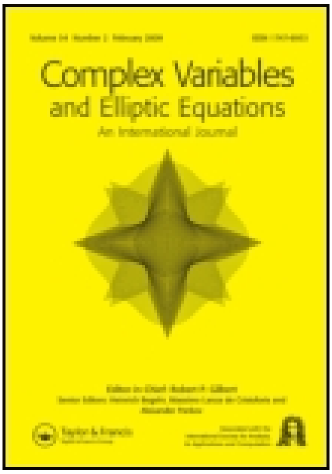

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Waring's theorem and the super Fermat problem for numbers and functions

W.K. Hayman [a]

[a] Department of Mathematics , Imperial College , London , SW7 2AZ , UK

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Waring's theorem and the super Fermat problem for numbers and functions

W.K. Hayman*

Department of Mathematics, Imperial College, London SW7 2AZ, UK

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Assume that x_j, x are positive intergers. Waring's problem, first proved by Hilbert, asserts that every sufficiently large x can be expressed as the sum of at most $K = G(n)$ nth powers.

$$x_1^n + \ldots + x_k^n = x. \qquad (*)$$

The Super-Fermat problem asks for the smallest $k = G_0(n)$ for which a solution of $(*)$ exists with x of the form x_0^n. Wiles has proved that $G_0(n) \geqslant 3$. We survey some known bounds for $G(n)$ and $G_0(n)$ and also for the corresponding quantities when the x_j and x are complex polynominals or entire or meromorphic functions in the plane.

Keywords: Waring's problem; Fermat's problem; integers; functions

AMS Subject Classifications: Primary: 11D25; 11G05; 30D35

1. The Super Fermat problem

Wiles [1] has proved an old conjecture of Fermat of 1637, that the equation

$$x_1^n + x_2^n = x^n,$$

for positive integers x, n, x_1, x_2 and $n > 2$ has no solutions. This makes it natural to ask, how large $k = k(n)$ must be for the equation

$$x_1^n + x_2^n + \ldots + x_k^n = x^n \qquad (1)$$

to have a solution in positive integers. We denote by $G_0(n)$ the smallest positive integer k, such that $k > 1$, and a solution of (1) exists. Wiles' Theorem [1] states that $G_0(n) \geqslant 3$ if $n > 2$.

Euler [2] conjectured that $G_0(n) \geqslant n$.

It took nearly two centuries for this conjecture to be disproved. It $n = 3$ then

$$3^3 + 4^3 + 5^3 = 6^3$$

*Email: k.taylor@imperial.ac.uk

and Wiles' theorem shows that $G_0(n) = 3$. It $n = 4$ an example found by Roger Frye and quoted by Elkies [3,p.834] is

$$95800^4 + 217519^4 + 414560^4 = 422481^4,$$

Elkies gives the example $2682440^4 + 15365634^4 + 18796760^4 = 201615673^4$ which shows that $G_0(4) = 3$ and disproves Euler's conjecture. A little earlier Lander and Parkin [4] had found the first counter example to Euler's conjecture namely,

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

Thus, $G_0(5) = 3$ or 4. For higher values of n, Euler's conjecture is still open. The best result for $n = 6$ is contained in Lander, Parkin and Selfridge [5];

$$74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6 = 1141^6,$$

and shows that $3 \leqslant G_0(6) \leqslant 7$. There are no corresponding examples for $n > 6$.

2. Large *n* and Waring's problem

Waring [6] conjectured and Hilbert [7] first proved, that every large positive integer z can be expressed as the sum of at most $G(n)$ nth powers (see [8,p.5]). Density considerations show that $G(n) \geqslant n$. On the other hand it is known that

$$G(n) \leqslant Cn \log n.$$

For $C > 2$, see Vinogradov [9]. The best result due to Wooley [10] is

$$G(n) < n(\log n + \log\log n + O(1)) \qquad \text{as} \qquad n \to \infty.$$

We see that,

$$G_0(n) \leqslant G(n) + 1. \tag{2}$$

If x is a large positive integer, then $x^n - 1$ is the sum of k nth powers, where $2 \leqslant k \leqslant G(n)$;

$$x^n - 1 = x_1^n + x_2^n + \ldots + x_k^n,$$

and so,

$$x^n = 1 + x_1^n + \ldots x_k^n$$

is the sum of at most $G(n) + 1$ nth powers of positive integers smaller than x.

However, (2) is a rather weak result. For instance, we have seen that $G_0(3) = 3$ while it is only known that $G(3) \leqslant 7$ (Linnik [11]). Again we saw that $G_0(4) = 3$, while Davenport [12] has shown that $G(4) = 16$. For further details we refer the reader to the beautiful survey paper on Waring's problem by Vaughan and Wooley [13].

3. The analogue for functions: lower bounds

Let C be a semi-ring and let n be an integer satisfying $n \geqslant 2$. We let $F_C(n)$ denote the smallest positive integer k, such that we have a non-trivial representation

$$x_1^n + x_2^n + \ldots + x_k^n = x^n, \tag{3}$$

where $x, x_1 \ldots, x_k \in C$. Thus, $G_0(n) = F_{\mathbb{Z}}(n)$ in this notation. Next let $W_C(n)$ be the smallest number k, such that every m in C can be expressed in the form,

$$m = m_1^n + m_2^n + \ldots + m_k^n. \tag{4}$$

The case $C = \mathbb{Z}$ is generally called Waring's problem and $W_{\mathbb{Z}}(n)$ is denoted by $g(n)$. If we only ask for (4) to hold for all sufficiently large integers m, the corresponding quantity is denoted by $G(n)$.[8,p.5]

It is of some interest to discuss $F_C(n)$ and $W_C(n)$ for some classes of non-constant analytic functions in the plane. Let us denote by M, R, E, P the rings of meromorphic, rational, entire functions in the plane and polynomials, respectively. The corresponding questions were treated in [14,15] and the results were more satisfactory than those for the more difficult problem of $\mathbb{Z}$. We quote [14, Theorem 4.1, p.439]

THEOREM 1 *If $n \geqslant 2$*

$$F_M(n) \geqslant \sqrt{(n+1)},$$
$$F_R(n) > \sqrt{(n+1)},$$
$$F_E(n) \geqslant \frac{1}{2} + \sqrt{\left(n + \frac{1}{4}\right)}$$

and

$$F_P(n) > \frac{1}{2} + \sqrt{\left(n + \frac{1}{4}\right)}.$$

For W it is in all cases enough to represent the identity function z.[14,p.442] The results for W are almost identical to those of F.

THEOREM 2 *If $n \geqslant 2$*

$$W_M(n) \geqslant \sqrt{(n+1)},$$
$$W_R(n) > \sqrt{(n+1)},$$
$$W_E(n) \geqslant \frac{1}{2} + \sqrt{\left(n + \frac{1}{4}\right)}.$$

Further, if $n \geqslant 3$,

$$W_P(n) > \frac{1}{2} + \sqrt{\left(n + \frac{1}{4}\right)}.$$

We also have $W_P(2) = 2$, since

$$\left(\frac{1}{2}(z+1)\right)^2 + \left(\frac{1}{2}i(z-1)\right)^2 = z.$$

But, $F_P(2) = 3$. If P_1, P_2, are polynomials such that

$$P_1^2 + P_2^2 = (P_1 + iP_2)(P_1 - iP_2) = 1,$$

it follows that $P_1 \pm iP_2$ have no zeros and so are constant and hence P_1, P_2 are constant, contrary to the hypothesis. Thus, $F_P(2) > 2$. On the other hand,

$$\left((1+z)/\sqrt{2}\right)^2 + \left((1-z)/\sqrt{2}\right)^2 + (iz)^2 = 1.$$

So, $F_P(2) = 3$. It turns out that for $C = E$, R or M we have [14, Lemma 5.2, p.442].

$$F_C(n) \leqslant W_C(n), \qquad n \geqslant 2.$$

It is an open question, whether

$$F_P(n) \leqslant W_P(n), \qquad \text{if} \qquad n > 2.$$

4. Upper bounds

For F we can use an interesting identity due to Molluzzo [16] to show that Theorem 1 gives the right order of magnitude. Let b and n be integers, satisfying $1 \leqslant b \leqslant n$, and set $\omega_\nu = \exp\{2\pi i\nu/b\}$, $1 \leqslant \nu \leqslant b$. Then,

$$\sum_{\nu=1}^{b} \left(1 + \omega_\nu z^n\right)^n = A_0 + A_1 z^{nb} + \ldots + A_{[n/b]} z^{nb[n/b]}. \tag{5}$$

Here each A_λ is a positive integer, $A_0 = b$ and $[n/b]$ is the integral part of n/b. Thus, A_0 is expressed as the sum of $b + n/b$ nth powers. It follows that

$$F_P(n) \leqslant b + [n/b].$$

Choosing b so as to minimize the right hand side we obtain

$$F_P(n) \leqslant \sqrt{(4n+1)}$$

with the same bound for the larger classes R, E and M. So we obtain the right values for F within a factor of 2. We also deduce that,

$$F_P(3) = F_R(3) = F_E(3) = 3.$$

Gross and Osgood [17] have shown that, there exist non-constant elliptic functions f, g, such that $f^3 + g^3 = 1$, so that $F_M(3) = 2$. We also find that,

$$F_C(4) = F_C(5) = 3$$

for $C = E$, R and M. $F_M(6) = 3$ and $F_E(14) = 5$. (see [14] and references there given.)

Unfortunately, the situation for W is much less satisfactory. Although the lower bounds for F and W in Theorems 1 and 2 are virtually identical, no analogue of Molluzzo's identity is known for z. Heilbronn [18,p.16] noted the following identity. If n is an integer, $n \geqslant 2$

$$\alpha = \exp(2\pi i/n)$$

and

$$f_j(z) = n^{-2/n} \quad \alpha^{-j/n}(1 + \alpha^j z)$$

for $1 \leqslant j \leqslant n$, then

$$\sum_{j=1}^{n} f_j(z)^n = z.$$

Thus, for $C = P$, E, R or M we have

$$W_C(n) \leqslant n, \quad n \geqslant 2. \tag{6}$$

Gross and Osgood [17] have also shown that, there exist meromorphic functions f, g satisfying

$$f^3 + g^3 = z,$$

so that $W_M(3) = 2$. Examples in [15, Satz 12] show that

$$W_M(n), \quad W_R(n) \leqslant \frac{1}{2}n + a,$$

where $a = 3/2$ when n is odd, $a = 2$, when 4 is a factor of n and $a = 3$, when 2 is a factor of n, but 4 is not. Thus, if $C = M$ or R we can improve slightly on (6) if $n = 5$, or $n \geqslant 7$.

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References

[1] Wiles A. Modular elliptic curves and Fermat's last theorem. Ann. Math. 1995;141:443–451.
[2] Euler L. Universal Arithmetic 1769; 2. Available from: http://archive.org/details/1769LEONHARDEULERUniversalArithmeticVol2.
[3] Elkies Noam D. On $A^4 + B^4 + C^4 = D^4$. Math. Comput. 1988;51:825–835.
[4] Lander LJ, Parkin TR. Counter-examples to Euler's conjecture on sums of like powers. Bull. Amer. Math. Soc. 1966;72:1079.
[5] Lander LJ, Parkin TR, Selfridge JL. A survey of equal sums of like powers. Math. Comput. 1967;21:446.
[6] Waring E. Meditationes algebraicae. Cantabrigiae: Typis Academicis excudebat J. Archdeacon; 1770. Available from: http://www.worldcat.org/title/meditationes-algebraicae/oclc/187480384
[7] Hilbert D. Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (waringsches Problem). Math. Ann. 1909;67:281–300.
[8] Vaughan RC. The Hardy-Littlewood method. Cambridge: Cambridge University Press; 1997.
[9] Vinogradov IM. The method of trigonometrical series in the theory of numbers. Trav. Inst. Steklov 23, 1947. Translated from the Russian, revised and annotated by Davenport A and Roth KF. New York Interscience; 1954.
[10] Wooley TD. Large improvements in Waring's problem. Ann. Math. 1992;135:131–164.
[11] Linnik JuV. On the representation of large numbers as sums of seven cubes. Mat. Sbornik N.S. 1943;12:218–224.
[12] Davenport H. On Waring's problem for fourth powers. Ann. Math. 1939;40:731–747.

[13] Vaughan RC, Wooley TD. Waring's problem: a survey Number theory for the millenium III. Natick (MA): A.K. Peters; 2002.

[14] Gundersen Gary G, Hayman Walter K. The strength of Cartan's version of Navanlinna theory. Bull. London Math. Soc. 2004;36:433–454.

[15] Hayman Walter K. Waring's Problem für analytische Funktionen. Bayer. Akad. Wiss. Math. Natur. Kl. Sitzungsber. 1985;1984:1–13.

[16] Molluzzo J. Monotonicity of quadrature formulae and polynominal representation [Doctoral thesis]. Yeshiva University; 1972.

[17] Gross F, Osgood CF. On the functional equation $f^n + g^n = h^n$, and a new approach to a certain class of more general functional equations. Indian J. Math. 1981;23:17–39.

[18] Hayman WK. Research problems in function theory. University of London, London: Athlone Press; 1967.