AN NOTE ON FRÖBERG'S CONJECTURE FOR FORMS OF EQUAL DEGREES

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Abstract. In this note by using very elementary consideration, we settle Fröberg's conjecture for a large number of cases, when all generators of ideals have the same degree.

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables, where $\mathbb{K}$ is an algebraicly closed field of characteristic zero. The number of variables n and the field $\mathbb{K}$ are fixed in the whole paper. Define S_d as the d-th graded component of S, i.e. the linear space of all homogeneous polynomials of degree d in n variables.

In [Fr] R. Fröberg formulated the following conjecture:

Conjecture 1. *Let $f_1, \dots, f_z$ be generic forms of degrees $a_1, \dots, a_z$ respectively. Set $I = <f_1, \dots, f_z>$. The Hilbert series of S/I is given by:*

$$HS_{S/I}(t) = \left[\frac{\prod_{i=1}^{z}(1 - t^{a_i})}{(1-t)^n} \right],$$

where [..] *means that we truncate formal power series at the first negative term.*

He proved Conjecture 1 for 2 variables, and noticed that the left-hand side is at least the right-hand side in the lexicographic sense. In [An] D. J. Anick proved Conjecture 1 for 3 variables.

In this paper we present some related results in the case when all degrees $a_1, \dots, a_z$ are the same.

Let $\mathcal{D}_d$ be any nonempty class of forms of degree d closed under the linear change of coordinates. For example: $\mathcal{D}_d = S_d$ or $\mathcal{D}_d$ is the set of d-th powers of linear forms.

We will work with the Hilbert function of an ideal, it is easy to convert to HF of the quotient algebra, because the sum of dimensions of m-th graded components of S/I and of I is the dimension of S_m. For $\mathcal{D}_d$ and z, denote by $HF_{(\mathcal{D}_d,z)}(m)$ the dimension of the m-th graded component of an ideal generated by z generic forms from $\mathcal{D}_d$, and by $HS_{(\mathcal{D}_d,z)}(t) = \sum HF_{(\mathcal{D}_d,z)}(m)t^m$ the Hilbert series of this ideal. In [HL] M. Hochster and D. Laksov found the value of $HF_{(S_d,z)}(d+1)$ for any d and z. We generalize their result for the $(d+k)$-th graded component. However we miss $2 \cdot dim(S_k)$ possible values of z.

Theorem 1. *Let d and k be positive integers. Then*

- *for $z \leqslant \frac{dim(S_{d+k})}{dim(S_k)} - dim(S_k)$, the dimension $HF_{(\mathcal{D}_d,z)}(d+k) = z \cdot dim(S_k)$;*
- *for $z \geqslant \frac{dim(S_{d+k})}{dim(S_k)} + dim(S_k)$, the dimension $HF_{(\mathcal{D}_d,z)}(d+k) = dim(S_{d+k})$.*

As a consequence of Theorem 1 we get the following statement:

Proposition 2. *Led d and z are positive integers and $\mathcal{D}_d$. If there exists r such that*

$$\frac{dim(S_{d+r+1})}{dim(S_{r+1})} + dim(S_{r+1}) \leqslant z \leqslant \frac{dim(S_{d+r})}{dim(S_r)} - dim(S_r),$$

then the Hilbert series of the ideal generated by z generic forms from $\mathcal{D}_d$ is given by

$$HS_{(\mathcal{D}_d,z)} = \sum_{k=0}^{\infty} min(z \cdot dim(S_k), dim(S_{d+k}))t^{d+k} = \frac{1}{(1-t)^n} - \left[\frac{(1-t^d)^z}{(1-t)^n}\right].$$

Of course, all interesting cases correspond to $z \leqslant dim(S_d)$; otherwise $HF_{(\mathcal{D}_d,z)}(m) = dim(S_m)$ for $m \geqslant d$. Define $p_d = \frac{\#\{\text{by criteria}\}}{dim(S_d)}$ as the probability, that for interesting z, the Hilbert series $HS_{(\mathcal{D}_d,z)}$ is given by Proposition 2.

Example 1. *For $n = 5$ and $d = 10$,*
$dim(S_d) = 1001$*;* $dim(S_1) = 5$ *and* $\frac{dim(S_{d+1})}{dim(S_1)} = 273$*;* $dim(S_2) = 15$ *and* $\frac{dim(S_{d+2})}{dim(S_2)} = \frac{364}{3} = 121\frac{1}{3}$*;* $dim(S_3) = 35$ *and* $\frac{dim(S_{d+3})}{dim(S_3)} = 68$*. Then the Hilbert series is given by Fröberg's conjecture if number of generators z belongs to one of the following intervals:*

- $z \geqslant 278$*;*
- $268 \geqslant z \geqslant 137$*;*
- $106 \geqslant z \geqslant 103$*.*

In other words, the Hilbert series is standard one except possible for $141 = 9+30+102$ cases. Thus

$$p_{10} = 1 - \frac{141}{1001} = 0,859..$$

For larger d: $p_{15} = 0,927..$*;* $p_{25} = 0,968..$*;* $p_{40} = 0,986..$

Proposition 3. *For any fixed number of variables, the probability p_d tends to 1 when $d \to +\infty$.*

Proposition 3 means that Proposition 2 gives a criterium, which covers a huge number of nontrivial cases for large d. As a consequence, we get that Fröberg's conjecture is true for many nontrivial cases for large d, when the degrees of all forms are the same.

Proofs

Proof of Theorem 1. Fix d, k and $\mathcal{D}_d$. For a given z, define a_z as the dimension of the $(d+k)$-th graded component of the intersection of two ideals generated by z forms from $\mathcal{D}_d$ and by one form from $\mathcal{D}_d$, which are generic. In other words, let $g_1, \dots, g_z, g$ be generic forms, then

$$a_z := dim\left(<g_1, \dots, g_z>_{d+k} \cap <g>_{d+k}\right).$$

Lemma 1. *If $a_{z+1} = a_z \neq 0$, then $a_z = dim(S_k)$ and $HF_{(\mathcal{D}_d,z)}(d+k) = dim(S_{d+k})$.*

Proof. Consider generic forms $g_1, \dots, g_z, g'_1, \dots, g'_z$ and g from $\mathcal{D}_d$.
We know that

$$dim\left(<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k}\right) = a_z =$$
$$= a_{z+1} = dim\left(<g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k}\right).$$

Hence

$$dim\left(<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k}\right) = dim\left(<g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k}\right).$$

The intersection in the left-hand side is a subspace of that in the right-hand side. Hence, they should coincide, i.e.

$$<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k} = <g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k}.$$

Similarly we have

$$<g_1, \dots g_{z-1}, g'_z>_{d+k} \cap <g>_{d+k} = <g_1, \dots g_{z-1}, g_z, g'_z>_{d+k} \cap <g>_{d+k},$$

which implies

$$<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k} = <g_1, \dots g_{z-1}, g'_z>_{d+k} \cap <g>_{d+k}.$$

Similarly, if we change g_{z-1} in right-hand side by form g'_{z-1}, we get the same space. Repeating this procedure with g_{z-2}, g_{z-3} and etc, we obtain

$$<g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k} = <g'_1, \dots g'_{z-1}, g'_z>_{d+k} \cap <g>_{d+k}.$$

Hence for generic $g_1, \dots, g_z, g \in \mathcal{D}_d$, the linear space $V_g := <g_1, \dots g_{z-1}, g_z>_{d+k} \cap <g>_{d+k}$ depends only on g.

Fix any generic g, and choose a function $h \in V_g$. Hence for any generic $g_1, \dots, g_z$, the form h belongs to the ideal. For a linear coordinate transformation A, define by h_A the form h after this coordinate transformation. Consider coordinate transformations $A_1, \dots, A_b$ (b is finite) such that the linear span of $h_{A_1}, \dots, h_{A_b}$ has the maximal dimension.

For generic $g_1, \dots, g_z$ (generic with these b coordinate transformations), the forms $h_{A_1}, \dots, h_{A_b}$ belong to the ideal I generated by $\{g_1, \dots, g_z\}$. Hence, the linear span of $h_{A_1}, \dots, h_{A_b}$ belongs to the ideal I. Because this linear space has the maximal dimension, it is closed under the change of coordinates.

Hence, there is a nonempty linear space $H \subset S_{d+k}$ closed under the change of coordinates, such that it belongs to any ideal generated by generic $\{g_1, \dots, g_z\}$. However S_{d+k} has only one such subspace, namely $H = S_{d+k}$. Therefore, the $(d+k)$-th graded component of the ideal is the whole S_{d+k}. This proves the lemma.

□

Let z_0 be the minimal z such that $a_z \neq 0$, and z_1 be the minimal z such that $a_{z_1} = dim(S_k)$. By Lemma 1, the dimension a_z is strictly growing between z_0 and z_1, thus

$$z_1 - z_0 \leqslant dim(S_k).$$

It is clear that

- for $z \leqslant z_0$, the dimension $HF_{(\mathcal{D}_d,z)}(d+k) = z \cdot dim(S_k)$;
- for $z \geqslant z_1$, the dimension $HF_{(\mathcal{D}_d,z)}(d+k) = dim(S_{d+k})$.

Since $z_0 \leqslant \frac{dim(S_{d+k})}{dim(S_k)}$ and $z_1 \geqslant \frac{dim(S_{d+k})}{dim(S_k)}$, we have

$$z_1 \leqslant z_0 + dim(S_k) \leqslant \frac{dim(S_{d+k})}{dim(S_k)} + dim(S_k);$$

$$z_0 \geqslant z_1 - dim(S_k) \geqslant \frac{dim(S_{d+k})}{dim(S_k)} - dim(S_k),$$

which gives the proof of the theorem.

□

Remark 1. *In fact, we proved that* $HF_{(\mathcal{D}_d,z)}(d+k) = min(z \cdot dim(S_k), dim(S_{d+k}))$ *except for at most* $dim(S_k)$ *cases for the values of* z*. However, we don't know these* $dim(S_k)$ *values exactly.*

Proof of Proposition 2. By Theorem 1, we know that $HF_{(\mathcal{D}_d,z)}(d+r+1) = dim(S_{d+r+1})$ and $HF_{(\mathcal{D}_d,z)}(d+r) = z \cdot dim(S_r)$. From the first claim we have that the $(d+r+1)$-th graded component of the ideal is S_{d+r+1}, hence for $k \geqslant r+1$, the $(d+k)$-th graded component of ideal is S_{d+k}.

From the second claim we know that for generic $g_1, \dots, g_z$ from $\mathcal{D}_d$, there are no $f_1, \dots, f_z \in S_r$ (not all zeroes), such that $g_1 f_1 + \ldots + g_z f_z = 0$. Hence, there are no such $f_1, \dots, f_z \in S_k$, for $k \leqslant r$. Then for $k \leqslant r$, we have $HF_{(\mathcal{D}_d,z)}(d+k) =$

$z \cdot dim(S_k)$. Hence, the whole Hilbert series is given by Fröberg's conjecture in this case.

□

Proof of Proposition 3. Take an integer k. Then for large d we know the Hilbert series for

$$\sum_{r=0}^{k}\left(\left(\frac{dim(S_{d+r})}{dim(S_r)} - dim(S_r)\right) - \left(\frac{dim(S_{d+r+1})}{dim(S_{r+1})} + dim(S_{r+1})\right)\right)$$

different values of z. Then we have

$$1-p_d \leqslant dim(S_d) - \frac{\sum_{r=0}^{k}\left(\left(\frac{dim(S_{d+r})}{dim(S_r)} - dim(S_r)\right) - \left(\frac{dim(S_{d+r+1})}{dim(S_{r+1})} + dim(S_{r+1})\right)\right)}{dim(S_d)},$$

$$1-p_d \leqslant \frac{\left(\frac{dim(S_{d+k+1})}{dim(S_{k+1})}\right)}{dim(S_d)} + \frac{\sum_{r=0}^{k}(dim(S_r)+dim(S_{r+1}))}{dim(S_d)}.$$

The fist summand tends to $\frac{1}{dim(S_{k+1})}$ and the second tends to zero, while d increases. Hence, limsup of $(1-p_d)$ is at most $\frac{1}{dim(S_{k+1})}$. Therefore, lim of $(1-p_d)$ is zero, because we have such a bound for any integer k.

□

References

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