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THE STRENGTH OF CARTAN'S VERSION OF NEVANLINNA THEORY

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Dedicated to Henri Cartan on his 100th birthday

Abstract

In 1933 Henri Cartan proved a fundamental theorem in Nevanlinna theory, namely a generalization of Nevanlinna's second fundamental theorem. Cartan's theorem works very well for certain kinds of problems. Unfortunately, it seems that Cartan's theorem, its proof, and its usefulness, are not as widely known as they deserve to be. To help give wider exposure to Cartan's theorem, the simple and general forms of the theorem are stated here. A proof of the general form is given, as well as several applications of the theorem.

1. *Introduction*

About eighty years ago, Rolf Nevanlinna extended the classical theorems of Picard and Borel, and developed the value distribution theory of meromorphic functions, which is called *Nevanlinna theory*. In many ways, Nevanlinna theory is a best possible theory for both meromorphic and entire functions, and it has been used to prove numerous important results about meromorphic and entire functions. Nevanlinna developed his theory in a series of papers from 1922–1925, and [**36**] is considered his most important paper; see [**14**, **16**, **28**, **37**, **47**]. In 1943, Weyl [**45**, p. 8] made the following comment about [**36**]: "The appearance of this paper has been one of the few great mathematical events in our century."

The main result in Nevanlinna theory is called the *second fundamental theorem*. In 1933, Cartan [**2**] proved a generalization of the second fundamental theorem, and for certain kinds of problems, Cartan's theorem seems to give better results than the second fundamental theorem.

Unfortunately, it appears that many people are not familiar with the details in Cartan's paper [**2**]. Specifically, it seems that the statement and proof of Cartan's theorem, and its potential applications, are not as widely known as they deserve to be. In this paper, we state the simple and general forms of Cartan's theorem, give a complete proof of the general form, and provide several applications of the theorem. We also give a proof that the second fundamental theorem is a corollary of Cartan's theorem. We hope this paper will help to give wider exposure to Cartan's theorem, and at the same time will help people to find new applications of both the simple and general forms of the theorem.

2. *The simple form of Cartan's theorem*

The general form of Cartan's theorem concerns q linear combinations $f_1, f_2, \ldots, f_q$ of p entire functions $g_1, g_2, \ldots, g_p$, where $q > p \geqslant 2$. The simplest case

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occurs when $q = p+1$ with $f_1 = g_1, f_2 = g_2, \dots, f_p = g_p$, and $f_{p+1} = g_1+g_2+\dots+g_p$. In this section we state Cartan's theorem in this simpler case, and then in Sections 3–6 we give some applications of the theorem.

In this paper, a *meromorphic function* means a function that is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the basic results and standard notation in Nevanlinna theory; see [**14**, **37**, **47**].

DEFINITION 2.1. For a meromorphic function f satisfying $f \not\equiv 0$ and a positive integer j, let $n_j(r,0,f)$ denote the number of zeros of f in $\{z : |z| \leqslant r\}$, counted in the following manner: a zero of f of multiplicity m is counted exactly k times where $k = \min\{m, j\}$. Then let $N_j(r,0,f)$ denote the corresponding integrated counting function; that is,

$$N_j(r,0,f) = n_j(0,0,f)\log r + \int_0^r \frac{n_j(t,0,f) - n_j(0,0,f)}{t}\,dt.$$

Regarding the well-known integrated counting functions $\bar{N}(r,0,f)$ and $N(r,0,f)$, we see that $\bar{N}(r,0,f) \leqslant N_j(r,0,f) \leqslant N(r,0,f)$, $N_j(r,0,f) \leqslant j\bar{N}(r,0,f)$ and $N_1(r,0,f) = \bar{N}(r,0,f)$.

We now state the simple form of Cartan's theorem [**2**], together with the observation made in [**17**], which is (2.5) below. We use the abbreviation *n.e.* (which stands for *nearly everywhere*) to mean 'everywhere in $0 < r < \infty$, except possibly for an exceptional set of finite linear measure'.

THEOREM 2.1 [**2**, **17**]. *Let $g_1, g_2, \dots, g_p$ be linearly independent entire functions, where $p \geqslant 2$. Suppose that for each complex number z we have $\max\{|g_1(z)|, |g_2(z)|, \dots, |g_p(z)|\} > 0$. For positive r, set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta - u(0), \quad \text{where } u(z) = \sup_{1\leqslant j\leqslant p} \log|g_j(z)|. \tag{2.1}$$

Set $g_{p+1} = g_1 + g_2 + \dots + g_p$. Then we have

$$T(r) \leqslant \sum_{j=1}^{p+1} N_{p-1}(r,0,g_j) + S(r) \leqslant (p-1)\sum_{j=1}^{p+1} \bar{N}(r,0,g_j) + S(r), \tag{2.2}$$

where $S(r)$ is a quantity satisfying

$$S(r) = O(\log T(r)) + O(\log r) \quad \text{as } r \to \infty \text{ n.e.} \tag{2.3}$$

If at least one of the quotients g_j/g_m is a transcendental function, then

$$S(r) = o(T(r)) \quad \text{as } r \to \infty \text{ n.e.}, \tag{2.4}$$

while if all the quotients g_j/g_m are rational functions, then

$$S(r) \leqslant -\frac{1}{2}p(p-1)\log r + O(1) \quad \text{as } r \to \infty. \tag{2.5}$$

Furthermore, if all the quotients g_j/g_m are rational functions, then there exist polynomials $h_1, h_2, \dots, h_{p+1}$, and an entire function ϕ, such that

$$g_j = h_j e^{\phi}, \qquad j = 1, 2, \dots, p+1, \tag{2.6}$$

and the identity $g_{p+1} = g_1 + g_2 + \dots + g_p$ reduces to the identity $h_{p+1} = h_1 + h_2 + \dots + h_p$.

Theorem 2.1 is a corollary of Theorem 7.1 in Section 7. We call Theorem 7.1 the *general form of Cartan's theorem*, and Theorem 2.1 the *simple form*. In Sections 7–9 we discuss and prove Theorem 7.1, and we show that the second fundamental theorem is a corollary of Theorem 7.1.

In Sections 3–5 we use Theorem 2.1 to prove some results for analytic functions that are analogues of well-known number-theoretic results and conjectures. In Section 6 we give an application of Theorem 2.1 to unique range sets for entire functions. For these applications of Theorem 2.1 in Sections 3–6, we also need the following result; this is a corollary of Lemma 8.1, which is stated and proved in Section 8.

LEMMA 2.2 [**2**]. *Assume that the hypotheses of Theorem* 2.1 *hold. Then for any j and m, we have*

$$T(r, g_j/g_m) \leqslant T(r) + O(1) \quad \text{as } r \to \infty, \tag{2.7}$$

and for any j, we have

$$N(r, 0, g_j) \leqslant T(r) + O(1) \quad \text{as } r \to \infty. \tag{2.8}$$

3. *Function theory analogues of the abc conjecture*

Lang said [**27**, p. 196]: "One of the most fruitful analogies in mathematics is that between the integers $\mathbb{Z}$ and the ring of polynomials $F[t]$ over a field F." We give an example of this.

The next result exhibits an interesting relationship between the degrees and distinct zeros of polynomials. For a polynomial Q satisfying $Q \not\equiv 0$, let $d(Q)$ denote the degree of Q, and let $\bar{d}(Q)$ denote the number of distinct zeros of Q.

THEOREM 3.1 (see [**5**], [**27**, p. 194], [**30**], [**31**], [**35**, p. 182] and [**42**]). *Let Q_1, Q_2 and Q_3 be three relatively prime polynomials that satisfy*

$$Q_1 + Q_2 = Q_3, \tag{3.1}$$

and suppose that Q_1, Q_2 and Q_3 are not all constants. Then

$$\max\{d(Q_1), d(Q_2), d(Q_3)\} \leqslant \bar{d}(Q_1 Q_2 Q_3) - 1. \tag{3.2}$$

If $Q_1(z) = (1+z)^2$, $Q_2(z) = -(1-z)^2$ and $Q_3(z) = 4z$, then the inequality (3.2) becomes the equality $2 = 2$. Thus the inequality (3.2) is sharp. For applications of Theorem 3.1, see [**31**] and [**35**, pp. 183–185].

In Theorem 3.1 we note [**35**, p. 182] that

$$\bar{d}(Q_1 Q_2 Q_3) = d(\operatorname{rad}(Q_1 Q_2 Q_3)),$$

where $\operatorname{rad}(Q_1 Q_2 Q_3)$ is the *radical of* $Q_1 Q_2 Q_3$, which is the product of the distinct linear factors of $Q_1 Q_2 Q_3$. On the other hand, the *radical of a nonzero integer* m, which we denote by $\mu(m)$, is the product of the distinct prime numbers that divide m. Distinct prime factors of an integer are often considered an appropriate analogy to distinct zeros of a polynomial [**5**, p. 1226]. After being influenced by Theorem 3.1, and by considerations of Szpiro and Frey, in 1985 Masser and Oesterlè formulated the *abc conjecture* for integers; see [**5**, **26**, **27**, **35**].

THE *abc* CONJECTURE. *For any given positive number ε, there exists a positive number $K(\varepsilon)$ such that, if a, b and c are any nonzero, relatively prime integers that satisfy $a + b = c$, then*

$$\max\{|a|, |b|, |c|\} \leqslant K(\varepsilon)\mu(abc)^{1+\varepsilon}.$$

To prove or disprove the *abc* conjecture would be an important contribution to number theory; see [**5**, **26**, **27**, **35**]. For instance, some results that would follow from the *abc* conjecture appear in [**35**, pp. 185–188].

An interesting discussion in [**5**] illustrates how naturally one is led from Theorem 3.1 to the formulation of the *abc* conjecture. As mentioned above, the distinct prime factors of an integer and the distinct zeros of a polynomial are often considered analogous; in addition, the absolute value of an integer is a measure of how 'large' the integer is, while the degree of a polynomial is a measure of how 'large' the polynomial is. Therefore, we see that Theorem 3.1 is a polynomial analogue of the *abc* conjecture.

Theorem 3.2 below extends Theorem 3.1 from polynomials to entire functions, and from three functions to any finite number of functions. We use Theorem 2.1 to prove Theorem 3.2.

THEOREM 3.2. *Let $g_1, g_2, \ldots, g_p$ be linearly independent entire functions, where $p \geqslant 2$. Suppose that for each complex number z we have*

$$\max\{|g_1(z)|, |g_2(z)|, \ldots, |g_p(z)|\} > 0.$$

Set $g_{p+1} = g_1 + g_2 + \ldots + g_p$.

We distinguish two cases.

(a) *Suppose that all the quotients g_j/g_m are rational functions. Then there exist polynomials $h_1, h_2, \ldots, h_{p+1}$, and an entire function ϕ, such that*

$$g_j = h_j e^{\phi}, \qquad j = 1, 2, \ldots, p+1. \tag{3.3}$$

Then $h_{p+1} = h_1 + h_2 + \ldots + h_p$ and

$$\max\{d(h_1), d(h_2), \ldots, d(h_p)\} \leqslant (p-1)\left\{\sum_{j=1}^{p+1} \bar{d}(h_j) - \frac{1}{2}p\right\}. \tag{3.4}$$

In particular, if all the functions $g_1, g_2, \ldots, g_p$ are polynomials, then

$$\max\{d(g_1), d(g_2), \ldots, d(g_p)\} \leqslant (p-1)\left\{\sum_{j=1}^{p+1} \bar{d}(g_j) - \frac{1}{2}p\right\}. \tag{3.5}$$

(b) *Suppose that at least one quotient g_j/g_m is a transcendental function. Set*

$$N(r) = \sup_{1\leqslant j\leqslant p+1} N(r, 0, g_j).$$

Then

$$\frac{N(r)}{\log r} \to \infty \quad \text{as } r \to \infty, \tag{3.6}$$

and we have

$$(1 + o(1))N(r) \leqslant (p-1)\sum_{j=1}^{p+1} \bar{N}(r, 0, g_j) \quad \text{as } r \to \infty \text{ n.e.} \tag{3.7}$$

Theorem 3.1 is a special case of Theorem 3.2(a). By using the left-hand inequality in (2.2), we can obtain stronger inequalities than (3.4), (3.5) and (3.7). Regarding (3.5), see [**20**]. For a non-Archimedean version of Theorem 3.2, see [**19**].

Proof of Theorem 3.2. First suppose that all the quotients g_j/g_m are rational functions. Then (3.3) follows from (2.6), and we have

$$h_{p+1} = h_1 + h_2 + \ldots + h_p. \tag{3.8}$$

Set

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta - u(0), \quad \text{where } u(z) = \sup_{1\leqslant j\leqslant p} \log|h_j(z)|. \tag{3.9}$$

Then we can apply Theorem 2.1 to (3.8), and this yields

$$T(r) \leqslant (p-1)\sum_{j=1}^{p+1} \bar{N}(r,0,h_j) - \frac{1}{2}p(p-1)\log r + O(1) \quad \text{as } r \to \infty. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$d\,\log r \leqslant (p-1)\left\{\sum_{j=1}^{p+1} \bar{d}(h_j) - \frac{1}{2}p\right\}\log r + O(1) \quad \text{as } r \to \infty,$$

where $d = \max\{d(h_1), d(h_2), \ldots, d(h_p)\}$. This gives (3.4).

In particular, if all the functions $g_1, g_2, \ldots, g_p$ are polynomials, then in (3.3) we can choose $\phi \equiv 0$ and $g_j \equiv h_j$ for $j = 1, 2, \ldots, p$. Thus, (3.5) follows from (3.4). This proves part (a).

Now suppose that at least one quotient g_j/g_m is a transcendental function. Set

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta - u(0), \quad \text{where } u(z) = \sup_{1\leqslant j\leqslant p} \log|g_j(z)|.$$

Then, from Theorem 2.1, we find that

$$(1+o(1))T(r) \leqslant (p-1)\sum_{j=1}^{p+1} \bar{N}(r,0,g_j) \quad \text{as } r \to \infty \text{ n.e.} \tag{3.11}$$

Since at least one quotient g_j/g_m is transcendental, we have [**14**, p. 24]

$$\frac{T(r, g_j/g_m)}{\log r} \to \infty \quad \text{as } r \to \infty.$$

Thus we see that (3.6) follows from (2.7) and (3.11). Also, by combining (2.8) and (3.11), we obtain (3.7). This proves part (b), and completes the proof of the theorem.

We now give an example that shows that the inequality (3.5) is asymptotically sharp as $p \to \infty$. Let b and n be integers satisfying $1 \leqslant b \leqslant n$, and set $\omega_\nu = \exp\{2\pi i\nu/b\}$ for $1 \leqslant \nu \leqslant b$. For a real number x, let $[x]$ denote the greatest integer that is less than or equal to x. Consider the following identity [**32**]:

$$\sum_{\nu=1}^{b}(1+\omega_\nu z^n)^n = A_0 + A_1 z^{nb} + A_2 z^{2nb} + \ldots + A_{[n/b]} z^{nb[n/b]}, \tag{3.12}$$

where each A_ν is a positive integer.

Now let p be an integer, where $p \geqslant 2$. Set $b = [(p/2)^{1/3}]$ and $n = b(p-b)$. Then $1 \leqslant b \leqslant n$ and $p = b + n/b$. Hence, from (3.12), we find that there exist polynomials $Q_1, Q_2, \ldots, Q_{p+1}$ such that

$$Q_1 + Q_2 + \ldots + Q_p = Q_{p+1},$$

where

$$\begin{array}{lll} d(Q_j) = n^2 & \text{and} \quad \bar{d}(Q_j) = n, & \text{for } 1 \leqslant j \leqslant b, \\ d(Q_j) = (j-b)nb & \text{and} \quad \bar{d}(Q_j) = 1, & \text{for } b+1 \leqslant j \leqslant p, \end{array}$$

and

$$d(Q_{p+1}) = \bar{d}(Q_{p+1}) = 0.$$

Then we have

$$d = n^2 = b^2p^2 - 2b^3p + b^4.$$

Set $d = \max\{d(Q_1), \ldots, d(Q_p)\}$.We also have $\bar{d}(Q_1) + \bar{d}(Q_2) + \ldots + \bar{d}(Q_{p+1}) = bn + p - b$, and this yields

$$(p-1)\left\{\sum_{j=1}^{p+1} \bar{d}(Q_j) - \frac{1}{2}p\right\} = p^2\left(b^2 + \frac{1}{2}\right) - p\left(b^3 + b^2 + b + \frac{1}{2}\right) + b^3 + b.$$

From these equations, we obtain the following three statements:

$$d = \bigl(2^{-2/3} + o(1)\bigr)p^{8/3} \quad \text{as } p \to \infty; \tag{3.13}$$

$$(p-1)\left\{\sum_{j=1}^{p+1} \bar{d}(Q_j) - \frac{1}{2}p\right\} = \bigl(2^{-2/3} + o(1)\bigr)p^{8/3} \quad \text{as } p \to \infty; \tag{3.14}$$

$$(p-1)\left\{\sum_{j=1}^{p+1} \bar{d}(Q_j) - \frac{1}{2}p\right\} - d < \frac{1}{2}p^2 + pb^3 \leqslant p^2. \tag{3.15}$$

From (3.13) and (3.14), it can be deduced that the inequality (3.5) is asymptotically sharp as $p \to \infty$. Furthermore, (3.15) shows that we cannot replace the term $-\frac{1}{2}p(p-1)$ in (3.5) with $-\frac{3}{2}p^2$ for any p. By using a similar argument, it can be shown that if α is any given constant satisfying $\alpha > 1$, then there exists an integer $p_0 = p_0(\alpha)$, where $p_0 \geqslant 2$, such that we cannot replace the term $-\frac{1}{2}p(p-1)$ in (3.5) with $-\alpha p^2$ whenever $p \geqslant p_0$.

4. *Fermat-type equations*

'Fermat's last theorem', which was proved by Wiles [**46**] and by Taylor and Wiles [**43**], states that there do not exist nonzero rational numbers x, y, and an integer n, where $n \geqslant 3$, such that

$$x^n + y^n = 1.$$

On the other hand, Ramanujan noted that $9^3 + (10)^3 + (-12)^3 = 1$. These facts can be expressed in the following way. Let C be a ring, and let n be an integer satisfying $n \geqslant 2$. We let $F_C(n)$ denote the smallest positive integer k such that we have a *nontrivial* representation

$$x_1^n + x_2^n + \ldots + x_k^n = 1,$$

where $x_j \in C$ for $j = 1, 2, \ldots, k$. From the above facts, we have $F_{\mathbb{Z}}(3) = 3$.

Now let M, R, E and P denote the rings of meromorphic functions, rational functions, entire functions and polynomials, respectively. Thus if C is equal to M, R, E or P, and n is an integer satisfying $n \geqslant 2$, then $F_C(n)$ denotes the smallest positive integer k such that the equation

$$f_1^n + f_2^n + \ldots + f_k^n = 1 \tag{4.1}$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C. Hence, k depends on n.

Theorem 4.1 below is a collection of results to be found in [**4**], [**6**], [**17**], [**38**] and [**44**]. Some special cases of Theorem 4.1 were proved in [**7**] and [**22**]. The four inequalities in Theorem 4.1 are, for every n, the best lower estimates that are known.

THEOREM 4.1. *We have the following results for equation* (4.1).
(a) $F_M(n) \geqslant \sqrt{n+1}$, *if* $n \geqslant 2$.
(b) $F_R(n) > \sqrt{n+1}$, *if* $n \geqslant 2$.
(c) $F_E(n) \geqslant 1/2 + \sqrt{n+1/4}$, *if* $n \geqslant 2$.
(d) $F_P(n) > 1/2 + \sqrt{n+1/4}$, *if* $n \geqslant 2$.

There is another way to express Theorem 4.1. For example, Theorem 4.1(a) can be expressed as follows. *There do not exist k nonconstant meromorphic functions $f_1, f_2, \ldots, f_k$ satisfying* (4.1) *when $n \geqslant k^2$*. Similarly, statements (b), (c) and (d) can be expressed as follows. *For C equal to R, E or P, there do not exist k nonconstant functions $f_1, f_2, \ldots, f_k$ in C satisfying* (4.1) *when $n \geqslant k^2 - 1$, $n \geqslant k^2 - k + 1$, or $n \geqslant k^2 - k$, respectively.*

A result that is stated in [**38**, p. 481] would give a stronger result than Theorem 4.1(d), but there appears to be an error in the reasoning in the proof, because from [**38**, p. 482, line 4] it seems that one can deduce only Theorem 4.1(d) and no better.

Cartan's theorem was used in [**17**] to prove all four parts of Theorem 4.1. We used Cartan's theorem to prove Theorem 3.2 above, and we now use Theorem 3.2 to prove all four parts of Theorem 4.1.

Proof of Theorem 4.1. Suppose that $f_1, f_2, \ldots, f_k$ are nonconstant meromorphic functions satisfying (4.1). Obviously, $k \geqslant 2$. We assume without loss of generality that the functions $f_1^n, f_2^n, \ldots, f_k^n$ are linearly independent.

First suppose that each f_j is a polynomial. Set $d = \max\{d(f_1^n), d(f_2^n), \ldots, d(f_k^n)\}$. Then $d > 0$. From (3.5) we see that

$$d \leqslant (k-1)\left\{\sum_{j=1}^{k} \bar{d}(f_j^n) - \frac{1}{2}k\right\} \leqslant (k-1)\left\{k\frac{d}{n} - \frac{1}{2}k\right\} < (k-1)k\frac{d}{n},$$

which yields $n < k^2 - k$. This proves part (d).

Now suppose that each f_j is an entire function, where at least one f_j is transcendental. It follows from (4.1) that at least one quotient f_j/f_m must be transcendental. Then from Theorem 3.2(b), we find that

$$(1 + o(1))N(r) \leqslant (k-1)\sum_{j=1}^{k} \bar{N}(r, 0, f_j^n) \quad \text{as } r \to \infty \text{ n.e.},$$

where

$$N(r) = \sup_{1 \leqslant j \leqslant k} N(r, 0, f_j^n).$$

Thus

$$(1+o(1))N(r) \leqslant (k-1)\frac{k}{n}N(r) \quad \text{as } r \to \infty \text{ n.e.} \tag{4.2}$$

From (3.6), we have $N(r) \to \infty$ as $r \to \infty$. Hence from (4.2), we see that $n \leqslant k^2 - k$, which, together with part (d), proves part (c).

Next suppose that at least one f_j is not an entire function. From (4.1), it follows that there exist entire functions $h_1, h_2, \ldots, h_{k+1}$ such that

$$h_1^n + h_2^n + \ldots + h_k^n = h_{k+1}^n. \tag{4.3}$$

If each f_j is a rational function, then in (4.3) we can assume that each h_j is a polynomial. In this case, from (3.5) we obtain

$$d \leqslant (k-1)\left\{\sum_{j=1}^{k+1} \bar{d}(h_j^n) - \frac{1}{2}k\right\}, \tag{4.4}$$

where $d = \max\{d(h_1^n), d(h_2^n), \ldots, d(h_k^n)\}$. This gives

$$d \leqslant (k-1)\left\{(k+1)\frac{d}{n} - \frac{1}{2}k\right\} < (k^2-1)\frac{d}{n},$$

which yields $n < k^2 - 1$. This proves part (b).

The last case occurs when each f_j is meromorphic, where at least one f_j is transcendental. In this case it follows from (4.1) that at least one quotient f_j/f_m is transcendental, which implies that at least one quotient h_j^n/h_m^n must be transcendental. Then we can apply Theorem 3.2(b) to (4.3), and this yields

$$(1+o(1))N(r) \leqslant (k-1)\sum_{j=1}^{k+1} \bar{N}(r,0,h_j^n) \quad \text{as } r \to \infty \text{ n.e.},$$

where

$$N(r) = \sup_{1 \leqslant j \leqslant k+1} N(r,0,h_j^n).$$

This gives

$$(1+o(1))N(r) \leqslant (k-1)\frac{k+1}{n}N(r) \quad \text{as } r \to \infty \text{ n.e.} \tag{4.5}$$

From (3.6) and (4.5) we obtain $n \leqslant k^2 - 1$, which, together with part (b), proves part (a). The proof of the theorem is complete.

We now give examples for equation (4.1). First note (see [**17**]) that

$$\left(\frac{1+z}{\sqrt{2}}\right)^2 + \left(\frac{1-z}{\sqrt{2}}\right)^2 + (iz)^2 = 1, \tag{4.6}$$

which shows that $F_P(2) \leqslant 3$. On the other hand (see [**22**]), if f and g are nonconstant entire solutions of $f^2 + g^2 = 1$, then for some entire function w, we have $f = \cos w$ and $g = \sin w$. It follows that $F_P(2) = 3$ and $F_E(2) = F_M(2) = 2$. Since (see [**7**]) $f(z) = 2z(z^2+1)^{-1}$ and $g(z) = (z^2-1)(z^2+1)^{-1}$ satisfy $f^2+g^2 = 1$, we have $F_R(2) = 2$. Below we give more exact values of $F_C(n)$ for certain values of n and classes C.

We now consider two identities, where the following notation is used. Let b and n be integers satisfying $1 \leqslant b \leqslant n$, and set $\omega_\nu = \exp\{2\pi i\nu/b\}$ for $1 \leqslant \nu \leqslant b$. From

(3.12), there exist $k = b + [n/b]$ nonconstant polynomials $f_1, f_2, \ldots, f_k$ satisfying equation (4.1). Since the minimum over all b of $k = b + [n/b]$ is $[\sqrt{4n+1}]$ (see [**38**]), it follows that $F_P(n) \leqslant \sqrt{4n+1}$, $n \geqslant 2$.

Next consider the following identity [**38**]:

$$\sum_{\nu=1}^{b} \frac{\omega_\nu (1+\omega_\nu z^n)^n}{z^{(b-1)n}} = B_0 + B_1 z^{bn} + B_2 z^{2bn} + \ldots + B_{[(n+1)/b]-1} z^{([(n+1)/b]-1)bn}, \quad (4.7)$$

where each B_ν is a positive integer. From (4.7) we see that there exist $k = b + [(n+1)/b] - 1$ nonconstant rational functions $f_1, f_2, \ldots, f_k$ satisfying equation (4.1). Since the minimum over all b of $k = b + [(n+1)/b] - 1$ is $[\sqrt{4n+5}] - 1$ (see [**17**]), we find that $F_C(n) \leqslant \sqrt{4n+5} - 1$, $n \geqslant 2$, for C equal to E, R or M. When $C = E$, we can see this by replacing z with e^z in (4.7).

For C equal to M, R, E or P, we observe that as $n \to \infty$, the above upper estimates for $F_C(n)$ are asymptotic to $2\sqrt{n}$, while the lower estimates for $F_C(n)$ in Theorem 4.1 are asymptotic to $\sqrt{n}$. Also, for fixed n, the upper estimates for M, R, E and P differ from each other by at most 1, and the lower estimates for M, R, E and P differ from each other by at most 1.

When $b = 2$ and $n = 3$ in (3.12), we obtain the following identity:

$$\tfrac{1}{2}(1+z^3)^3 + \tfrac{1}{2}(1-z^3)^3 - 3(z^2)^3 = 1. \quad (4.8)$$

On the other hand, f and g are nonconstant meromorphic solutions of $f^3 + g^3 = 1$ if and only if f and g are certain nonconstant elliptic functions composed with an entire function; see [**1**]. Combining this result with (4.8) gives $F_M(3) = 2$ and $F_P(3) = F_R(3) = F_E(3) = 3$.

When $b = 3$ and $n = 4$ in (4.7), we obtain the following identity:

$$\frac{1}{18}\left(\frac{1+z^4}{z^2}\right)^4 + \frac{e^{2\pi i/3}}{18}\left(\frac{1+e^{2\pi i/3}z^4}{z^2}\right)^4 + \frac{e^{4\pi i/3}}{18}\left(\frac{1+e^{4\pi i/3}z^4}{z^2}\right)^4 = 1. \quad (4.9)$$

By combining Theorem 4.1 and (4.9), we find that $F_M(4) = F_R(4) = F_E(4) = 3$. When $C = E$, we can see this by replacing z with e^z in (4.9). Examples in [**9**, **10**, **13**, **39**], together with Theorem 4.1, show that $F_E(5) = F_M(5) = F_R(5) = 3$, $F_M(6) = 3$, and $F_E(14) = 5$. For more examples and references concerning equation (4.1), see [**11**].

5. *Waring's problem for analytic functions*

Waring's problem in number theory can be stated as follows: "For a given integer n satisfying $n \geqslant 2$, what is the smallest positive integer k such that any positive integer m can be expressed in the form

$$m = m_1^n + m_2^n + \ldots + m_k^n$$

for some choice of positive integers $m_1, m_2, \ldots, m_k$?" Let $W(n)$ denote this smallest positive integer k (which depends on n). Hilbert [**18**] first proved that $W(n) < \infty$, $n \geqslant 2$.

Analogously, if C is equal to M, R, E or P, then Waring's problem for the ring C is the following question: "For a given integer n satisfying $n \geqslant 2$, what is the smallest positive integer k such that any function f in C can be expressed in the form $f = f_1^n + f_2^n + \ldots + f_k^n$ for some choice of functions $f_1, f_2, \ldots, f_k$ in C?"

Let $W_C(n)$ denote this smallest positive integer k (which depends on n). To answer this question for each of the four rings M, R, E and P, it is enough to consider the function $f(z) = z$; that is, we need only to consider the equation

$$z = f_1(z)^n + f_2(z)^n + \ldots + f_k(z)^n. \tag{5.1}$$

To see this, suppose first that C is equal to P, E or R, and that equation (5.1) is satisfied by k functions $f_1, f_2, \ldots, f_k$ in C. Then for any f in C, we have

$$f(z) = f_1(f(z))^n + f_2(f(z))^n + \ldots + f_k(f(z))^n.$$

Now suppose that (5.1) holds for k functions $f_1, f_2, \ldots, f_k$ in M. If $f \in M$, then there exist entire functions g and h such that $f = g/h^n$. Then, from (5.1),

$$f(z) = \left(\frac{f_1(g(z))}{h(z)}\right)^n + \left(\frac{f_2(g(z))}{h(z)}\right)^n + \ldots + \left(\frac{f_k(g(z))}{h(z)}\right)^n.$$

Thus to answer Waring's problem for M, R, E and P, it is enough to consider equation (5.1).

The following result gives, for every n indicated, the best lower estimates that are known.

Theorem 5.1 [**17**]. *We have the following results for equation* (5.1).

(a) $W_M(n) \geqslant \sqrt{n+1}$, *if* $n \geqslant 2$.
(b) $W_R(n) > \sqrt{n+1}$, *if* $n \geqslant 2$.
(c) $W_E(n) \geqslant 1/2 + \sqrt{n+1/4}$, *if* $n \geqslant 2$.
(d) $W_P(n) > 1/2 + \sqrt{n+1/4}$, *if* $n \geqslant 3$.

A result that is stated in [**38**, p. 481] would give a stronger result than Theorem 5.1(d), but there appears to be an error in the reasoning in the proof, because from [**38**, p. 482, line 4] it seems that one can deduce only Theorem 5.1(d) and no better.

The corresponding inequalities in Theorems 4.1 and 5.1 are identical, except when $n = 2$ in part (d). This case is different because Theorem 5.1(d) does not hold when $n = 2$, since

$$\left(\frac{z+1}{2}\right)^2 + \left(\frac{z-1}{2i}\right)^2 = z;$$

see [**17**]. This shows that $W_P(2) = 2$. On the other hand, $F_P(2) = 3$; see Section 4.

As with Theorem 4.1 above, there is another way to state Theorem 5.1; see the paragraph after the statement of Theorem 4.1. Cartan's theorem was used in [**17**] to prove Theorem 5.1. We used Cartan's theorem to prove Theorems 3.2 and 4.1 above, and we now use Theorems 3.2 and 4.1, together with the next lemma, to prove Theorem 5.1.

Lemma 5.2 [**17**]. *If* $n \geqslant 2$, *we have*

$$F_C(n) \leqslant W_C(n) \quad \text{for } C = E,\ C = R \text{ or } C = M. \tag{5.2}$$

Proof. Let C be equal to E, R or M, and suppose that there exist k functions $f_1, f_2, \ldots, f_k$ in C satisfying (5.1). Then by replacing z in (5.1) with either z^n or e^{nz}, we deduce that there exist k nonconstant functions $h_1, h_2, \ldots, h_k$ in C satisfying $h_1^n + h_2^n + \ldots + h_k^n = 1$. This proves (5.2).

We showed above that $W_P(2) = 2$ and $F_P(2) = 3$. Hence, from Lemma 5.2, there remains the following open question: "Is $F_P(n) \leqslant W_P(n)$ when $n \geqslant 3$?"

Proof of Theorem 5.1. Parts (a), (b) and (c) follow from Theorem 4.1 and Lemma 5.2, and so it remains to prove part (d). Let n be an integer satisfying $n \geqslant 3$, and suppose that $f_1, f_2, \ldots, f_k$ are polynomials satisfying (5.1). We assume without loss of generality that the functions $f_1^n, f_2^n, \ldots, f_k^n$ are linearly independent. Set $d = \max\{d(f_1^n), d(f_2^n), \ldots, d(f_k^n)\}$. Then $d > 0$. Since $n \geqslant 3$, we find that $k \geqslant 3$, from part (b). From (3.5),

$$d \leqslant (k-1)\left\{\sum_{j=1}^{k} \bar{d}(f_j^n) + 1 - \frac{1}{2}k\right\} \leqslant (k-1)\left\{k\frac{d}{n} + 1 - \frac{1}{2}k\right\} < (k-1)k\frac{d}{n},$$

which yields $n < k^2 - k$. This proves part (d). The proof is complete.

We now give examples of (5.1). Heilbronn [**15**, p. 16] gave the following example. If n is an integer satisfying $n \geqslant 2$, $\alpha = \exp(2\pi i/n)$ and

$$f_j(z) = n^{-2/n}\alpha^{-j/n}(1 + \alpha^j z), \qquad j = 1, 2, \ldots, n, \tag{5.3}$$

then $f_1, f_2, \ldots, f_n$ satisfy equation (5.1) where $k = n$. Thus for C equal to P, E, R or M, we have $W_C(n) \leqslant n$, $n \geqslant 2$. Hence, from Theorem 5.1, $W_P(3) = W_E(3) = W_R(3) = 3$. On the other hand [**8**], there exist meromorphic functions f and g satisfying $f^3 + g^3 = z$, which shows that $W_M(3) = 2$. Examples in [**17**, Satz 12] show that if $n \geqslant 2$, then $W_M(n) \leqslant W_R(n) \leqslant n/2 + a$, where $a = 3/2$ when n is odd, $a = 2$ when n divides 4, and $a = 3$ when n divides 2 but not 4. Thus if C equals M or R, we can do a little better than Heilbronn's examples above, in the cases when $n = 5$ and $n \geqslant 7$. On the other hand [**23**], we cannot improve Heilbronn's examples by replacing the $f_j(z)$ in (5.3) with other linear polynomials.

In the above examples, the four upper estimates for $W_C(n)$ (for $C = M$, $C = R$, $C = E$ and $C = P$) have the property that k and n have the same order as $n \to \infty$, while the four lower estimates for $W_C(n)$ in Theorem 5.1 have the property that k and $\sqrt{n}$ have the same order as $n \to \infty$. In view of Section 4, it is natural to ask the following question: "If C is equal to M, R, E or P, then for all large n, do there exist k functions $f_1, f_2, \ldots, f_k$ in C that satisfy equation (5.1), such that k and $\sqrt{n}$ have the same order as $n \to \infty$?"

6. *Unique range sets for entire functions*

In [**11**] it was shown that Cartan's theorem can be used to prove the best known theorems concerning the existence of unique range sets for entire functions. We now discuss this.

DEFINITION 6.1. Let S be a set of distinct complex numbers. We say that two nonconstant entire functions f and g *share the set S CM* (counting multiplicities), provided that $f(z) \in S$ if and only if $g(z) \in S$, where corresponding roots have the same multiplicity.

DEFINITION 6.2. Let S be a set of distinct complex numbers that has the following property: if f and g are nonconstant entire functions that share the set S CM, then $f \equiv g$. Then S is called a *unique range set for entire functions* (*URSE*).

We have the following results.

THEOREM 6.1 [**48**]. *The set $S_1 = \{z : z^7 + z + 1 = 0\}$ is a URSE with seven elements.*

THEOREM 6.2 [**29**]. *The set $S_2 = \{z : z^7 + z^6 + 1 = 0\}$ is a URSE with seven elements.*

In the other direction, there exist examples [**21**] that show that a URSE must have at least five elements. This leaves the following open questions:

(i) "Does there exist a URSE with five elements?", and

(ii) "Does there exist a URSE with six elements?"

Cartan's theorem was used in [**11**] to give proofs of Theorems 6.1 and 6.2. In this connection, see also [**40**] and [**41**].

Remarks on the proof of Theorem 6.1 *given in* [**11**]. Let f and g be nonconstant entire functions that share the set S_1 CM. Then, for some entire function w, we have

$$f^7 + f + 1 = e^w(g^7 + g + 1).$$

Set $g_1 = f^7$, $g_2 = f+1$, $g_3 = -e^w g^7$ and $g_4 = e^w(g+1) = g_1+g_2+g_3$. If g_1, g_2 and g_3 are linearly independent, then we use Theorem 2.1 and Lemma 2.2 to obtain a contradiction. If g_1, g_2 and g_3 are linearly dependent, then we use other arguments to show that $f^7(g+1) \equiv g^7(f+1)$, from which we obtain $f \equiv g$.

Remark on the proof of Theorem 6.2 *given in* [**11**]. Let f and g be nonconstant entire functions that share the set S_2 CM. Then, for some entire function w, we have

$$f^7 + f^6 + 1 = e^w(g^7 + g^6 + 1).$$

Set $g_1 = f^7 + f^6$, $g_2 \equiv 1$, $g_3 = -e^w(g^7 + g^6)$ and $g_4 = e^w = g_1 + g_2 + g_3$. If g_1, g_2 and g_3 are linearly independent, then we use Theorem 2.1 and Lemma 2.2 to obtain a contradiction. If g_1, g_2 and g_3 are linearly dependent, then we use other arguments to show that $f^7 + f^6 \equiv g^7 + g^6$, from which we obtain $f \equiv g$.

Observation. In these proofs of Theorems 6.1 and 6.2, we needed the inequality

$$T(r) \leqslant \sum_{j=1}^{p+1} N_{p-1}(r, 0, g_j) + S(r)$$

from (2.2), in order to deal with: (i) $g_2 = f + 1$ and $g_4 = e^w(g+1)$ in the proof of Theorem 6.1, and (ii) $g_1 = f^7 + f^6$ and $g_3 = -e^w(g^7 + g^6)$ in the proof of Theorem 6.2. On the other hand, in the above proofs of Theorems 4.1 and 5.1, we needed only the weaker inequality

$$T(r) \leqslant (p-1) \sum_{j=1}^{p+1} \bar{N}(r, 0, g_j) + S(r)$$

from (2.2).

7. *The general form of Cartan's theorem*

Theorem 2.1 is a special case of the following result, which is the fundamental theorem that Cartan stated in his paper [**2**], together with additional observations to be found in [**17**] and [**25**].

THEOREM 7.1 [**2**, **17**, **25**]. *Let $g_1, g_2, \ldots, g_p$ be linearly independent entire functions, where $p \geqslant 2$. Suppose that for each complex number z, we have $\max\{|g_1(z)|, |g_2(z)|, \ldots, |g_p(z)|\} > 0$. For positive r, set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta - u(0), \quad \text{where } u(z) = \sup_{1\leqslant j\leqslant p} \log|g_j(z)|. \tag{7.1}$$

Let $f_1, f_2, \ldots, f_q$ be q linear combinations of the p functions $g_1, g_2, \ldots, g_p$, where $q > p$, such that any p of the q functions $f_1, f_2, \ldots, f_q$ are linearly independent. Let H be the meromorphic function defined by

$$H = \frac{f_1 f_2 \ldots f_q}{W(g_1, g_2, \ldots, g_p)}, \tag{7.2}$$

where $W(g_1, g_2, \ldots, g_p)$ is the Wronskian of $g_1, g_2, \ldots, g_p$. Then

$$(q-p)T(r) \leqslant N(r, 0, H) - N(r, H) + S(r), \qquad r > 0, \tag{7.3}$$

where $S(r)$ is a quantity satisfying

$$S(r) = O(\log T(r)) + O(\log r) \quad \text{as } r \to \infty \text{ n.e.} \tag{7.4}$$

We have

$$N(r, 0, H) \leqslant \sum_{j=1}^{q} N_{p-1}(r, 0, f_j), \tag{7.5}$$

and this gives

$$(q-p)T(r) \leqslant \sum_{j=1}^{q} N_{p-1}(r, 0, f_j) - N(r, H) + S(r). \tag{7.6}$$

If at least one of the quotients g_j/g_m is a transcendental function, then

$$S(r) = o(T(r)) \quad \text{as } r \to \infty \text{ n.e.}, \tag{7.7}$$

whereas if all the quotients g_j/g_m are rational functions, then

$$S(r) \leqslant -\frac{1}{2}p(p-1)\log r + O(1) \quad \text{as } r \to \infty. \tag{7.8}$$

Furthermore, if all the quotients g_j/g_m are rational functions, then there exist polynomials $h_1, h_2, \ldots, h_p$, and an entire function ϕ, such that

$$g_j = h_j e^{\phi}, \qquad j = 1, 2, \ldots, p. \tag{7.9}$$

It can be verified that the assumption in Theorem 7.1 that 'any p of the q functions $f_1, f_2, \ldots, f_q$ are linearly independent' is equivalent to the determinant condition that Cartan assumed in [**2**, p. 11, lemma]. By inspecting the particular form of the quantity $S(r)$ in Cartan's proof, it was observed in [**17**] that (7.8) holds. As Kim [**23**] has also noted, this original proof in [**17**] of (7.8) was incorrect. In [**25**, Chapter VII, Section 6] it was observed that Cartan's proof can be easily adjusted so that the term $-N(r,H)$ appears in (7.3). This additional term $-N(r,H)$ is needed in the proof that Theorem 7.1 is a generalization of the full statement of the second fundamental theorem; see Corollary 7.4 below.

Thus every application of the second fundamental theorem is also an application of Theorem 7.1. In Sections 3–6 we discussed some applications of Theorem 2.1, which is the simple form of Theorem 7.1. For other applications of Theorem 7.1, see [**4**], [**12**], [**25**], [**33**], [**34**], [**40**] and [**41**].

Theorem 7.1 has several similar features to the second fundamental theorem. For example, in Theorem 7.1 the function $T(r)$ is a measure of the growth of the entire functions $g_1, g_2, \dots, g_p$, while in the second fundamental theorem, the *Nevanlinna characteristic function* $T(r,f)$ is a measure of the growth of a meromorphic function f. Thus we call $T(r)$ the *Cartan characteristic function*. We also note that the error term $S(r)$ in Theorem 7.1 satisfies (7.4), while the error term $S(r,f)$ in the second fundamental theorem satisfies $S(r,f) = O(\log T(r,f)) + O(\log r)$ as $r \to \infty$ *n.e.* Last, we note that the estimate (7.6) in Theorem 7.1 has a similar nature to estimates in the second fundamental theorem. Regarding Theorem 7.1 and the second fundamental theorem, see also [**3**].

The next result (and Corollary 7.3 below) give a relationship between the Cartan characteristic function and the Nevanlinna characteristic function.

THEOREM 7.2 [**2**]. *Let h_1 and h_2 be two linearly independent entire functions that have no common zeros, and set $f = h_1/h_2$. For positive r, set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta - v(0), \quad \text{where } v(z) = \sup\{\log|h_1(z)|, \log|h_2(z)|\}.$$

Then

$$T(r) = T(r,f) + O(1) \quad \text{as } r \to \infty. \tag{7.10}$$

Proof. Let E denote the set of positive r with the property that f has at least one zero or pole on $|z| = r$. If $r \notin E$, then $v(z) = \log^+|f(z)| + \log|h_2(z)|$ for all z satisfying $|z| = r$. Thus, from integration,

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|h_2(re^{i\theta})|\,d\theta - v(0), \qquad r \notin E.$$

From Jensen's formula,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|h_2(re^{i\theta})|\,d\theta + C_1 = N(r,0,h_2) = N(r,f),$$

where C_1 is a real constant. Thus

$$T(r) = m(r,f) + N(r,f) - v(0) - C_1 = T(r,f) + O(1) \quad \text{as } r \to \infty,\ r \notin E.$$

Since $T(r)$ and $T(r,f)$ are continuous functions of r, it follows that (7.10) holds as $r \to \infty$ through all values of r. This proves the theorem.

Theorem 7.2 immediately implies the following result.

COROLLARY 7.3 [**2**]. *In the case when $p = 2$ in Theorem 7.1, we have*

$$T(r) = T(r, g_1/g_2) + O(1) \quad \text{as } r \to \infty.$$

We now show that the second fundamental theorem is a particular example of Theorem 7.1. The following statement is a well-known general form of the second fundamental theorem; see [**14**, **37**, **47**].

Corollary 7.4 (Second fundamental theorem). *Let f be a nonconstant meromorphic function, and let $a_1, a_2, \dots, a_q$ be distinct constants, where $q \geqslant 3$. Then*

$$(q-2)T(r,f) \leqslant \sum_{j=1}^{q} \bar{N}(r,a_j,f) - N_0(r,0,f') + S(r,f), \tag{7.11}$$

where $N_0(r,0,f')$ refers only to those roots of $f'(z)=0$ that satisfy $f(z) \neq a_j$ for $j=1,2,\dots,q$.

Regarding the proof that Corollary 7.4 is a corollary of Theorem 7.1, we note that the proof in [**2**] gives (7.11) without the term $-N_0(r,0,f')$. We can obtain this additional term $-N_0(r,0,f')$ by using the term $-N(r,H)$ in (7.3), and we now give this complete proof.

Proof of Corollary 7.4 (*using Theorem* 7.1). We know that there exist two linearly independent entire functions g_1 and g_2 that have no common zeros, such that $f = g_1/g_2$. Set $f_j = g_1 - a_j g_2$ for $j = 1,2,\dots,q$. Then by applying Theorem 7.1 with g_1, g_2 ($p=2$) and $f_1, f_2, \dots, f_q$ ($q \geqslant 3$), we obtain

$$(q-2)T(r) \leqslant \sum_{j=1}^{q} \bar{N}(r,0,f_j) - N(r,H) + S(r), \tag{7.12}$$

where $T(r)$ and $S(r)$ satisfy the conditions in Theorem 7.1, and

$$H = \frac{f_1 f_2 \dots f_q}{g_1 g_2' - g_1' g_2}.$$

Since

$$f' = -\frac{g_1 g_2' - g_1' g_2}{g_2^2},$$

we see that $N_0(r,0,f') \leqslant N(r,H)$. Then from Corollary 7.3 and (7.12), we obtain (7.11). This proves Corollary 7.4. Thus Theorem 7.1 is a generalization of the second fundamental theorem.

Remark. In the above proof it was shown that $N_0(r,0,f') \leqslant N(r,H)$, which is all that was needed. More precisely, it can be shown that

$$N(r,H) = N_0(r,0,f') + N(r,f) - \bar{N}(r,f).$$

8. *Lemmas*

In this section we prove two lemmas that we use in the proof of Theorem 7.1 in Section 9. In Theorem 7.1, Cartan observed that $T(r)$ is an upper bound on the growth of the quotients g_j/g_m and f_μ/f_ν, and these observations are included in the next lemma. For two meromorphic functions ψ_1 and ψ_2 (where $\psi_1 \not\equiv 0$ and $\psi_2 \not\equiv 0$), let $N(r,0;\psi_1,\psi_2)$ denote the counting function of the common zeros of ψ_1 and ψ_2, counted in the following manner. If z_0 is a zero of ψ_1 of order k_1 and a zero of ψ_2 of order k_2, then $N(r,0;\psi_1,\psi_2)$ counts z_0 exactly k times, where $k = \min\{k_1,k_2\}$.

Lemma 8.1 [**2**]. *Assume that the hypotheses of Theorem* 7.1 *hold. Then*

$$T(r) \to \infty \quad \text{as } r \to \infty. \tag{8.1}$$

If at least one quotient g_j/g_m is a transcendental function, then

$$\frac{T(r)}{\log r} \to \infty \quad \text{as } r \to \infty. \tag{8.2}$$

Furthermore, for any j and m, we have

$$T(r, g_j/g_m) + N(r, 0; g_j, g_m) \leqslant T(r) + O(1) \quad \text{as } r \to \infty, \tag{8.3}$$

and, for any μ and ν, we have

$$T(r, f_\mu/f_\nu) + N(r, 0; f_\mu, f_\nu) \leqslant T(r) + O(1) \quad \text{as } r \to \infty. \tag{8.4}$$

Also, for any j, we have

$$N(r, 0, g_j) \leqslant T(r) + O(1) \quad \text{as } r \to \infty. \tag{8.5}$$

Proof. Let g_j and g_m be any two functions in Theorem 7.1. Then there exist entire functions h_j, h_m and w_{jm}, where h_j and h_m are linearly independent and have no common zeros, such that

$$g_j = h_j w_{jm} \quad \text{and} \quad g_m = h_m w_{jm}, \tag{8.6}$$

where $N(r, 0, w_{jm}) = N(r, 0; g_j, g_m)$. Set

$$v(z) = \sup\{\log |h_j(z)|, \log |h_m(z)|\}. \tag{8.7}$$

By applying Theorem 7.2 to h_j and h_m, we obtain

$$T(r, g_j/g_m) = T(r, h_j/h_m) = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta + O(1) \quad \text{as } r \to \infty. \tag{8.8}$$

We note that $\sup\{\log |g_j(z)|, \log |g_m(z)|\} \leqslant u(z)$ whenever $z \in \mathbb{C}$, where $u(z)$ is given in (7.1). Hence, from (8.8), (8.7), (8.6) and (7.1), we obtain that

$$T(r, g_j/g_m) \leqslant T(r) - \frac{1}{2\pi} \int_0^{2\pi} \log |w_{jm}(re^{i\theta})|\, d\theta + O(1) \quad \text{as } r \to \infty.$$

Then from Jensen's formula,

$$T(r, g_j/g_m) \leqslant T(r) - N(r, 0, w_{jm}) + O(1) \quad \text{as } r \to \infty.$$

Since $N(r, 0, w_{jm}) = N(r, 0; g_j, g_m)$, we obtain (8.3).

Now suppose that f_μ and f_ν are any two of the functions $f_1, f_2, \ldots, f_q$. Since f_μ and f_ν are linear combinations of the functions $g_1, g_2, \ldots, g_p$, there exists a positive constant C_0 such that $\sup\{\log |f_\mu|, \log |f_\nu|\} \leqslant u(z) + C_0$ whenever $z \in \mathbb{C}$, where $u(z)$ is given in (7.1). Therefore, by repeating the above process with g_j and g_m replaced by f_μ and f_ν, it can be deduced that (8.4) holds.

If $j \neq m$, then g_j/g_m is not a constant, and $T(r, g_j/g_m) \to \infty$ as $r \to \infty$. Thus (8.1) follows from (8.3). If at least one quotient g_j/g_m is a transcendental function, then (see [**14**, p. 24])

$$\frac{T(r, g_j/g_m)}{\log r} \to \infty \quad \text{as } r \to \infty.$$

Thus (8.2) also follows from (8.3).

Finally, by Jensen's formula and (7.1), there exist real constants C_1 and C_2 such that

$$N(r,0,g_j) = \frac{1}{2\pi}\int_0^{2\pi} \log|g_j(re^{i\theta})|\,d\theta + C_1 \leqslant T(r) + C_2, \qquad j = 1,2,\ldots,p.$$

This proves (8.5), and completes the proof of the lemma.

LEMMA 8.2 [**2**]. *Assume that the hypotheses of Theorem* 7.1 *hold. Suppose that* $z \in \mathbb{C}$, *and arrange the moduli of the function values* $|f_j(z)|$ *in a weakly decreasing order; that is,*

$$|f_{m_1}(z)| \geqslant |f_{m_2}(z)| \geqslant \ldots \geqslant |f_{m_q}(z)|, \tag{8.9}$$

where the integers $m_1, m_2, \ldots, m_q$ *depend on* z.

Then there exists a positive constant A *that does not depend on* z, *such that*

$$|g_j(z)| \leqslant A|f_{m_\nu}(z)| \quad \text{whenever } 1 \leqslant j \leqslant p \text{ and } 1 \leqslant \nu \leqslant q-p+1. \tag{8.10}$$

Thus, for any z, *there exist at least* $q-p+1$ *functions* f_j *that do not vanish at* z.

Proof. Since each f_k is a linear combination of the functions $g_1, g_2, \ldots, g_p$,

$$f_k = \sum_{j=1}^{p} a_{jk} g_j, \qquad k = 1,2,\ldots,q, \tag{8.11}$$

for some constants $\{a_{jk}\}$. For each z, let $m_1, m_2, \ldots, m_q$ be the integers in (8.9), which depend on z. Now let ν be any fixed integer satisfying $1 \leqslant \nu \leqslant q-p+1$, and then let $h_1, h_2, \ldots, h_p$ denote the p functions $f_{m_\nu}, f_{m_{q-p+2}}, f_{m_{q-p+3}}, \ldots, f_{m_q}$. Hence

$$|h_n(z)| \leqslant |f_{m_\nu}(z)|, \qquad n = 1,2,\ldots,p. \tag{8.12}$$

From (8.11),

$$h_n(z) = \sum_{j=1}^{p} b_{jn} g_j(z), \qquad n = 1,2,\ldots,p, \tag{8.13}$$

for some constants $\{b_{jn}\}$ that make up a subset of the constants $\{a_{jk}\}$ in (8.11). From the hypotheses of Theorem 7.1, the p functions $h_1, h_2, \ldots, h_p$ are linearly independent. Hence, if D_p denotes the $p \times p$ coefficient determinant of the constants $\{b_{jn}\}$ in the $p \times p$ system of equations in (8.13), then we deduce that $D_p \neq 0$. It follows that

$$g_j(z) = \sum_{n=1}^{p} c_{nj} h_n(z), \qquad j = 1,2,\ldots,p, \tag{8.14}$$

where the constants $\{c_{nj}\}$ depend only on the constants $\{b_{jn}\}$ in (8.13). Thus the constants $\{c_{nj}\}$ depend only on the constants $\{a_{jk}\}$ in (8.11). Therefore, from (8.14) and (8.12), it follows that there exists a positive constant A that depends only on the constants $\{a_{jk}\}$ in (8.11), such that $|g_j(z)| \leqslant A|f_{m_\nu}(z)|$, $j = 1,2,\ldots,p$. Since ν is any fixed integer satisfying $1 \leqslant \nu \leqslant q-p+1$, this proves (8.10), which proves the lemma.

9. *Proof of Theorem* 7.1

Since $g_1, g_2, \ldots, g_p$ are linearly independent, we have $W(g_1, g_2, \ldots, g_p) \not\equiv 0$. Thus H in (7.2) is a well-defined meromorphic function.

For each complex number z, set

$$v(z) = \max_{\{k_j\}} \log |f_{k_1}(z) f_{k_2}(z) \ldots f_{k_{q-p}}(z)|, \tag{9.1}$$

where this maximum is taken over all possible sets of $q - p$ distinct integers $k_1, k_2, \ldots, k_{q-p}$ in the set $\{1, 2, \ldots, q\}$. By Lemma 8.2, if $z \in \mathbb{C}$, then there exist at least $q - p + 1$ functions f_k that do not vanish at z. Thus $v(z)$ in (9.1) is a finite real number whenever $z \in \mathbb{C}$.

Now suppose that $a_1, a_2, \ldots, a_{q-p}$ are any $q - p$ distinct integers in the set $\{1, 2, \ldots, q\}$. Let $b_1, b_2, \ldots, b_p$ denote the remaining integers in the set $\{1, 2, \ldots, q\}$. From the hypotheses, we see that $f_{b_1}, f_{b_2}, \ldots, f_{b_p}$ are linearly independent. Thus $W(f_{b_1}, f_{b_2}, \ldots, f_{b_p}) \not\equiv 0$. Since each f_k is a linear combination of the functions $g_1, g_2, \ldots, g_p$, we have a matrix equation of the form

$$\begin{pmatrix} f_{b_1} & f_{b_2} & \cdots & f_{b_p} \\ f'_{b_1} & f'_{b_2} & \cdots & f'_{b_p} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(p-1)}_{b_1} & f^{(p-1)}_{b_2} & \cdots & f^{(p-1)}_{b_p} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_p \\ g'_1 & g'_2 & \cdots & g'_p \\ \vdots & \vdots & \ddots & \vdots \\ g^{(p-1)}_1 & g^{(p-1)}_2 & \cdots & g^{(p-1)}_p \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pp} \end{pmatrix},$$

for some constants $\{\alpha_{jm}\}$. By taking determinants, we see that

$$W(g_1, g_2, \ldots, g_p) = K(b_1, b_2, \ldots, b_p) W(f_{b_1}, f_{b_2}, \ldots, f_{b_p}),$$

where $K = K(b_1, b_2, \ldots, b_p)$ is a nonzero constant. Hence, by (7.2),

$$H = \frac{f_1 f_2 \ldots f_q}{K(b_1, b_2, \ldots, b_p) W(f_{b_1}, f_{b_2}, \ldots, f_{b_p})}. \tag{9.2}$$

Thus

$$H = \frac{f_{a_1} f_{a_2} \cdots f_{a_{q-p}}}{K(b_1, b_2, \ldots, b_p) G}, \tag{9.3}$$

where

$$G = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f'_{b_1}/f_{b_1} & f'_{b_2}/f_{b_2} & \cdots & f'_{b_p}/f_{b_p} \\ f''_{b_1}/f_{b_1} & f''_{b_2}/f_{b_2} & \cdots & f''_{b_p}/f_{b_p} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(p-1)}_{b_1}/f_{b_1} & f^{(p-1)}_{b_2}/f_{b_2} & \cdots & f^{(p-1)}_{b_p}/f_{b_p} \end{vmatrix}. \tag{9.4}$$

For each complex number z, set

$$w(z) = \max_{\{b_j\}} \log |K(b_1, b_2, \ldots, b_p) G(z)|, \tag{9.5}$$

where this maximum is taken over all possible sets of p distinct integers $b_1, b_2, \dots, b_p$ in the set $\{1, 2, \dots, q\}$. Here we observe that for some z, it is possible to have either $w(z) = +\infty$ or $w(z) = -\infty$.

We now show that

$$\int_0^{2\pi} v(re^{i\theta})\, d\theta = \int_0^{2\pi} \log|H(re^{i\theta})|\, d\theta + \int_0^{2\pi} w(re^{i\theta})\, d\theta, \qquad r > 0. \tag{9.6}$$

From (9.1), (9.3) and (9.5), we have $v(z) = \log|H(z)| + w(z)$ for any z satisfying $H(z) \neq 0, \infty$. Thus (9.6) holds for those positive r for which H has no zeros or poles on $|z| = r$. Now suppose that H has a finite number of zeros and poles on $|z| = r$ (where $r > 0$). For these r, we integrate the three integrands in (9.6) around a curve $\gamma = \gamma(r, \delta)$ consisting of arcs of $|z| = r$ and small indentations of radius δ about each zero and pole of H on $|z| = r$. In this case, (9.6) holds when the path of integration is γ instead of $|z| = r$. As $\delta \to 0$, on each small indentation the two integrands on the right-hand side of (9.6) are $O(-\log\delta)$, and the length of the indentation is $O(\delta)$, and so the corresponding integrals around each indentation tend to zero. Since the whole curve γ approaches the circle $|z| = r$ as $\delta \to 0$, we see that (9.6) holds on $|z| = r$. Hence, (9.6) holds for all positive r.

We now consider, separately, each of the three integrals in (9.6). Since H is a meromorphic function, Jensen's formula gives

$$\frac{1}{2\pi}\int_0^{2\pi} \log|H(re^{i\theta})|\, d\theta = N(r, 0, H) - N(r, H) + B, \qquad r > 0, \tag{9.7}$$

where B is a real constant.

We now make a lower estimate for the integral of v in (9.6). To this end, for each z, we choose the integers $a_1, a_2, \dots, a_{q-p}$ in (9.3) to be a particular set of integers $c_1, c_2, \dots, c_{q-p}$ satisfying

$$f_{c_1}(z) f_{c_2}(z) \dots f_{c_{q-p}}(z) \neq 0 \tag{9.8}$$

and

$$v(z) = \log|f_{c_1}(z) f_{c_2}(z) \dots f_{c_{q-p}}(z)| \tag{9.9}$$

in (9.1). The choice of these integers $c_1, c_2, \dots, c_{q-p}$ depends on z. From (9.8), (9.9) and Lemma 8.2, it follows that there exists a positive constant A, such that for all z we have $|g_j(z)| \leqslant A|f_{c_\nu}(z)|$ and $\log|g_j(z)| \leqslant \log A + \log|f_{c_\nu}(z)|$ whenever $1 \leqslant j \leqslant p$ and $1 \leqslant \nu \leqslant q - p$. Then from (7.1), $u(z) \leqslant \log A + \log|f_{c_\nu}(z)|$ whenever $1 \leqslant \nu \leqslant q - p$. Thus from (9.9) and (7.1), we obtain

$$(q - p)T(r) \leqslant \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\, d\theta + O(1) \quad \text{as } r \to \infty. \tag{9.10}$$

We next make an upper estimate for the integral of w in (9.6). From (9.5), there exists a positive constant D, such that for all z,

$$w(z) \leqslant D + \max_{\{b_j\}} \log|G(z)|, \tag{9.11}$$

where this maximum is taken over all possible sets of p distinct integers $b_1, b_2, \dots, b_p$ in the set $\{1, 2, \dots, q\}$. Since the function G in (9.4) has the form

$$G = \frac{W(f_{b_1}, f_{b_2}, \dots, f_{b_p})}{f_{b_1} f_{b_2} \dots f_{b_p}},$$

and since (see [**24**, p. 12])

$$(1/f_1)^p W(f_{b_1}, f_{b_2}, \ldots, f_{b_p}) = W(f_{b_1}/f_1, f_{b_2}/f_1, \ldots, f_{b_p}/f_1),$$

it can be seen that the function G does not change if we replace each function $f_{b_j}^{(k)}/f_{b_j}$ in (9.4) with $(f_{b_j}/f_1)^{(k)}/(f_{b_j}/f_1)$. From this observation, (9.11), Milloux's result [**14**, p. 55] and Nevanlinna's fundamental estimate of the logarithmic derivative, it can be deduced that

$$\int_0^{2\pi} w(re^{i\theta})\, d\theta \leqslant O(\log r) + \sum_{j=1}^{q} O\left(\log T\left(r, \frac{f_j}{f_1}\right)\right) \quad \text{as } r \to \infty \text{ n.e.}$$

Thus from (8.4),

$$\int_0^{2\pi} w(re^{i\theta})\, d\theta = O(\log r) + O(\log T(r)) \quad \text{as } r \to \infty \text{ n.e.} \tag{9.12}$$

We now combine (9.6), (9.7), (9.10) and (9.12), to show that (7.3) holds, where $S(r)$ is a quantity satisfying (7.4). In particular, $S(r)$ is a quantity satisfying

$$S(r) = \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\theta})\, d\theta + O(1) \quad \text{as } r \to \infty. \tag{9.13}$$

We now show that (7.5) holds. To this end, let z_0 be a zero of H of multiplicity μ. Then from (7.2), at least one of the functions $f_1, f_2, \ldots, f_q$ has a zero at z_0. As in (9.8) and (9.9), we choose a particular set of integers $a_1, a_2, \ldots, a_{q-p}$ satisfying

$$f_{a_1}(z_0) f_{a_2}(z_0) \ldots f_{a_{q-p}}(z_0) \neq 0,$$

and we use these integers in (9.3). Then from (9.3) and (9.4), z_0 is a pole of G of multiplicity μ. For each $j = 1, 2, \ldots, p$, z_0 is a zero of f_{b_j} of multiplicity m_j, where $m_j \geqslant 0$. Here, for convenience, we set $m_j = 0$ whenever $f_{b_j}(z_0) \neq 0$. From inspection of the form of G in (9.4), we deduce that

$$\mu \leqslant \sum_{j=1}^{p} \min\{m_j, p-1\}.$$

It follows that (7.5) holds. Now (7.6) follows from (7.3) and (7.5).

To complete the proof, it remains to prove (7.7), (7.8) and (7.9). If at least one of the quotients g_j/g_m is a transcendental function, then from (8.2) and (7.4), we obtain (7.7).

Now suppose that all the quotients g_j/g_m are rational functions. Since for each complex number z we have $\max\{|g_1(z)|, |g_2(z)|, \ldots, |g_p(z)|\} > 0$, it follows that each g_j has only a finite number of zeros, and we see that (7.9) holds. Since each f_j is a linear combination of the functions $g_1, g_2, \ldots, g_p$, it follows from (7.9) that all the quotients f_j/f_m are rational functions. As we observed above, the function G in (9.4) does not change if we replace each function $f_{b_j}^{(k)}/f_{b_j}$ in (9.4) with $(f_{b_j}/f_1)^{(k)}/(f_{b_j}/f_1)$. Since f_{b_j}/f_1 is a rational function for all j, it follows that G in (9.4) is a rational function; furthermore, G is equal to a finite sum of rational functions $R_1, R_2, \ldots, R_n$ satisfying

$$|R_\nu(z)| = O(|z|^{-p(p-1)/2}) \quad \text{as } z \to \infty,\ \nu = 1, 2, \ldots, n.$$

Therefore, from (9.5) and (9.13), it can be deduced that (7.8) holds. This completes the proof of Theorem 7.1.

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