JOURNAL OF ALGEBRA 174, 1080–1090 (1995)

Inverse System of a Symbolic Power, I

J. EMSALEM*

Université de Paris VII, 75005 Paris, France

AND

A. IARROBINO[†‡]

Mathematics Department, Northeastern University, Boston, Massachusetts 02115

Communicated by David Buchsbaum

Received July 15, 1993

The authors determine Macaulay's inverse system to a symbolic power of the graded ideal of functions vanishing at a union of irreducible varieties in projective space. © 1995 Academic Press, Inc.

1. INTRODUCTION

We fix r and an infinite field k, and consider two polynomial rings, $\mathcal{R} = k[X_1, \ldots, X_r]$ and $R = k[x_1, \ldots, x_r]$ as well as the divided power ring $\mathcal{D} = k[D_1, \ldots, D_r]$. We suppose that P is a sequence of points $P = [p(1), \ldots, p(s)]$ of $\mathbb{P}^{r-1}$, that $N = (n_1, \ldots, n_s)$ with $n_1 \geq \cdots \geq n_s$ is a sequence of positive integers, and that $m_{p(u)}$ is the graded ideal in R of functions vanishing at the point $p(u)$. We define the *vanishing ideal* $K(P, N)$ as the ideal of functions vanishing to orders at least N at the points P,

$$K(P, N) = m_{p(1)}^{n_1} \cap \cdots \cap m_{p(s)}^{n_s}. \tag{1}$$

When $s = 1$, the ordinary ath power of the ideal $m_{p(1)}$ of R is the same as the symbolic ath power. But when $s > 1$ and $N = (\mathbf{a}) = (a, \ldots, a)$, the vanishing ideal $K(P, \mathbf{a})$ equals $K(P, \mathbf{1})^{(a)}$, the symbolic ath power of

* E-mail address: emsalemj@mathp7.jussieu.fr.

† Supported in part by the National Science Foundation, and the Bulgarian Foundation "Scientific Research" under the U.S.–Bulgarian project "Algebra and Algebraic Geometry."

‡ E-mail address: iarrobin@neu.edu.

0021-8693/95 $6.00

$K(P, \mathbf{1})$, and is not in general the same as the usual ath power of $K(P, \mathbf{1})$ (see [IK] for examples).

We will define an action of R on $\mathscr{R}$ (respectively, $\mathscr{D}$) in which a monomial μ of R acts as a higher partial differential operator on $\mathscr{R}$ (respectively, as a contraction operator on $\mathscr{D}$). The inverse system K^{-1} in $\mathscr{R}$ (or $\mathscr{D}$) of an ideal K of R is the submodule of $\mathscr{R}$ (or $\mathscr{D}$) of all elements annihilated by I.

We first determine the inverse system of the vanishing ideal $K(P, N)$ at the points P (Theorem I). We then determine the inverse system of a vanishing ideal at a union of irreducible varieties of arbitrary dimension in $\mathbb{P}^{r-1}$ (Theorem II).

In further articles, the second author applies Theorem I in the special case $N = \mathbf{(2)} = (2, \dots, 2)$ to show that the solution to the Waring problem for forms is a consequence of a theorem of A. Alexander and A. Hirschowitz [I1], and uses Theorem I to study the Hilbert functions of the algebras $R/K(P, N)$ [I2]. The authors also apply Theorem I to the study of simultaneous decompositions of forms [EI2]. V. Kanev and the second author use Theorem I in a study of the stratification of the space of homogeneous forms by the number of linearly independent derivates [IK].

2. Inverse System of a Vanishing Ideal at s Points of $\mathbb{P}^{r-1}$

We suppose that $P = [p(1), \dots, p(s)]$ are s points of $\mathbb{P}^{r-1}$, and we choose elements $p_{uv} \in k$ such that the point $p(i) = (p_{u1}, \dots, p_{ur})$ up to k^* multiple. We let $L_i = L_{p(i)} = \sum p_{iu} X_u$, a homogeneous degree one element of $\mathscr{R}$. We first define the vector spaces $\mathscr{V}(P, N, i)$ in $\mathscr{R}$ (respectively, $\mathscr{D}(P, N, i)$ in $\mathscr{D}$), which will turn out to be the degree-i pieces of the inverse system $K(P, N)^{-1}$ to $K(P, N)$ in $\mathscr{R}$ (respectively, $\mathscr{D}$). For each nonnegative integer i we define an ideal of the polynomial ring $\mathscr{R} = k[X_1, \dots, X_r]$,

$$\mathscr{I}(P, N, i) = \left((L_1)^{i+1-n_1}, \dots, (L_s)^{i+1-n_s}\right); \tag{2}$$

and we denote by $\mathscr{V}(P, N, i)$ the vector space of degree-i forms in $\mathscr{I}(P, N, i)$,

$$\begin{aligned} \mathscr{V}(P, N, i) &= \mathscr{I}(P, N, i) \cap \mathscr{R}_i \\ &= \mathscr{R}_{n_1-1}(L_1)^{i+1-n_1} + \cdots + \mathscr{R}_{n_s-1}(L_s)^{i+1-n_s}. \end{aligned} \tag{2a}$$

If $i + 1 - n_i$ is negative for any i, then we take $V_i(P, N) = \mathscr{R}_i$.

If $K = (k_1, \ldots, k_r)$ is a sequence of nonnegative integers, we let $p^K = p_1^{k_1} \cdot \ldots p_r^{k_r}$, $Y^K = Y_1^{k_1} \cdot \ldots Y_r^{k_r}$. If $L = \sum p_i Y_i$ in $\mathscr{D}$, we let $L^{[n]} = \sum_{|K|=n} p^K Y^K$.

We denote by $\mathscr{I}_{\mathscr{D}}(P, N, i)$ the ideal

$$\mathscr{I}_{\mathscr{D}}(P, N, i) = \left((L_1)^{[i+1-n_1]}, \ldots, (L_s)^{[i+1-n_s]}\right) \tag{3}$$

of the divided power ring $\mathscr{D}$, and by $\mathscr{D}(P, N, i)$ the intersection

$$\mathscr{D}(P, N, i) = \mathscr{I}_{\mathscr{D}}(P, N, i) \cap \mathscr{D}_i. \tag{3a}$$

DEFINITION 1 (apolarity action). We let R act on $\mathscr{R}$ as higher order partial differential operators: thus if $h \in R$, and $f \in \mathscr{R}$,

$$h \cdot f = h\left(\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_r}\right) \circ f. \tag{4}$$

We let R act on $\mathscr{D}$ as contraction: thus if U, V are multiindices

$$x^U \cdot Y^{[V]} = \begin{cases} Y^{[V-U]} & \text{if } V \geq U, \\ 0 & \text{otherwise.} \end{cases} \tag{4a}$$

The contraction action for monomials extends bilinearly to the contraction of R on $\mathscr{D}$, and can be viewed as a variation of (4), where if $h = \sum a_U X^U$ and $f \in \mathscr{D}$ then

$$h \cdot f = \sum a_U x^U \circ f = \sum a_U \frac{\partial^{[U]}}{\partial Y^{[U]}} \circ f. \tag{4b}$$

We call both the partial differential operator (PDO) action of R on $\mathscr{R}$ and the contraction action of R on $\mathscr{D}$ *apolarity* actions. Classical algebraic geometers studied the notion of apolarity between hypersurfaces in $\mathbb{P}^r$ and enveloping hypersurfaces in $\check{\mathbb{P}}^r$ (see Sections 1, 2 of [DK] for a modern discussion). Macaulay studied the space of apolar forms to an ideal I defining a subscheme of $\mathbb{P}^r$ as an algebraic object, the "inverse system" of the ideal I.

DEFINITION 2 (inverse system). If I is a graded ideal of R, then the inverse system I^{-1} of I in $\mathscr{R}$ is the R-submodule $\operatorname{Ann}(I)$ of $\mathscr{R}$, namely the annihilator,

$$I^{-1} = \operatorname{Ann}(I) = \{f \in \mathscr{R} \mid h \in I \Rightarrow h \cdot f = 0\}. \tag{5}$$

The degree i-piece $[I^{-1}]_i \subset \mathscr{R}_i$ satisfies

$$[I^{-1}]_i = \operatorname{Ann}(I) \cap \mathscr{R}_i = \langle I_i \rangle^{\perp}, \tag{5a}$$

in the exact pairing $R_i \times \mathscr{R}_i \to k$. The inverse system I^{-1} of I in $\mathscr{D}$ is defined similarly, and is Macaulay's inverse system [Mac].

Note that the inverse system is not an ideal of $\mathscr{R}$ or $\mathscr{D}$. See [DK], [EhR], [Em], [EI1], [Mat], [N] for discussions of apolarity, inverse systems, or the related Matlis duality and injective envelopes.

The following result is well known and is easily shown. Recall that if $p = (p_1, \dots, p_r)$ is a point of $\mathbb{A}^r$ then $L_p \in \mathscr{R}$ is the linear form $L_p = p_1 X_1 + \cdots + p_r X_r$; if $h \in R_t$, $h(p)$ denotes the value of h at p. We let $i_t = i!/(i-t)!$.

LEMMA. *When* char $k = 0$, *or* char $k > i$, *the apolarity action* $R_i \times \mathscr{R}_i \to \mathscr{R}_0 = k$ *is exact, so exhibits* $\mathscr{R}_i$ *as the dual vector space of* R_i. *In all characteristics, the apolarity action* $R_i \times \mathscr{D}_i \to \mathscr{D}_0 = k$ *is exact. Furthermore, if* $h \in R_t$, $t \le i$, *and* $L_p = p_1 X_1 + \cdots + p_r X_r \in \mathscr{R}_1$, *then*

$$h \cdot \left((L_p)^i\right) = i_t (L_p)^{i-t} h(p). \tag{6}$$

If $L_p = p_1 Y_1 + \cdots + p_r Y_r \in \mathscr{D}_1$, *then*

$$h \cdot \left(L_p^{[i]}\right) = L_p^{[i-t]} h(p). \tag{6a}$$

COROLLARY. *If* $h \in R_t$, $t \le i$, *and* $p = (p_1, \dots, p_r)$ *is a point of* $\mathbb{A}^r$, *then* $h \cdot L_p^i = 0$ *iff* $h(p) = 0$.

We define a differential dp from the polynomial ring $\mathscr{R}[p_1, \dots, p_r; dp_1, \dots, dp_r]$ to itself such that $k[dp_1, \dots, dp_r]$ are differential constants, and if $f \in \mathscr{R}[p_1, \dots, p_r]$ implies

$$df = \sum (\partial f_i / \partial p_i) dp_i.$$

We define dp similarly on $R[p_1, \dots, p_r; dp_1, \dots, dp_r]$.

The following result may be classical, although we have not found a reference. We give several proofs. The case $N = \mathbf{1}$ is found in the proof of Lemma 4.2.1 of [DK]. When char $k = 0$, the case $N = \mathbf{2}$ was shown by Terracini [T], and is equivalent to Proposition 4.1 of [EhR].

THEOREM I (inverse system of a vanishing ideal at s points). *When* char $k = 0$ *or is larger than* i, *the annihilator* $[I^{-1}]_i$ *in* $\mathscr{R}_i$ *of the vector* *space $I = K(P, N)_i$ of R is $\mathscr{V}(P, N, i)$. In all characteristics, the annihilator of $K(P, N)_i$ in $\mathscr{D}_i$ is $\mathscr{D}(P, N, i)$.*

Proof. We give the proof for the apolarity pairing $(R, \mathscr{R})$; we then give the changes needed for the pairing $(R, \mathscr{D})$. By the exactness of the duality between R_i and $\mathscr{R}_i$, it suffices to show that if $p = (p_1, \ldots, p_r)$ is a point of $\mathbb{P}^{r-1}$, then

$$\mathscr{R}_u L_p^{i-u} = \left(\operatorname{Ann}\left[(m_p)^{u+1}\right]\right) \cap \mathscr{R}_i. \tag{7}$$

The dimension of the left side of (7) is $\dim_k \mathscr{R}_u$. If we take for the point $p = e(1) = (1, 0, \ldots, 0)$ and $\mathscr{R}' = k[X_2, \ldots, X_r]$ we readily see that the right side of (7) is $\bigoplus_{v \le u} \mathscr{R}'_v X_1^{i-v}$, which is also the left side. It follows that for arbitrary points p,

Claim. If char $k = 0$ or is greater than i, the vector space dimension of each side of (7) is $\Sigma_{v \le u} \dim_k \mathscr{R}'_v = \dim_k \mathscr{R}_u$.

This claim is false for $(R, \mathscr{R})$ if $\operatorname{char}(k) \le i$, but the analogous claim for the pair $(R, \mathscr{D})$ is easily seen to be true in arbitrary characteristic.

We now apply an invertible linear transformation $[X'_1, \ldots, X'_r] = A[X_1, \ldots, X_r]$ to $\mathscr{R}$ or $\mathscr{D}$, and the contragradient transformation $A\hat{}$: $[x_1, \ldots, x_r] \to [x'_1, \ldots, x'_r]$ to R, for which $[x_1, \ldots, x_r] = A[x'_1, \ldots, x'_r]$ on R. Since the apolarity pairing is preserved under these transformations (see the end of Section 64, pp. 71–73, of [Mac]), we conclude from the special case $p = e(1)$ of (7) that

$$\mathscr{R}_u A\left(L_{e(1)}^{i-u}\right) = \operatorname{Ann}\left[A\hat{}\left(m_{e(1)}^{u+1}\right)\right] \cap \mathscr{R}_i. \tag{8}$$

Consider a point p with $p_1 \ne 0$. We may assume $p_1 = 1$, and we take $A = (a_{ij})$, $a_{ii} = 1$, $a_{1j} = p_j$, $a_{ij} = 0$ otherwise. Then $A[X_1] = X_1 + p_2 X_2 + \cdots + p_r X_r$, while $[x'_1, \ldots, x'_r] = [x_1, x_2 - p_2 x_1, \ldots, x_r - p_r x_1]$, so $m_p = (x'_2, \ldots, x'_r) = (x_2 - p_1 x_1, \ldots, x_r - p_r x_r)$. Then (8) becomes (7), which is now shown whenever the point p satisfies $p_1 \ne 0$. To complete the proof, one may begin with the special case $p = e(i) = (0, \ldots, 0, p_i = 1, 0, \ldots, 0)$ and similarly transform to an arbitrary p, where $p_i \ne 0$. This completes the proof of (7) for R, $\mathscr{R}$ when characteristic $k = 0$ or is greater than i; it proves also the analog of (7) for R, $\mathscr{D}$ in arbitrary characteristic.

We give a second proof of (7) for the pairing R, $\mathscr{R}$. By the claim, the dimensions of both sides of (7) are equal. Since the duality between $\mathscr{R}_i$ and R_i is exact, it suffices to show that (m_p^{u+1}) is zero on $\mathscr{R}_u L_p^{i-u}$. We can verify this by explicit calculation, or by applying the differential operator $D_p = \Sigma(\partial/\partial p_i) \cdot dp_i$ repeatedly—u times—to the identity (6). A single application gives

$$h \cdot \left[\left(jL_p^{j-1} \right) \left(\sum X_i dp_i \right) \right] = \left[j_t (j-t) L_p^{j-t} \left(\sum X_i dp_i \right) h(p) \right] + \left[j_t L_p^{j-t} \left(D_p h(p) \right) \right]. \tag{9}$$

Since $D_p h(p) = D_x h(x)|_{x=p}$, we have $h \in m_p^2$ implies $D_p(h(p)) = 0$, as the partial derivatives of h vanish at p; thus, the right side of (9) is zero, implying that h is zero on $\mathscr{R}_1 L_p^{j-1}$. Successive application of D_p to (9) yields similar identities, whence we conclude (6).

Alternative proof of Theorem I. One may show that (m_p^{u+1}) is zero on $\mathscr{R}_u L_p^{i-u}$ by explicit calculation. The point is that any element of $(m_p^{u+1})_i$ can be written as a linear combination $\sum g_j h(j)$ with coefficients g_j in $R_{i-(u+1)}$ of degree $u+1$ monomials $h(j)$ in the degree 1 elements of m_p. Consider the term $gh = g_j h(j)$ of the sum. If the monomial $h = h_1 \cdot \ldots \cdot h_{u+1}$ with $h_i \in m_p \cap R_1$, if $g \in R_{i-(u+1)}$ and $b \in \mathscr{R}_u$ then

$$(gh_1 h_2 \ldots h_u) \cdot bL_p^{j-u} = g \cdot \left[(h_1 h_2 \ldots h_{u+1}) \cdot bL_p^{j-u} \right]. \tag{9a}$$

We may assume wolog that b is a product $b = b_1 \cdot \ldots \cdot b_u$ of linear terms, and repeatedly use the product rule for a linear partial differential operator to expand $[(h_1 h_2 \ldots h_{u+1}) \cdot bL_p^{j-u}]$ into a new sum; each term of the new sum is the product of h_1 to h_{u+1} acting on a choice of $u+1$ linear factors of bL_p^{j-u}, multiplied by the product of the remaining $j-1-u$ linear factors of bL_p^{j-u}. But each such term contains a factor $h_i \cdot L_p = 0$. This completes an alternative proof of (7) and of Theorem I when the pairing is between R and $\mathscr{R}$.

Another proof of (7) for the apolar pair $(R, \mathscr{D})$ can be obtained by applying a contraction operator $D_p^u = \sum_{|U|=u} (\partial^{[U]}/\partial p^{[U]}) dp^{[U]}$ to (6a).

If $\mathscr{L} = (L_{p(1)}, \ldots, L_{p(s)}) \subset \mathscr{R}_1$ or $\mathscr{D}_1$ are homogeneous linear elements of $\mathscr{R}$ or $\mathscr{D}$ we let $\mathscr{L}^u$ denote the vector subspace $\mathscr{L}^u = \langle L_{p(1)}^u, \ldots, L_{p(s)}^u \rangle$ of $\mathscr{R}_u$, and $\mathscr{L}^{[u]} = \langle L_{p(1)}^{[u]}, \ldots, L_{p(s)}^{[u]} \rangle \subset \mathscr{D}_u$. We let $(\mathbf{a}) = (a, \ldots, a)$, a sequence of length s.

COROLLARY 3. *Let* $N = \mathbf{a}$. *The inverse system* I^{-1} *in* $\mathscr{R}$ *of* $I = K(P, \mathbf{a}) = m_{p(1)}^a \cap \cdots \cap m_{p(s)}^a$ *satisfies, when* char $k = 0$ *or* char $k = p > i$,

$$[I^{-1}]_i = \mathscr{R}_{a-1} \mathscr{L}^{i+1-a}. \tag{10}$$

In arbitrary characteristic, the inverse system I^{-1} *in* $\mathscr{D}$ *to* $K(P, \mathbf{a})$ *satisfies*

$$[I^{-1}]_i = \mathscr{D}_{a-1} \mathscr{L}^{[i+1-a]} \tag{10a}$$

Remark. By Corollary 3, the problem of determining the Hilbert function of $K(P, \mathbf{a})$ for P a generically chosen set of s points of $\mathbb{P}^{r-1}$ is equivalent to determining the dimensions of the spaces in (10) or (10a) for a set $\mathscr{L}$ of s generically chosen linear forms of $\mathscr{R}_1$ or of $\mathscr{D}_1$. We discuss this problem in the case $\mathbf{a} = 2$ in [I1], and for arbitrary $K(P, \mathbf{a})$ in [I2].

3. Inverse System of a Vanishing Ideal on a Union of Varieties

We now extend Theorem I to symbolic powers of ideals defining reduced subschemes of $\mathbb{P}^{r-1}$ (Theorem II).

We suppose that k is algebraically closed and that Z is a subvariety of $\mathbb{P}^{r-1}(k)$: we permit Z to have several components $Z = Z(1) \cup \cdots \cup Z(s)$, but no embedded components, so Z is a reduced scheme. We let $I = I(Z)$ (respectively, $I(i)$) be the homogeneous ideal of functions in $R = k[x_1, \ldots, x_r]$ vanishing on Z (respectively, on $Z(i)$): thus, $I(Z) = I(1) \cap \cdots \cap I(s)$ is a radical ideal in R. We denote by $I^{(u)}$ the uth symbolic power of I. When $s = 1$, $I = \mathscr{P}$ is prime and $\mathscr{P}^{(u)}$ is defined as the $\mathscr{P}$-primary component of $\mathscr{P}^u$; it is the inverse image in R of the uth power $m_{\mathscr{P}}^u$ of the maximal ideal $m_{\mathscr{P}}$ in the local ring $R_{\mathscr{P}}$. When $s > 1$, $I^{(u)}$ is the intersection $I(Z)^{(u)} = I(1)^{(u)} \cap \cdots \cap I(s)^{(u)}$; it is also the ideal of forms f vanishing to order at least u at each point of Z (see [Z], [EiH]). When $\operatorname{char}(k)$ is zero, $f \in I(Z)^{(u)}$ iff f and its partial derivatives of order up to $u - 1$ vanish at all points of Z. If $N = (n_1, \ldots, n_s)$ is a sequence of positive integers, then we denote by $K(Z, N)$ the ideal

$$K(Z, N) = I(Z_1)^{(n_1)} \cap \cdots \cap I(Z_s)^{(n_s)}$$

in R consisting of all functions vanishing at each Z_u to order at least n_u. Recall that if $p = (p_1, \ldots, p_r)$ is a point of $\mathbb{A}^r$, we define $L_p = \sum p_u X_u \in \mathscr{R}_1$; if p up to k^*-multiple is a point of $\mathbb{P}^{r-1}$, then L_p is defined up to k^*-multiple in the polynomial ring $\mathscr{R}$.

Definition. If Z is a reduced subscheme of $\mathbb{P}^{r-1}$, we let $\mathscr{L} = \mathscr{L}(Z)$ denote the family

$$\mathscr{L}(Z) = \{L_p \mid p \in Z,\ p \text{ a closed point of } Z\},$$

and we denote by $\mathscr{L}^i$ (respectively, $\mathscr{L}^{[i]}$ the vector space span of the ith powers $(L_p)^i$ in $\mathscr{R}_i$ (respectively, of the ith divided powers $(L_p)^{[i]}$ in $\mathscr{D}_i$) of elements in $\mathscr{L}(Z)$,

$$\mathscr{L}^i = \left\langle \left\{ (L_p)^i, p \in Z \right\} \right\rangle.$$

We denote by $(\mathscr{L}(Z)^u)$ the ideal of $\mathscr{R}$ generated by $\mathscr{L}(Z)^u$, and by

$(\mathscr{L}(Z)^{[u]})$ the ideal of $\mathscr{D}$ generated by $\mathscr{L}(Z)^{[u]}$. For each nonnegative integer i we define the following ideal of $\mathscr{R}$,

$$\mathscr{I}(Z, N, i) = \left(\mathscr{L}(Z_1)^{i+1-n_1}, \ldots, \mathscr{L}(Z_s)^{i+1-n_s}\right),$$

and we set

$$\mathscr{V}(Z, N, i) = \mathscr{I}(Z, N, i) \cap \mathscr{R}_i.$$

We define the ideal $\mathscr{I}_{\mathscr{D}}(Z, N, i)$ of $\mathscr{D}$ and the vector subspace $\mathscr{D}(N, Z, i)$ of $\mathscr{D}_i$ similarly (see (3) and (3a) for the special case $Z = P$).

THEOREM IIA (inverse system of a symbolic power). *Suppose that Z is a possibly reducible variety of $\mathbb{P}^{r-1}$, having no embedded components. If* char $k = 0$ *or is greater than i, then for each integer $u > 0$, the annihilator in $\mathscr{R}_i$ of $(I(Z)^{(u)})_i$ satisfies*

$$\left\langle \left(I(Z)^{(u)}\right)_i \right\rangle^{\perp} = \mathscr{R}_{u-1}\mathscr{L}(Z)^{i+1-u}$$
$$= \left(\mathscr{L}(Z)^{i+1-u}\right) \cap \mathscr{R}_i. \qquad (11)$$

Similarly, in arbitrary characteristic, the annihilator in $\mathscr{D}_i$ of $(I(Z)^{(u)})_i$ is $\mathscr{D}_{u-1}\mathscr{L}(Z)^{[i+1-u]}$.

THEOREM IIB (inverse system of a vanishing ideal). *If $Z = Z(1) \cup \cdots \cup Z(S)$ is the union of irreducible varieties $Z(i)$ of $\mathbb{P}^{r-1}$, none of which contains another, if $N = (n_1, \ldots, n_s)$ is a sequence of positive integers, and if* char $k = 0$ *or is greater than i, then the inverse system $K(Z, N)^{-1}$ in $\mathscr{R}$ to $K(Z, N)$ has degree-i piece $[K(Z, N)^{-1}]_i = \langle K(Z, N)_i \rangle^{\perp}$ satisfying*

$$\langle K(Z, N)_i \rangle^{\perp} = \mathscr{V}(Z, N, i) \subset \mathscr{R}. \qquad (12a)$$

Likewise, in arbitrary characteristic, the inverse system in $\mathscr{D}$ to $K(Z, N)$ satisfies

$$\langle K(Z, N)_i \rangle^{\perp} = \mathscr{D}(Z, N, i) \subset \mathscr{D}. \qquad (12b)$$

Proof of A. If m_p denotes as above the ideal in R of functions vanishing at p, we have, since $I(Z)$ is by assumption radical,

$$I(Z) = \bigcap_{p \in Z} m_p,$$

and the symbolic power $I^{(u)} = \cap_{p \in Z} m_p^u$, by Zariski's Main Lemma on holomorphic functions (see [Z], [EiH]). By the exactness of the duality and

Theorem I, we see that the annihilator of $I^{(u)}$ satisfies

$$\left\langle \left(I(Z)^{(u)}\right)_i \right\rangle^{\perp} = \left\langle \left\{ \mathscr{R}_{u-1} L_p^{i+1-u} \mid p \in Z \right\} \right\rangle$$
$$= \mathscr{R}_{u-1} \mathscr{L}(Z)^{i+1-u}.$$

This is (11a). The proof of the analogue for $\mathscr{D}_i$ is similar.

Proof of B. This follows from part A applied to each Z_i and the exactness of the pairings $R_i \times \mathscr{R}_i \to k$, and $R_i \times \mathscr{D}_i \to k$.

EXAMPLE 4A (line). We suppose $r > 3$, and let Z be the projective line $(x_3 = \cdots = x_r = 0)$ in $\mathbb{P}^{r-1}$. Then $I(Z) = (x_3, \ldots, x_r)$, while $I(Z)^{(u)} = (x_3, \ldots, x_r)^u$, and

$$\mathscr{L}(Z)^j = \left\langle \left\{ (aX_1 + bX_2)^j \mid a, b \in k \right\} \right\rangle$$
$$= k[X_1, X_2]_j \subset \mathscr{R}.$$

The annihilator of $I_i^{(u)}$ in $\mathscr{R}_i$ is evidently $\mathscr{R}_{u-1} k[X_1, X_2]_{i+1-u}$.

EXAMPLE 4B (curve). We suppose $r = 3$, that $f = x_1 x_2^2 - x_3^3$, and let Z be the projective curve $f = 0$. Then $I(Z) = (f)$, and $I^{(u)} = (f^u)$. Then

$$\mathscr{L}(Z)^j = \left\langle \left\{ (aX_1 + bX_2 + cX_3)^j \mid a, b, c \in k \text{ and } ab^2 = c^3 \right\} \right\rangle,$$

and the annihilator of $(f^u)_i = R_{i-u} f^u$ in $\mathscr{R}_i$ is $\mathscr{R}_{u-1} \mathscr{L}(Z)^{i+1-u}$.

EXAMPLE 4C (determinantal ideal). Suppose $r = 9$, let $R = k[\{x_{ij} \mid 1 \le i, j \le 3\}]$ and consider the zeroes Z in $\mathbb{P}^8 = \mathbb{P}(R)$ of the prime ideal J_2 of 2×2 minors of the 3×3 matrix $M_x = (x_{ij})$. Then $I(Z)$ is J_2, and $I^{(2)} = (I^2, \det(M_x))$ (see [Ei, Sect. 3.9.1] or Theorem 10.4 of [BV]) while by definition

$$\mathscr{L}(Z)^j = \left\langle \left\{ \left(\sum a_{ij} X_{ij} \right)^j \mid a_{ij} \in k \text{ and } M_a \text{ has rank one} \right\} \right\rangle.$$

By Theorem II the annihilator of $I_i^{(2)}$ in $\mathscr{R}_i$ is $\mathscr{R}_1 \mathscr{L}(Z)^{i-1}$. Thus, when $i = 3$, then $(I^{(2)})_3 = \langle \det(M_x) \rangle$ and is the annihilator in R_3 of $\mathscr{R}_1 \mathscr{L}(Z)^2$. We now show directly that $\det(M_x)$ annihilates $\mathscr{R}_1 \mathscr{L}(Z)^2$. If $L_a = \sum a_{ij} X_{ij}$, if M_a has rank one, and M_{uv} denotes the minor of x_{uv} in M_x, then in the action $R \cdot \mathscr{R} \to \mathscr{R}$, we have

$$\begin{aligned} \det(M_x) \cdot X_{uv} L_a^2 &= (X_{uv} \cdot \det(M_x)) \cdot L_a^2 \\ &= M_{uv} \cdot L_a^2 = 2\, M_{uv|a} = 0. \end{aligned} \tag{13}$$

In (13), since M_{uv} consists of mixed terms, $M_{uv} \cdot L_a^2$ is twice the minor of a_{uv} in M_a, which is zero.

Macaulay studied primary decompositions of ideals by considering additive decompositions of the corresponding inverse systems [Mac]. We give an example concentrated at $p = (1, 0, 0)$ of $\mathbb{P}^2$ to illustrate; here $R = k[x, y, z]$.

EXAMPLE 5. We consider $I = (z, xy, y^2, yz)$, $m_p = (y, z)$ and note that $m_p \supset I \supset m_p^2$. We have by Theorem 1, $m_p^\perp = \langle k[X] \rangle$, and $(m_p^2)^\perp = \langle Yk[X], Zk[X], k[X] \rangle = \mathscr{R}_1 k[X]$ (we interpret $\mathscr{R}_1 k[X]$ as including the space $\langle 1 \rangle$). We have $(m_p^2)^\perp \supset I^\perp \supset m_p^\perp$, and in fact $I^\perp = \langle Y, k[X] \rangle$. *To study subspaces of forms in I_j, so vanishing on Spec R/I, it is equivalent to studying dually the vector spaces V in $\mathscr{R}_j$ that include the perpendicular space $(I^\perp)_j$.*

Remark. If $V \subset \mathscr{R}_{i+1-u}$ is a generically chosen vector subspace of dimension greater than r, then there is no scheme Z in $\mathbb{P}^{r-1}$ such that the inverse system to $\mathscr{I}(Z)^{(u)}$ has degree-i piece equal to $\mathscr{R}_{u-1}V$: the annihilator of $\mathscr{R}_{u-1}V$ is primary to the irrelevant ideal of R. Thus, the most generic vector subspaces $V \subset \mathscr{R}_{i+1-u}$ of fixed dimension occurring as the inverse systems to vanishing ideals for schemes Z in $\mathbb{P}^{r-1}$ occur for punctual subschemes P.

ACKNOWLEDGMENTS

This work began from a collaboration of the authors, and a suggestion by V. Kanev. The authors warmly thank Vassil Kanev whose collaboration concerning determinantal ideals of catalecticants [IK] led to this work. The second author thanks the Bulgarian Academy of Sciences, and the Mathematics Department of the University of Nice, in particular the Nice geometry group of J. Briançon, A. Galligo, A. Hirschowitz, P. Maisonobe, M. Merle, and J. Yameogo, for their hospitality and discussions during the preparation of this article.

REFERENCES

[BV] W. BRUNS AND U. VETTER, "Determinantal Rings," Lecture Notes in Mathematics, Vol. 1327, Springer-Verlag, Berlin/Heidelberg/New York, 1988.

[DK] I. DOLGACHEV AND V. KANEV, Polar covariants of plane cubics and quartics, *Adv. in Math.* **98** (1993), 216–301.

[Ei] D. EISENBUD, "Commutative Algebra with a View toward Algebraic Geometry," Graduate Tests in Mathematics **150**, Springer-Verlag, Berlin/Heidelberg/New York, 1995.

[EiH] D. EISENBUD AND M. HOCHSTER, A Nullstellensatz with nilpotents and Zariski's Main Lemma on holomorphic functions, *J. Algebra* **38** (1979), 157–161.

[EhR] R. EHRENBORG AND G.-C. ROTA, Apolarity and canonical forms for homogeneous polynomials, *European J. Combin.* **14** (1993), 157–182.

[Em] J. EMSALEM, Géomètrie des points épais., *Bull. Soc. Math. France* **106** (1978), 399–416.

[EI1] J. Emsalem and A. Iarrobino, Some zero-dimensional generic singularities; finite algebras having small tangent space, *Compositio Math.* **36** (1978), 145–188.

[EI2] J. Emsalem and A. Iarrobino, Simultaneous Waring problem for forms, preprint, 1993.

[I1] A. Iarrobino, Inverse system of a symbolic power. II. The Waring problem for forms, *J. Algebra* **174** (1995), 1091–1110.

[I2] A. Iarrobino, Inverse system of a symbolic power. III. Thin algebras, and fat points, preprint, 1994.

[IK] A. Iarrobino and V. Kanev, The length of a homogeneous form, determinantal loci of catalecticant matrices, and Gorenstein algebras, preprint, 1995.

[Mac] F. H. S. Macaulay, "The Algebraic Theory of Modular Systems," Cambridge Univ. Press, London/New York, 1916; reprinted 1994.

[Mat] E. Matlis, Injective modules over Noetherian rings, *Pacific J. Math.* **8** (1958), 511–528.

[N] D. G. Northcott, Injective envelopes and inverse polynomials, *J. London Math. Soc. (2)* **8** (1974), 290–296.

[T] A. Terracini, Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari, *Ann. Mat. Pura Appl.*, Serie III, t.24, p. 1–10, 1915.

[Z] O. Zariski, A fundamental lemma from the theory of holomorphic functions on an algebraic variety, *Ann. Mat. Pura Appl. (4)* **29** (1949), 187–198.