WARING LOCUS FOR POLYNOMIALS

May 22nd, 2015

§ 1 INTRODUCTION

Let $S = \mathbb{C}[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$ be the polynomial ring in $n+1$ variables and complex coefficients with standard gradation, i.e. S_d is the $\mathbb{C}$-vector space of degree d homogeneous polynomials.

1.1 Waring problems.

We look to additive decompositions of (homogeneous) polynomials as sum of powers. Given a polynomial $F \in S_d$, a *Waring decomposition* is an espression of the type

$$F = L_1^d + \dots + L_s^d, \qquad L_i \in S_1.$$

The *Waring rank* of F is the length of the shortest Waring decomposition of F ($rk(F)$).

Questions. [i] What is the Waring rank of the *general* polynomial in S_d? ($rk_{gen}(d,n)$)

[ii] What is the Waring rank of a *given* polynomial $F \in S_d$? How many decompositions does it have?

[iii] What is the maximal rank among all polynomials in S_d? ($rk_{max}(d,n)$)

Known. [i] Completely solved by J. Alexander & A. Hirschowitz (1995)

[ii] Binary forms by J.J. Sylvester (1861)

Monomials by E. Carlini, M.V. Catalisano & A.V. Geramita (2012)

Partial algorithms by Buczyńska - Buczyński, Brachat - Comon - Mourrain - Tsigaridas, Oeding - Ottaviani

[iii] Results in low degrees and variables by Segre, De Paris, Kleppe

In general, by Blekherman - Teitler

$$rk_{max}(d,n) \leq 2\, rk_{gen}(d,n) - 1.$$

Remark. In 2012, R. Fröberg, G. Ottaviani and B. Shapiro introduced a Waring problem for *higher degrees forms*; namely, given $F \in S_{kd}$ a *k-th Waring decomposition* is an expression of the type

$$F = G_1^k + \dots + G_s^k, \qquad G_i \in S_d.$$

Then we can define the *k-th Waring rank* of F and ask the same questions.

1.2 Apolarity.

Consider also the polynomial ring $T = \mathbb{C}[\partial_0, \dots, \partial_n] = \bigoplus_{d \geq 0} T_d$ acting on S as partial derivatives; namely, we define

$$\partial_i \circ x_j := \frac{\partial}{\partial x_i}(x_j) = \delta_{i,j}, \qquad \forall\, i,j;$$

and we extend with the usual properties of differentials.

Given a polynomial $F \in S_d$, we define its *perp ideal* as

$$F^\perp := \{ \partial \in T \mid \partial F = 0 \}$$

APOLARITY LEMMA. GIVEN $F \in S_d$, THE FOLLOWING ARE EQUIVALENT

[i] $F = L_1^d + \dots + L_s^d$, $L_i \in S_1$;

[ii] $F^\perp \supset I_{\mathbb{X}}$ WHERE $\mathbb{X}$ IS A SET OF s DISTINCT SIMPLE POINTS IN $\mathbb{P}^n$.

EXAMPLE. $F = xyz \in \mathbb{C}[x,y,z]$; THEN $F^\perp = (x^2, y^2, z^2)$ WHICH CONTAINS

$I = (x^2 - y^2, x^2 - z^2)$ WHICH IS THE IDEAL

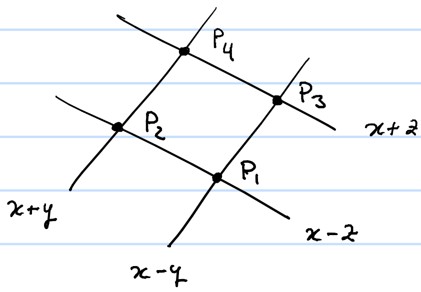
DEFINING FOUR SIMPLE POINTS IN $\mathbb{P}^2$:

$P_1 = [1:1:1]$

$P_2 = [1:-1:1]$

$P_3 = [1:1:-1]$

$P_4 = [1:-1:-1]$



AND CONSEQUENTLY

$$24\,xyz = (x+y+z)^3 - (x-y+z)^3 - (x+y-z)^3 + (x-y-z)^3$$

§2 WARING LOCUS & FORBIDDEN POINTS

QUESTION. HOW DO MINIMAL WARING DECOMPOSITIONS LOOK LIKE?

GIVEN $F \in S_d$, WE DEFINE THE FOLLOWING LOCI:

WARING LOCUS. $\mathcal{W}_F := \{ [L] \in \mathbb{P}(S_1) \mid L \text{ APPEARS IN A MINIMAL WARING DECOMPOSITION OF } F \}$

FORBIDDEN POINTS. $\mathcal{F}_F := \mathcal{W}_F^c$.

REMARK. GIVEN A POLYNOMIAL $F \in S_d$, WE SAY THAT F essentially involves $m+1$ variables IF THERE EXISTS A CHANGE OF VARIABLES SUCH THAT $F \in \mathbb{C}[\ell_0 \dots \ell_m]$, $\ell_i \in S_1$.

LEMMA. IF $F \in S_d$ ESSENTIALLY INVOLVES $m+1$ VARIABLES, THEN EVERY MINIMAL WARING DECOMPOSITION INVOLVES LINEAR FORMS ESSENTIALLY IN $m+1$ VARIABLES.

CONSEQUENTLY, WE WILL ALWAYS LOOK WARING LOCUS AND FORBIDDEN POINTS IN THE SMALLEST SET OF VARIABLES POSSIBLE; i.e. IF F ESSENTIALLY INVOLVES $m+1$ VARIABLES, THEN $\mathcal{W}_F, \mathcal{F}_F \subseteq \mathbb{P}^m = \mathbb{P}(\mathbb{C}[\ell_0 \dots \ell_m]_1)$.

2.1 QUADRICS

PROPOSITION [CARLINI - O., 2015]

GIVEN $Q \in S_2$, THEN THE FORBIDDEN POINTS ARE THE ZEROES OF Q; i.e.

$$\mathcal{F}_Q = V(Q) \subseteq \mathbb{P}^n.$$

2.2 BINARY FORMS

IT IS A WELL-KNOWN FACT THAT, GIVEN $F \in \mathbb{C}[x,y]_d$,

$$F^\perp = (G_1, G_2), \text{ WHERE } \deg G_i = d_i, \quad d_1 \le d_2 \text{ AND } d_1 + d_2 = d+2;$$

FROM THERE, WE HAVE AN EXPLICIT WAY TO DETERMINE THE RANK OF F WHICH GOES BACK TO J.J. SYLVESTER:

IF G_1 IS SQUARE-FREE, THEN $rk(F) = \deg G_1$;

OTHERWISE, $rk(F) = \deg G_2$.

THEOREM [CARLINI - O., 2015] LET $F \in \mathbb{C}[x,y]_d$ WITH $F^\perp$ AS ABOVE; THEN,

[i] IF $rk(F) < \lceil \frac{d+1}{2} \rceil$, THEN $\mathcal{W}_F = V(G_1)$;

[ii] IF $rk(F) > \lceil \frac{d+1}{2} \rceil$, THEN $\mathcal{F}_F = V(G_1)$;

[iii] IF $rk(F) = \lceil \frac{d+1}{2} \rceil$, THEN

IF d ODD, $\mathcal{W}_F = V(G_1)$;

IF d EVEN, $\mathcal{F}_F$ IS FINITE AND NOT EMPTY.

proof: [i] IN THAT CASE, G_1 IS SQUARE-FREE AND $G = L_1 \cdots L_{d_1}$; HENCE,

$$F^\perp \supset (L_1 \cdots L_{d_1}) = (L_1) \cap \ldots \cap (L_{d_1}) = I_{\mathbb{X}}$$

IDEAL OF d_1 DISTINCT POINTS IN $\mathbb{P}^1$; MOREOVER, THE DECOMPOSITION IS UNIQUE HENCE,

$$\mathcal{W}_F = \mathbb{X} = [L_1^\vee] + \ldots + [L_{d_1}^\vee]$$

($[L_i^\vee]$ DENOTES THE SOLUTION OF L_i)

[ii] IN THIS CASE, G_1 IS NOT SQUARE-FREE AND $\deg G_1 < \deg G_2$.

CONSIDER $[L^\vee] \in V(G_1)$, NAMELY L DIVIDES G_1.

SINCE G_1 AND G_2 DO NOT HAVE COMMON FACTORS ($F^\perp$ IS ARTINIAN), THE ONLY MULTIPLES OF L IN $F^\perp$ MUST BE MULTIPLES OF G_1 AND, IN PARTICULAR, THEY CANNOT BE SQUARE-FREE AND CONSEQUENTLY $[L^\vee] \in \mathcal{F}_F$.

VICEVERSA, CONSIDER $[L^\vee] \notin V(G_1)$;

THEN $$F^\perp : (L) = (L \circ F)^\perp = (H_1, H_2)$$

WHERE ACTUALLY $H_1 = G_1$ SINCE L DOES NOT DIVIDE G_1. THUS,

$$rk(L \circ F) = rk(F) - 1$$

AND THERE EXISTS A POLYNOMIAL SQUARE-FREE $H \in (L \circ F)^\perp$ OF DEGREE $d_2 - 1$; HENCE, HL HAS DEGREE d_2, IS SQUARE-FREE AND CONTAINED IN $F^\perp$.

CONSEQUENTLY, $[L^\vee] \in \mathcal{W}_F$.

[iii] IF d ODD: $d_2 = d_1 + 1$ AND $rk(F) = d_1$; NAMELY,

G_1 IS SQUARE-FREE AND THE DECOMPOSITION IS AGAIN UNIQUE.

IF d EVEN: $d_1 = d_2$ AND THEN WE LOOK AT THE LINE

$$\mathbb{P}(\mathcal{L}) = \{ \alpha G_1 + \beta G_2 \mid [\alpha : \beta] \in \mathbb{P}^1 \} \cong \mathbb{P}^1.$$

FOR EACH POINT $[L^\vee] \in \mathbb{P}^1$, WE GET A UNIQUE POINT $[G] \in \mathbb{P}(\mathcal{L}) \subseteq \mathbb{P}(S_{d_1})$:

IF $[G]$ IS SQUARE-FREE, THEN $[L^\vee] \in W_F$

OTHERWISE, $[L^\vee] \in \mathcal{F}_F$.

THE NON-SQUARE FREE ELEMENTS $[G] \in \mathbb{P}(\mathcal{L})$ ARE GIVEN BY THE (FINITE AND NON-EMPTY) INTERSECTION BETWEEN $\mathbb{P}(\mathcal{L})$ AND THE HYPERSURFACE Δ_{d_1} GIVEN BY THE DISCRIMINANT.

$\square$

REMARK. BY ITERATING THIS RESULT WE CAN CONSTRUCT A MINIMAL WARING DECOMPOSITION OF OF A GIVEN POLYNOMIAL F. ASSUME THAT THE RANK IS LARGE AND WE DON'T HAVE A UNIQUE DECOMPOSITION: PICK $[L] \notin \mathcal{F}_F$ AND LOOK AT $F - L^d$; IT HAS RANK SMALLER AND WE CAN USE AGAIN THE RESULT SINCE $\mathcal{F}_{F-L^d} = \mathcal{F}_F \cup [L]$. IN CONCLUSION, GIVEN F WITH $rk(F) = r$:

FOR ANY $L_1 \dots L_s \notin \mathcal{F}_F$ WITH $\begin{cases} s = r - \lceil \frac{d+1}{2} \rceil & \text{IF } d \text{ ODD} \\ s = r - \lfloor \frac{d+1}{2} \rfloor & \text{IF } d \text{ EVEN} \end{cases}$, THERE IS A (UNIQUE) MINIMAL WARING DECOMPOSITION OF F.

2.3 MONOMIALS

CONSIDER $M = x_0^{a_0} \cdots x_n^{a_n}$ WITH $0 < a_0 = \dots = a_m < a_{m+1} \leq \dots \leq a_n$; THEN, $M^\perp = (x_0^{a_0+1} \dots x_n^{a_n+1})$

THEOREM [CARLINI - CATALISANO - GERAMITA, 2012]

$$rk(M) = \frac{1}{a_0+1} \prod_{i=0}^{n} (a_i+1)$$

THEOREM [CARLINI - O., 2015]

IN THE SAME NOTATION AS ABOVE,

$$\mathcal{F}_M = V(x_0 \dots x_m) = \{ [c_0 x_0 + \dots + c_m x_m] \in \mathbb{P}(S_1) \mid c_0 \cdots c_m = 0 \}$$

proof (sketch): TAKE $[c_0 : \dots : c_n] \in \mathbb{P}(S_1) \setminus V(x_0 \cdots x_m)$, WE MAY ASSUME $c_0 = 1$; DEFINE

$$H_i = \begin{cases} x_i^{d_i+1} - c_i x_0^{d_i+1} & \text{if } c_i \neq 0 \\ x_i^{d_i+1} - x_0^{d_i+1} & \text{if } c_i = 0 \end{cases}$$

THEN, $(H_1, \dots, H_n) \subseteq M^\perp$ IS THE IDEAL DEFINING THE SET

$$\mathbb{X} = \{ [1 : p_{1,j_1} : \dots : p_{n,j_n}] \mid 1 \leq j_i \leq a_i + 1, \quad i = 1 \dots n \}$$

WHERE

$$p_{i,j} = \begin{cases} \xi_i^j c_i & \text{if } c_i \neq 0 \\ \xi_i^j & \text{if } c_i = 0 \end{cases}$$

HENCE, $\mathbb{X}$ IS A SET OF SIMPLE POINTS GIVING A MINIMAL WARING DECOMPOSITION OF M CONTAINING $[c_0 : \dots : c_n]$; CONSEQUENTLY, $\mathcal{F}_M \subseteq V(x_0 \dots x_n)$.

FOR THE OTHER INCLUSION, WE GIVE A VERY SIMPLE ARGUMENT THAT WORKS FOR $a_0 \geq 2$.

CONSIDER $i = 1 \dots m$: BY [CCG] FORMULA,

$rk(\partial_{x_i} M) = rk(M) =: r$. ASSUME $M = \sum_{i=1}^{r} L_i^d$ WHERE $L_r = c_0 x_0 + \dots + c_n x_n$ WITH $c_i = 0$; THEN

$$\partial_{x_i} M = \sum_{i=1}^{r-1} L_i^{d-1} \quad ; \quad \text{CONTRADICTION.} \qquad \square$$

3.4 PLANE CUBICS

WE CONSIDER NOW $F \in \mathbb{C}[x,y,z]_3$; NAMELY CUBICS IN $\mathbb{P}^2$.

THERE IS A COMPLETE CHARACTERIZATION OF PLANE CUBICS WITH RESPECT TO WARING RANK, SEE LANDSBERG - TEITLER.

[1]	TRIPLE LINE	x^3	$rk = 1$
[2]	THREE CONCURRENT LINES	$xy(x+y)$	2
[3]	DOUBLE LINE + LINE	x^2y	3
[4]	SMOOTH	$x^3+y^3+z^3$	3
[5]	THREE NON CONCURRENT LINES	xyz	4
[6]	LINE + CONIC (TRANSVERSALLY)	$x(yz+x^2)$	4
[7]	NODAL	$xyz-(y+z)^3$	4
[8]	CUSP	x^3-y^2z	4
[9]	GENERAL	$x^3+y^3+z^3+a\,xyz \quad (a^3 \neq 0,6)$	4
[10]	LINE + CONIC (TANGENT)	$x(yz+z^2)$	5

- [1], [3] & [5] : MONOMIALS
- [2] : BINARY FORM

- [4] : $F = x^3+y^3+z^3$ HAS A UNIQUE DECOMPOSITION AND THEN

$$\mathcal{W}_F = \{[1:0:0],[0:1:0],[0:0:1]\}$$

INDEED, WE KNOW THAT HILBERT FUNCTION OF $S/F^\perp$ IS $1 \quad 3 \quad 3 \quad 1$

AND THEN, IF $I_{\mathbb{X}} \subset F^\perp$ WITH $\mathbb{X}$ IS SET OF 3 POINTS,

IT MUST BE

$$HF(S/I_{\mathbb{X}}) : \quad 1 \quad 3 \quad 3 \quad 3 \quad 3$$

(HILBERT FUNCTION OF SET OF POINTS IS INCREASING AND EVENTUALLY EQUAL TO THE NUMBER OF POINTS)

IN PARTICULAR, $[I_{\mathbb{X}}]_2 = [F^\perp]_2$. IT IS EASY TO CHECK THAT

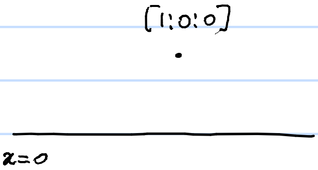
$$F^\perp = (xy,\, xz,\, yz,\, x^3-y^3,\, x^3-z^3).$$

HENCE,

$$I_{\mathbb{X}} = (xy,\, xz,\, yz) \quad \text{AND} \quad \mathbb{X} = \{[1:0:0],[0:1:0],[0:0:1]\}.$$

• [8] : LET'S START WITH THE CUSP, $F = x^3 - y^2 z$; THEN

$$W_F = \{[1:0:0]\} \cup \{[0:a:1] \mid a \in \mathbb{C}\}$$



IN OTHER WORDS, ALL THE MINIMAL WARING DECOMPOSITIONS ARE OF TYPE $x^3 + L^3 + M^3 + N^3$

WHERE L, M, N DO NOT INVOLVE THE VARIABLE x.

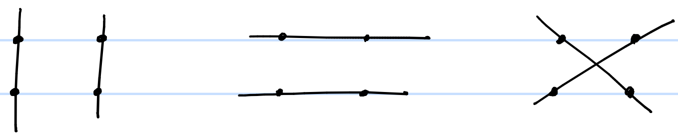
proof : $F^\perp = (xy, xz, z^2, x^3 + 3y^2 z)$

CLAIM : IF $\mathbb{X}$ IS A SET OF 4 POINTS WITH $I_{\mathbb{X}} \subset F^\perp$ THEN THREE ARE COLLINEAR.

FIRST WE OBSERVE THAT IF $\mathbb{X}$ IS A SET OF 4 POINTS, THEN

$HF(S/I_{\mathbb{X}})$: 1 3 4 4 4 ...

THEN WE HAVE A PENCIL OF CONICS ($\dim [I_{\mathbb{X}}]_2 = 2$) THROUGH $\mathbb{X}$ AND, IF NO THREE THEM ARE COLLINEAR, THE PENCIL CONTAINS EXACTLY THREE REDUCIBLE CONICS



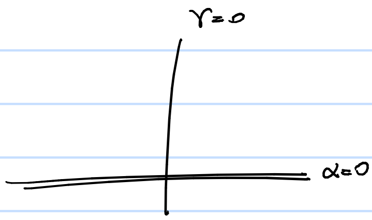
CONSIDER $\mathcal{L} = (F^\perp)_2$ AND $\mathbb{P}(\mathcal{L}) \cong \mathbb{P}^2$; LET Δ BE THE LOCUS OF REDUCIBLE CONICS NAMELY

$$\mathcal{L} = \{\alpha xy + \beta xz + \gamma z^2 \mid [\alpha : \beta : \gamma] \in \mathbb{P}^2\} \cong \mathbb{P}^2_{\alpha,\beta,\gamma}$$

AND

$$\Delta = \left\{ \det \begin{bmatrix} 0 & \frac{1}{2}\alpha & \frac{1}{2}\beta \\ \frac{1}{2}\alpha & 0 & 0 \\ \frac{1}{2}\beta & 0 & \gamma \end{bmatrix} = -\frac{1}{4}\alpha^2\gamma = 0 \right\}$$

THEN, Δ is



IF WE CONSIDER THE PENCIL OF CONICS THROUGH $\mathbb{X}$, SAY $\mathcal{L}' := \mathcal{L}(-\mathbb{X})$

WE GET A LINE $\mathbb{P}(\mathcal{L}') \subset \mathbb{P}(\mathcal{L})$ WHICH INTERSECT Δ IN TWO POINTS UNLESS IT IS $\alpha = 0$ OR $\beta = 0$.

HENCE, THE CLAIM IS PROVEN.

FURTHER, WE CAN CHECK THAT $\mathbb{X}$ CONTAINS ALWAYS $[1:0:0]$ AND THREE POINTS ON THE LINE $\{x=0\}$.

IDEA : (1) $I_{\mathbb{X}}$ HAS TO BE OF THE FORM (G_1, G_2, H) WITH

$\deg G_i = 2$ & $\deg H = 3$

AND G_i'S ARE BOTH DIVISIBLE BY A LINEAR FORM L.

(2) $\mathcal{L} = [F^\perp]_2 = \langle xy, xz, z^2 \rangle$

HAS TWO BASE POINTS $[1:0:0]$ & $[0:1:0]$

AND THE LINE L DOESN'T PASS THROUGH BOTH POINTS

(3) $[0:1:0]$ IS FORBIDDEN ;

$$\operatorname{rank}(F - y^3) = \operatorname{rank}(x^3 - (y^2z - y^3)) = 4$$

SINCE y^3 IS FORBIDDEN FOR y^2z.

HENCE,

$$\mathbb{X} = [1:0:0] + \text{THREE POINTS ON } L$$

(4) $F - x^3 = y^2z$ HENCE THE OTHER THREE LINEAR INVOLVE ONLY y, z.

$\square$

LEMMA. LET F A PLANE CUBIC OF RANK 4 AND $\mathbb{X}$ A SET OF FOUR POINTS S.T. $I_{\mathbb{X}} \subset F^\perp$; THEN,

$\mathbb{X}$ HAS THREE COLLINEAR POINTS IF AND ONLY IF F IS CUSP.

- [6], [7] & [9] :

IDEA : (1) $[F^\perp]_2 = \langle C_1, C_2, C_3 \rangle =: \mathcal{L}$; SINCE F NOT A CUSP, ANY $I_{\mathbb{X}} \subset F^\perp$ WITH $\#(\mathbb{X}) = 4$ IS A COMPLETE INTERSECTION OF TWO CONICS.

(2) WE LOOK FOR TWO DIMENSIONAL $\mathcal{L}' \subset \mathcal{L}$ WITH 4 DISTINCT BASE POINTS ; IN PARTICULAR,

TAKE ANY POINT $P \in \mathbb{P}^2$ AND LOOK AT THE LINE

$$\mathbb{P}(\mathcal{L}(-P)) \subset \mathbb{P}^2_{\alpha,\beta,\gamma} = \{\alpha C_1 + \beta C_2 + \gamma C_3 \mid [\alpha:\beta:\gamma] \in \mathbb{P}^2\}$$

IF $\mathcal{L}(-P)$ HAS 4 DISTINCT BASE POINTS $\Rightarrow P \in \mathcal{W}_F$

OTHERWISE, $P \in \mathcal{F}_F$ (IT MEANS THAT $\mathbb{X}$ WAS NOT REDUCED)

(3) THIS IS EQUIVALENT TO LOOK FOR LINES $\mathbb{P}(\mathcal{L}') \subset \mathbb{P}(\mathcal{L})$ WHICH INTERSECT THE CUBIC OF REDUCIBLE CONICS Δ IN PRECISELY THREE POINTS ;

IN THAT CASE, $\mathcal{L}' = \mathcal{L}(-P)$ WITH $P \in \mathcal{W}_F$.

IN OTHER WORDS,

$$\mathcal{F}_F = \{ P \in \mathbb{P}^2 \mid \mathbb{P}(\mathcal{L}(-P)) \text{ IS TANGENT TO } \Delta \}$$

THEOREM. [6] $F = x^3 + xyz$; $F^\perp = (x^2 - 6yz, y^2, z^2)$

THEN $\Delta \subset \mathbb{P}^2_{\alpha,\beta,\gamma} = \mathbb{P}([F^\perp]_2)$ IS $9\alpha^3 - \alpha\beta\gamma = 0$

NAMELY, CONIC AND SECANT LINE ; THUS,

$$\mathcal{F}_F = V(yz(x^2 - 12yz)) = \{\text{CONIC AND TWO TANGENT LINES}\}$$

[7] IN THIS CASE, Δ IS IRREDUCIBLE CUBIC

AND THEN

$$\mathcal{F}_F = \text{SEXTIC} + \text{CUBIC MEETING IN 9 POINTS}$$

[9] IN THIS CASE,

$$\Delta = \left\{ \alpha^3 + \beta^3 + \gamma^3 - \left(\frac{a^3 - 54}{9}\right)\alpha\beta\gamma = 0 \right\}$$

THEN,

- $\left(\frac{a^3 - 54}{9}\right) \neq 27$, $\mathcal{F}_F$ IS GIVEN BY THE DUAL CURVE AS [7]
- OTHERWISE, Δ = THREE NON-CONCURRENT LINES

  AND THEN

  $$\mathcal{F}_F = \text{THREE NON-CONCURRENT LINES}$$

- [10] : $F = x(xy + z^2)$ THEN $\mathcal{F}_F = \{[1:0:0]\}$

CONSIDER $F' = F - L^3$:

$L \in \mathcal{F}_F$ IF AND ONLY IF $rk(F') = 5$

BUT, FOLLOWING AN IDEA BY B. REZNICK, WE STUDY HESSIAN; INDEED

IF HESSIAN IS A CUBE OF LINEAR FORM THEN $rk(F) = 5$.

BY MACAULAY2, $F - L^3$ HAS HESSIAN EQUAL TO THE CUBE OF A LINEAR FORM IF AND ONLY IF $L = x$.

§4 FUTURE WORK

4.1 CAN WE CONSIDER OTHER FAMILIES OF POLYNOMIALS ?

FOR EXAMPLE, WHAT IF $F^\perp$ IS COMPLETE INTERSECTION ?

4.2 TENSORS. GIVEN $T \in V_1 \otimes \cdots \otimes V_s$ (V_i VECTOR SPACES)

WARING PROBLEMS FOR TENSORS LOOK AT DECOMPOSITIONS AS SUM OF DECOMPOSABLE TENSORS, NAMELY

$$T = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{s,i}.$$