

# ON COMMUTATIVE ALGEBRA ASSOCIATED TO $t$ -LABELED SUBFORESTS OF A GRAPH

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ABSTRACT. For a given graph  $G$ , we construct an associated commutative algebra, whose dimension is equal to the number of  $t$ -labeled forests of  $G$ .

We show that the dimension of the  $k$ -th graded component of this algebra also has a combinatorial meaning and that its Hilbert polynomial can be expressed through the Tutte polynomial of  $G$ .

## 1. INTRODUCTION

The famous matrix-tree theorem of Kirchhoff (see [Kir] and p. 138 in [Tut]) claims that the number of spanning trees of a given graph  $G$  equals to the determinant of the Laplacian matrix of  $G$ . It is also well known that the number of spanning forests of  $G$  or equivalently trees for connected  $G$  equals to  $T_G(1, 1)$  and the number of all subforests of  $G$  equals to  $T_G(2, 1)$ , where  $T_G$  is the Tutte polynomial of  $G$  (see e.g. 237 in [Tut]).

There exist many generalization of the matrix-tree theorem, e.g. for directed graphs, matrix-forest theorems, etc (see e.g. [ChK]). In particular, in [PSh] A. Postnikov and B. Shapiro constructed several algebras associated to  $G$  whose dimensions are equal to the number of either spanning trees or forests of  $G$ . Below we extend construction of [PSh] to a larger class of algebras.

Given a graph  $G$ ; let us associate commuting variables  $\phi_e, e \in G$  to all edges of  $G$ . For a given positive integer  $t \geq 1$ , let  $\Phi_G^{F_t}$  be the algebra generated by  $\{\phi_e : e \in G\}$  with relations  $\phi_e^{t+1} = 0$ , for any  $e \in G$ .

Take any linear order of vertices of  $G$ . For  $i = 1, \dots, n$ , Set

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

where  $c_{i,e} = \pm 1$  for vertices incident to  $e$  (for the smaller vertex,  $c_{i,e} = 1$ , for the bigger vertex, is  $c_{i,e} = -1$ ) and 0 otherwise. Denote by  $\mathcal{C}_G^{F_t}$  the subalgebra of  $\Phi_G^{F_t}$  generated by  $X_1, \dots, X_n$ .

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Let  $\mathbb{K}$  be some field of characteristic 0. Consider the ideal  $J_G^{F_t}$  in the ring  $\mathbb{K}[x_1, \dots, x_n]$  generated by

$$p_I^{F_t} = \left( \sum_{i \in I} x_i \right)^{tD_I+1},$$

where  $I$  ranges over all nonempty subsets of vertices, and  $D_I$  is the total number of edges from vertices in  $I$  to vertices outside the subset  $I$ . Define the algebra  $\mathcal{B}_G^{F_t}$  as the quotient  $\mathbb{K}[x_1, \dots, x_n]/J_G^{F_t}$ .

**Notation 1.** Fix some linear order on the edges of  $G$ . Let  $F$  be any a subforest in  $G$ . By  $\text{act}_G(F)$  denote the number of all externally active edges of  $F$ , i.e. the number of edges  $e \in G \setminus F$  such that subgraph  $F + e$  has a cycle and  $e$  is the minimal edge in this cycle in the above linear order.

Denote by  $F^+$  the set of edges of the forest  $F$  together with externally active edges, and denote by  $F^- = G \setminus F^+$  the set of nonactive edges.

It is well known that the number of spanning trees and subforests with fixed external activity is independent of the linear order on the set of edges of  $G$ .

For  $t = 1$  these algebras (denoted by  $\mathcal{B}_G^F$  and  $\mathcal{C}_G^F$ ) were introduced in [PSh] where the following result was proved.

**Theorem 1** (cf. [PSh]). *The algebras  $\mathcal{B}_G^F$  and  $\mathcal{C}_G^F$  are isomorphic. Their total dimension as vector spaces over  $\mathbb{K}$  is equal to the number of subforests in the graph  $G$ .*

*The dimension of the  $k$ -th graded component of these algebras equals the number of subforests  $T$  of  $G$  with external activity  $|G| - |T| - k$ .*

Below we generalize this result for  $t > 1$ , and show that the corresponding dimension coincides with the number of the so-called  $t$ -labeled trees. In Theorem 3 we prove that the Hilbert polynomial of  $\mathcal{B}_G^{F_t}$  can be expressed in terms of the Tutte polynomial of  $G$ . And conversely, in Proposition 6 we show that the Tutte polynomial of  $G$  can be restored from the Hilbert series of the algebra  $\mathcal{B}_G^{F_t}$  for any sufficiently large  $t$ .

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## 2. $t$ -LABELED FORESTS

Consider a finite labelling set containing  $t$  different labels; each label corresponds to a number from 1 to  $t$ .

**Definition 2.** *A spanning forest of the graph  $G$  with a label on each edge is called a  $t$ -labeled forest. The weight of a  $t$ -labeled forest  $F$ , denoted by  $\omega(F)$ , is the sum of labels of all its edges.*

**Theorem 2.** (I) For any graph  $G$  and a positive integer  $t$ , algebras  $\mathcal{B}_G^{F_t}$  and  $\mathcal{C}_G^{F_t}$  are isomorphic, their total dimension over  $\mathbb{K}$  is equal to the number of  $t$ -labeled forests in  $G$ .

(II) The dimension of the  $k$ -th graded component of the algebra  $\mathcal{B}_G^{F_t}$  is equal to the number of  $t$ -labeled forests  $F$  of  $G$  with the weight  $t \cdot (e(G) - \text{act}_G(F)) - k$ .

*Proof.* Denote by  $\widehat{G}$  the graph on  $n$  vertices and  $t \cdot e(G)$  edges such that each edge of  $G$  corresponds to  $t$  clones in the graph  $\widehat{G}$ , i.e. each edge is substituted by its  $t$  copies with labels  $1, 2, \dots, t$ . For each edge  $e \in G$ , its clones  $e_1, \dots, e_t \in \widehat{G}$  are ordered according to their numbers; clones of different edges have the same linear order as the original edges.

Consider the following bijection between  $t$ -labeled forests in  $G$  and forests in  $\widehat{G}$ : each  $t$ -labeled forest  $F \in G$  corresponds to the forest  $F' \in \widehat{G}$ , such that for each edge  $e \in F$ , the forest  $F'$  has the clone of the edge  $e$  whose number is identical with the label of edge  $e$  in the forest  $F$ .

Obviously,

$$\text{act}_{\widehat{G}}(F') = t \cdot \text{act}_G(F) + \omega(F) - F,$$

and  $e(\widehat{G}) = t \cdot e(G)$ . Since  $\mathcal{B}_G^{F_t}$  and  $\mathcal{B}_{\widehat{G}}^F$  are the same, the Hilbert series of the algebra  $\mathcal{B}_G^{F_t}$  coincides with the Hilbert series of the algebra  $\mathcal{B}_{\widehat{G}}^F$ , which settles the second part of Theorem 2.

To prove the first part of the theorem, observe that  $\mathcal{B}_G^{F_t}$  and  $\mathcal{B}_{\widehat{G}}^F$  are the same, and algebras  $\mathcal{C}_{\widehat{G}}^{F_t}$  and  $\mathcal{B}_{\widehat{G}}^F$  are isomorphic. Thus we must show that algebras  $\mathcal{C}_G^F$  and  $\mathcal{C}_{\widehat{G}}^{F_t}$  are isomorphic. It is indeed true, because for every edge  $e \in G$ , the elements  $\phi_e, \dots, \phi_e^t$  are linearly independent in the algebra  $\Phi_G^{F_t}$  with coefficients containing no  $\phi_e$ . Also elements  $(\phi_{e_1} + \dots + \phi_{e_t}), \dots, (\phi_{e_1} + \dots + \phi_{e_t})^t$  are linearly independent in the algebra  $\Phi_{\widehat{G}}^F$  with coefficients containing no  $\phi_{e_1}, \dots, \phi_{e_t}$ , and  $(\phi_{e_1} + \dots + \phi_{e_t})^{t+1} = 0$ . Moreover elements  $\phi_{e_i}$  only occur in the sum  $(\phi_{e_1} + \dots + \phi_{e_t})$  in the algebra  $\Phi_{\widehat{G}}^F$ .  $\square$

Denote by  $c(G)$  the number of connected components of the graph  $G$ .

**Theorem 3.** Dimension of the  $k$ -th graded component of  $\mathcal{B}_G^{F_t}$  is equal to the coefficient of the monomial  $y^{t \cdot e(G) - c(G) + v(G) + 1 - k}$  in the polynomial

$$\left( \frac{y^t - 1}{y - 1} \right)^{v(G) - c(G)} \cdot T_G \left( \frac{y^{t+1} - 1}{y^{t+1} - y}, y^t \right).$$

*Proof.* Consider the graph  $\widehat{G}$  constructed in the proof of Theorem 2. Set  $J_G(x, y) := T_{\widehat{G}}(x, y)$ ; we will use the follow deletion–contraction recurrence for  $J_G$ , where  $G - e$  denote the graph obtained by deleting  $e$  from  $G$  and  $G \cdot e$  is the contraction of  $G$  by  $e$ .

**Lemma 4.** *Polynomial  $J_G(x, y)$  satisfies the following:*

- (1) *If  $G$  is empty, then  $J_G(x, y) = 1$ .*
- (2) *If  $e$  is a loop in  $G$ , then  $J_G(x, y) = y^t J_{G-e}(x, y)$ .*
- (3) *If  $e$  is a bridge in  $G$ , then  $J_G(x, y) = (y^{t-1} + \dots + 1) \cdot J_{G-e}(x, y) + (x - 1) \cdot J_{G-e}(x, y)$ .*
- (4) *If  $e$  is not a loop or a bridge, then  $J_G(x, y) = (y^{t-1} + \dots + 1) \cdot J_{G-e}(x, y) + J_{G-e}(x, y)$ .*

*Proof.* We prove these relations by using the deletion–contraction recurrence for the usual Tutte polynomial

- (1) If  $G$  is empty, then  $T_G(x, y) = 1$ .
- (2) If  $e$  is a loop in graph  $G$ , then  $T_G(x, y) = y \cdot T_{G-e}(x, y)$ .
- (3) If  $e$  is a bridge in graph  $G$ , then  $T_G(x, y) = x \cdot T_{G-e}(x, y)$ .
- (4) If  $e$  is not a loop or a bridge, then  $T_G(x, y) = T_{G-e}(x, y) + T_{G-e}(x, y)$ .

1. Graph  $\widehat{G}$  is also empty, hence  $J_G(x, y) = T_{\widehat{G}}(x, y) = 1$ .
2. Clones  $e_1, \dots, e_t$  is also loops in graph  $\widehat{G}$ , therefore  $T_{\widehat{G}}(x, y) = y^t \cdot T_{\widehat{G}-\{e_1, \dots, e_t\}}(x, y) = y^t \cdot T_{\widehat{G-e}}(x, y)$ , hence  $J_G(x, y) = y^t \cdot T_{\widehat{G-e}}(x, y) = J_{G-e}(x, y)$ .
3. We calculate our polynomial using the deletion–contraction recurrence for the Tutte polynomial.

$$\begin{aligned}
J_G(x, y) &= T_{\widehat{G}}(x, y) = T_{\widehat{G-e_k}}(x, y) + T_{\widehat{G-e_t}}(x, y) = \\
&= y^{t-1} \cdot T_{\widehat{G-e}}(x, y) + T_{\widehat{G-e_t}}(x, y) = \\
&= y^{t-1} \cdot T_{\widehat{G-e}}(x, y) + T_{(\widehat{G-e_t})e_{t-1}}(x, y) + T_{\widehat{G-e_t-e_{t-1}}}(x, y) = \\
&= (y^{t-1} + y^{t-2}) \cdot T_{\widehat{G-e}}(x, y) + T_{\widehat{G-e_t-e_{t-1}}}(x, y) = \\
&\quad \dots \\
&= (y^{t-1} + y^{t-2} + \dots + y) \cdot T_{\widehat{G-e}}(x, y) + T_{\widehat{G-e_t-\dots-e_2}}(x, y) = \\
&= (y^{t-1} + y^{t-2} + \dots + y) \cdot T_{\widehat{G-e}}(x, y) + x \cdot T_{\widehat{G-e_t-\dots-e_1}}(x, y) = \\
&= (y^{t-1} + y^{t-2} + \dots + y) \cdot T_{\widehat{G-e}}(x, y) + x \cdot T_{\widehat{G-e}}(x, y) = \\
&= (y^{t-1} + y^{t-2} + \dots + y + 1) \cdot T_{\widehat{G-e}}(x, y) + (x - 1) \cdot T_{\widehat{G-e}}(x, y) = \\
&= (y^{t-1} + \dots + 1) \cdot J_{G-e}(x, y) + (x - 1) \cdot J_{G-e}(x, y).
\end{aligned}$$

4. It is similar to 3, but now we have

$$\begin{aligned}
T_{\widehat{G-e_k-\dots-e_2}}(x, y) &= T_{(\widehat{G-e_k-\dots-e_2})e_1}(x, y) + T_{\widehat{G-e_k-\dots-e_2-e_1}}(x, y) = \\
&= T_{\widehat{G-e}}(x, y) + T_{\widehat{G-e}}(x, y),
\end{aligned}$$

since in this case edge  $e_1$  is not a bridge in  $\widehat{G} - e_k - \dots - e_2$ .

□

Now let us rewrite  $J_G(x, y)$  in terms of  $t$ -labeled forests using the deletion–contraction recurrence for  $J_G(x, y)$  in the above fixed linear order of edges of  $G$ . Obviously, the edges by which we contract the graph constitute a forest. Therefore,  $J_G(x, y) = \sum_{F_u} a(F_u)$ , where  $a(F_u)$  depends only on  $G$  and the forest  $F_u$ . Now rewrite the latter equality in terms of  $t$ -labeled forests. When we contract edge  $e$  in  $G$ , the term  $y^{k-1}$  in the factor  $(y^{t-1} + y^{t-2} + \dots + y + 1)$  corresponds to the choice the  $k$ -th label for edge  $e$ , i.e. we have  $J_G(x, y) = \sum_F y^{\omega(F)-|F|} b(F)$ . It remains to calculate  $b(F)$ . An edge for  $t$ -labeled forest  $F$  is a loop if and only if it is active, and the number of edges which are bridges in our recursion equals to  $c(F) - c(G) = (v(G) - 1 - |F|) - c(G) = (v(G) - c(G)) - 1 - |F|$ . Therefore, we have

$$\begin{aligned} J_G(x, y) &= \sum_F y^{\omega(F)-|F|} \cdot y^{t \cdot \text{act}_G(F)} \cdot (x-1)^{(v(G)-c(G))-1-|F|}, \\ J_G(x, y) &= \sum_F y^{\omega(F)-|F|+\text{act}_G(F)} \cdot (x-1)^{(c(G)-v(G))-1-|F|}, \\ J_G\left(1 + \frac{1}{y}, y\right) &= \sum_F y^{\omega(F)-|F|+t \cdot \text{act}_G(F)} \cdot \left(\frac{1}{y}\right)^{(c(G)-v(G))-1-|F|}, \\ J_G\left(1 + \frac{1}{y}, y\right) &= \sum_F y^{\omega(F)-|F|+t \cdot \text{act}_G(F)-((c(G)-v(G))+1+|F|)}, \\ J_G\left(1 + \frac{1}{y}, y\right) &= \sum_F y^{\omega(F)+t \cdot \text{act}_G(F)-c(G)+v(G)+1}. \end{aligned} \quad (*)$$

By Theorem 2 the dimension of the  $k$ -th graded component of algebra  $\mathcal{B}_G^{F_t}$  equals the number of  $t$ -labeled forests  $F$  of  $G$  with weight  $t \cdot (e(G) - \text{act}_G(F)) - k$ . Then the dimension of the  $k$ -th graded component is equal to the coefficient of the monomial  $y^{t \cdot e(G) - k - c(G) + v(G) + 1}$  in polynomial  $J_G\left(1 + \frac{1}{y}, y\right)$ .

**Lemma 5.**

$$J_G\left(1 + \frac{1}{y}, y\right) = \left(\frac{y^t - 1}{y - 1}\right)^{v-c(G)} \cdot T_G\left(\frac{y^{t+1} - 1}{y^{t+1} - y}, y^t\right).$$

*Proof.* Conditions 1, 2 and 4 of Lemma 4 hold for polynomial  $\left(\frac{y^t - 1}{y - 1}\right)^{v-c(G)}$ .

$T_G\left(\frac{y^{t+1} - 1}{y^{t+1} - y}, y^t\right)$ . Now we can check the 3-rd condition. Set

$$z := (y^{t-1} + \dots + 1) = \frac{y^t - 1}{y - 1},$$

then,  $\frac{y^{t+1} - 1}{y^{t+1} - y} = \frac{1}{zy} + 1$ . We have

$$z^{v(G)-c(G)} \cdot T_G\left(\frac{1}{zy} + 1, y^t\right) =$$

$$\begin{aligned}
& z^{v(G)-c(G)} \cdot T_{G \cdot e} \left( \frac{1}{zy} + 1, y^t \right) + \frac{1}{zy} \cdot z^{v(G)-c(G)} \cdot T_{G-e} \left( \frac{1}{zy} + 1, y^t \right) = \\
& (y^{t-1} + \dots + 1) \cdot z^{v(G \cdot e)-c(G \cdot e)} \cdot T_{G \cdot e} \left( \frac{1}{zy} + 1, y^t \right) + \\
& \frac{1}{y} \cdot z^{v(G-e)-c(G-e)} \cdot T_{G-e} \left( \frac{1}{zy} + 1, y^t \right).
\end{aligned}$$

Hence, the 3-rd condition holds as well. Therefore, if we calculate these polynomials using the recursion method we get the same results, hence, these polynomials coincide.  $\square$

This settles Theorem 3.  $\square$

**Proposition 6.** *For any positive integer  $t \geq n$ , it is possible to restore the Tutte polynomial of any connected graph  $G$  on  $n$  vertices knowing only the dimensions of each graded component of the algebra  $\mathcal{B}_G^t$ .*

*Proof.* Choose a integer  $t \geq n$ . By Theorem 2 we know that the degree of the maximal non empty graded component of  $\mathcal{B}_G^{F_t^n}$  equals to the maximum of  $t \cdot (e(G) - act_G(F)) - \omega(F)$  taken over  $F$ . It attains its maximal value for the empty forest (i.e.  $F = \emptyset$ ). Then we know the value of  $t \cdot e(G)$ , hence, we know the number of edges of the graph  $G$ .

By Theorem 3 we also know the polynomial

$$\left( \frac{y^t - 1}{y - 1} \right)^{v(G)-c(G)} \cdot T_G \left( \frac{y^{t+1} - 1}{y^{t+1} - y}, y^t \right),$$

because  $G$  is connected (i.e.  $c(G) = 1$ ). This polynomial equals to

$$\sum_F y^{\omega(F)+t \cdot act_G(F)-c(G)+v(G)+1} = \sum_F y^{\omega(F)+t \cdot act_G(F)+v(G)},$$

where the summation is taken over all  $t$ -labeled forests (see eq. (\*) and Lemma 5). Rewriting it in terms of the usual subforests, we can calculate

$$\sum_{F_u} (y + \dots + y^t)^{|F_u|} \cdot y^{t \cdot act_G(F_u)+v(G)},$$

Hence, we also know the sum

$$\sum_{F_u} (1 + \dots + y^{(t-1)})^{|F_u|} \cdot y^{|F_u|+t \cdot act_G(F_u)}. \quad (**)$$

Since  $|F_u| < t$ , then we can compute the number of usual subforests with a fixed pair of parameters  $|F_u|$  and  $act_G(F_u)$ . Consider the monomial of minimal degree in polynomial  $(\oplus)$ , and present it in the form  $s \cdot y^m$ . Observe that  $s$  is the number of subforests  $F_u$  s.t.  $F_u \equiv m \pmod{t}$  and with  $act_G(F_u) = \lfloor \frac{m}{t} \rfloor$ . Remove from the polynomial (\*\*) all summands for these subforests, and repeat this operation until we get 0.

It is well known that  $T_G(x, y) = \sum_{a,b} \#\{F_u : |F_u| = a, \text{act}(F_u) = b\} \cdot (x-1)^{n-1-a} \cdot y^b$ . Therefore since we know the number of usual subforests with any fixed number of edges and any fixed extrenal activity, we know the whole Tutte polynomial. □

**Remark 1.** *It is possible to obtain similar results for t-labeled trees except for Proposition 6. But in our opinion such results are not very interesting, because the number of edges in every tree is the same.*

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