in each column of the determinant separately without increasing this degree. Using the diagonal rule to evaluate the determinant, and noting that each column has a common denominator we see that the degree of the numerator and denominator of the determinant are each bounded above by $2k(k-1)A$. Thus the sum of the degrees of the numerator and denominator of the top determinant

$$\leqslant 4k(k-1)A.$$

Noting that the bottom determinant can have the same effect on the degrees and that the bottom determinant $\neq 0$ because of the linear independence of the $\{R_i\}$ we have finally that

$$(n-k+1)A \leqslant 8k(k-1)A$$

or

$$k > \frac{2^{1/2}n^{1/2}}{4}$$

and this completes the proof. We also get an identical bound for representing 1 as a sum of non-constant nth powers of rational functions.

The question of whether or not any polynomial can possibly be represented more efficiently by nth powers of rational functions rather than by nth powers of polynomials might arise at this point. This can be answered in the affirmative.

Certainly $(x^5 + 1/x)^6 - 1/x^6$ is a polynomial. Suppose that there existed polynomials P and Q such that

$$P^6 + Q^6 = \left(x^5 + \frac{1}{x}\right)^6 - \frac{1}{x^6}.$$

Upon multiplying through by x^6 we would have 1 as the sum of 3 sixth powers of polynomials. This is a contradiction since we have already established that 1 is the sum of no fewer than 4 sixth powers.

We can show that x is not the sum of two nth powers of rational functions for $n > 2$. We note that our two rational functions can have no common zero for if they did $R_1^n + R_2^n$ would have an nth order zero whereas x has only a simple zero. Our problem is now equivalent to demonstrating that $P^n + Q^n = xM^n$, where P, Q and M are polynomials and P and Q have no common zeros, has no solutions.

Let M be the polynomial of lowest order which is a solution to such an equation. Clearly M is not a constant since $P^n + Q^n = cx$ is not solvable in polynomials.

We can factorize $P^n + Q^n = xM^n$ as

$$\prod_{\omega_i \text{ an } n\text{th root of } -1} (P + \omega_i Q) = xM^n.$$

Now no two factors on the left side have a common zero. If x_1 were a common zero we would have $P(x_1) = -\omega Q(x_1)$, $P(x_1) = -\omega_2 Q(x_1)$ and this is possible only if $P(x_1) = Q(x_1) = 0$ which would contradict the fact that P and Q have no common zeros.

We are forced to conclude that each factor on the left except one is a perfect nth power. The remaining factor must be x times an nth power. Thus we have since $n > 2$

$$P + \omega_1 Q = A^n, \tag{1}$$

$$P + \omega_2 Q = B^n, \tag{2}$$

$$P + \omega_3 Q = xC^n. \tag{3}$$

(1) and (2) yields that

$$Q = \frac{A^n - B^n}{\omega_1 - \omega_2}, \qquad P = \frac{\omega_2 A^n - \omega_1 B^n}{\omega_2 - \omega_1}.$$

Substitution into (3) yields that

$$c_1 A^n + c_2 B^n = xC^n.$$

Since either A or B is certainly non-constant the degree of C is less than the degree of M and this contradicts our assumption that M is of minimum degree.

In [4] Molluzo showed that 1 can be represented as the sum of $[(4n + 1)^{1/2}]$ nth powers of polynomials. He considered

$$\sum_{\omega_i \text{ a } k\text{th root of unity}} (1 + \omega_i x^n)^n = 1 + a_1 x^{kn} + a_2 x^{2kn} + \cdots + a_{[n/k]} x^{[n/k]n}.$$

This gives 1 as the sum of $k + [n/k]$ nth powers. The min of $k + [n/k]$ for all k is $[(4n + 1)^{1/2}]$[1].

[1] $[n/k] + k$ is minimized for $k = [n^{1/2}]$ or $[n^{1/2}] + 1$.

Every number n is of the form $k^2 + a (0 \leqslant a \leqslant 2k)$ and the minimizing k is either k or $k + 1$.

Using k to minimize

$$\begin{matrix} \text{for } 0 \leqslant a < k \\ k \leqslant a < 2k \\ a = 2k \end{matrix} \quad [n/k] + k = \begin{cases} 2k \\ 2k + 1 \\ 2k + 2 \end{cases}$$

Using $k + 1$

$$\begin{matrix} \text{for } 0 \leqslant a < k \\ k \leqslant a \leqslant 2k \end{matrix} \quad [n/k] + k = \begin{cases} 2k \\ 2k + 1 \end{cases}$$

and these minimum values $= [(4n + 1)^{1/2}]$.

We can usually save one nth power with the use of rational functions.

$$\sum_{\omega_i \text{ a } k\text{th root of unity}} \frac{\omega_i(1 + \omega_i x^n)^n}{x^{(k-1)n}} = 1 + x^{kn} + \cdots + x^{([(n+1)/k]-1)kn}$$

and this is a clear saving of 1 nth power unless $n \equiv -1 \bmod k$ for the minimizing k. That is for numbers of the form $k^2 - 1$ and $k^2 + k - 1$.

5. Representation of nth Powers and Fermat's Problem for Polynomials

It is conjectured that there are no positive integral solutions to the equation

$$a^n + b^n = c^n, \qquad n > 2.$$

The same question can be posed for polynomials. Are there non-constant polynomials with no common factors P, Q and R such that

$$P^n + Q^n = R^n, \qquad n > 2?$$

It is a classical result that this equation has no solutions in non-constant polynomials with no common factors. With our method we can prove the following

(1) $P^n + Q^n = R^n$ has no solutions for $n \geqslant 32$,

(2) $P_1^n + P_2^n + \cdots + P_k^n = R^n$ has no solutions for $k \leqslant (n/8)^{1/2}$,

(3) Let A^n be any nth power polynomial then A^n is representable as a sum of $[(4n + 1)^{1/2}]$ nth powers of polynomials with no common factor.

The first statement follows from the second. The second statement follows from the theorem on representing 1 as a sum of nth powers of rational functions. Dividing both sides by R^n we have

$$\left(\frac{P_1}{R}\right) + \left(\frac{P_2}{R}\right)^n + \cdots + \left(\frac{P_k}{R}\right)^n = 1$$

and this was shown to be impossible for $k \leqslant (n/8)^{1/2}$.

For the third result we examine

$$\sum_{\omega_i \text{ a } k\text{th root of unity}} \omega_i^{k-1}(1 + \omega_i x^n)^n$$

$$= a_1 x^n + a_2 x^{(k+1)n} + \cdots a_{[(n-1)/k]+1} x^{([(n-1)/k]k+1)n}$$

and minimizing over all k^1 as before we get $a_1 x^n$ or x^n as a sum of $[(4n + 1)^{1/2}]$ nth powers. Now in the identity for x^n substitute $A(x)$ for x and this gives $A^n(x)$ as a sum of $[(4n + 1)]$ polynomials. These polynomials have no common factor since A^{cn} and $1 + \omega A^{bn}$ can have no common roots.

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