

# THE ANALOGUE OF HILBERT'S 1888 THEOREM FOR EVEN SYMMETRIC FORMS

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**ABSTRACT.** Hilbert proved in 1888 that a positive semidefinite (psd) real form is a sum of squares (sos) of real forms if and only if  $n = 2$  or  $d = 1$  or  $(n, 2d) = (3, 4)$ , where  $n$  is the number of variables and  $2d$  the degree of the form. We study the analogue for even symmetric forms. We establish that an even symmetric  $n$ -ary  $2d$ -ic psd form is sos if and only if  $n = 2$  or  $d = 1$  or  $(n, 2d) = (n, 4)_{n \geq 3}$  or  $(n, 2d) = (3, 8)$ .

## 1. INTRODUCTION

A real form (homogeneous polynomial)  $f$  is called *positive semidefinite* (psd) if it takes only non-negative values and it is called a *sum of squares* (sos) if there exist other forms  $h_j$  so that  $f = h_1^2 + \cdots + h_k^2$ . Let  $\mathcal{P}_{n,2d}$  and  $\Sigma_{n,2d}$  denote the cone of psd and sos  $n$ -ary  $2d$ -ic forms (i.e. forms of degree  $2d$  in  $n$  variables) respectively.

In 1888, Hilbert [9] gave a celebrated theorem that characterizes the pairs  $(n, 2d)$  for which every  $n$ -ary  $2d$ -ic psd form can be written as a sos of forms. It states that every  $n$ -ary  $2d$ -ic psd form is sos if and only if  $n = 2$  or  $d = 1$  or  $(n, 2d) = (3, 4)$ . Hilbert demonstrated that  $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$  for  $(n, 2d) = (4, 4), (3, 6)$ , thus reducing the problem to these two basic cases.

Almost ninety years later, Choi and Lam [1] returned to this subject. In particular, they considered the question of when a symmetric psd form is sos. A form  $f(x_1, \dots, x_n)$  is called *symmetric* if  $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$  for all  $\sigma \in S_n$ . As an analogue of Hilbert's approach, they reduced the problem to finding symmetric psd not sos  $n$ -ary  $2d$ -ics for the pairs  $(n, 4)_{n \geq 4}$  and  $(3, 6)$ . They asserted the existence of psd not sos symmetric quartics in  $n \geq 5$  variables; contingent on these examples, the answer is the same as that found by Hilbert. In [6], we constructed these quartic forms.

A form is *even symmetric* if it is symmetric and in each of its terms every variable has even degree. Let  $S\mathcal{P}_{n,2d}^e$  and  $S\Sigma_{n,2d}^e$  denote the set of even symmetric psd and even symmetric sos  $n$ -ary  $2d$ -ic forms respectively. Set  $\Delta_{n,2d} := S\mathcal{P}_{n,2d}^e \setminus S\Sigma_{n,2d}^e$ . In this paper, we investigate the following question:

$Q(S^e)$  : For what pairs  $(n, 2d)$  is  $\Delta_{n,2d} = \emptyset$  ?

The current answers to this question in the literature are  $\Delta_{n,2d} = \emptyset$  if  $n = 2, d = 1, (n, 2d) = (3, 4)$  by Hilbert's Theorem,  $(n, 2d) = (3, 8)$  due to Harris [7], and  $(n, 2d) = (n, 4)_{n \geq 4}$ . The result  $\Delta_{n,4} = \emptyset$  for  $n \geq 4$  was attributed to Choi, Lam and Reznick in [7]; a proof can be found in [5, Proposition 4.1]. Further,  $\Delta_{n,2d} \neq \emptyset$  for  $(n, 2d) = (n, 6)_{n \geq 3}$  due to Choi, Lam and Reznick [3], for  $(n, 2d) = (3, 10), (4, 8)$

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due to Harris [8] and for  $(n, 2d) = (3, 6)$  due to Robinson [10]. Robinson's even symmetric psd not sos ternary sextic is the form

$$R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2.$$

Thus the answer to  $\mathcal{Q}(S^e)$  in the literature can be summarized by the following chart:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	o	o
10	✓	×	o	o	o
12	✓	o	o	o	o
14	✓	o	o	o	o
⋮	⋮	o	o	o	o

where, a tick (✓) denotes a positive answer to  $\mathcal{Q}(S^e)$ , a cross (×) denotes a negative answer to  $\mathcal{Q}(S^e)$ , and a circle (o) denotes “undetermined”. Indeed to get a complete answer to  $\mathcal{Q}(S^e)$ , we need to investigate the question in these remaining cases, namely  $(n, 8)$  for  $n \geq 5$ ,  $(3, 2d)$  for  $d \geq 6$  and  $(n, 2d)$  for  $n \geq 4, d \geq 5$ .

**Main Theorem.** An even symmetric  $n$ -ary  $2d$ -ic psd form is sos if and only if  $n = 2$  or  $d = 1$  or  $(n, 2d) = (n, 4)_{n \geq 3}$  or  $(n, 2d) = (3, 8)$ .

In other words, every “o” in the chart can be replaced by “×”.

The article is structured as follows. In Section 2, we develop the tools (Theorem 2.3 and Theorem 2.4) we need to prove our Main Theorem. These tools allow us to reduce to certain basic cases, in the same spirit as Hilbert and Choi-Lam. In Section 3 and Section 4 we resolve those basic cases by producing explicit examples for  $(n, 2d); n \geq 4, d = 4, 5, 6$ . For ease of reference we summarize these examples in Section 5. Finally in Section 6, we interpret our Main Theorem in terms of preorderings.

## 2. REDUCTION TO BASIC CASES

The following Lemma will be used in Theorem 2.3.

**Lemma 2.1.** For  $n \geq 3$ , the even symmetric real forms

$$p_n(x_1, \dots, x_n) = 4 \sum_{j=1}^n x_j^4 - 17 \sum_{1 \leq i < j \leq n} x_i^2 x_j^2;$$

$$q_n(x_1, \dots, x_n) = \sum_{j=1}^n x_j^6 + 3 \sum_{1 \leq i \neq j \leq n} x_i^4 x_j^2 - 100 \sum_{1 \leq i < j < k \leq n} x_i^2 x_j^2 x_k^2$$

are irreducible over  $\mathbb{R}$ .

*Proof.* First observe that if a form  $g$  has a factorization

$$g(x_1, \dots, x_n) = \prod_{r=1}^u f_r(x_1, \dots, x_n),$$

then the same holds when  $x_{k+1} = \dots = x_n = 0$ , hence it suffices to show that  $p_3$  and  $q_3$  are irreducible over  $\mathbb{R}$ . Second, observe that if (in addition)  $g$  is even and symmetric, then for all  $\sigma \in S_n$  and choices of sign,  $f_r(\pm_1 x_{\sigma_1}, \dots, \pm_n x_{\sigma_n})$  is also a factor of  $g$ . We call distinct (non-proportional) forms of this kind *cousins* of  $f_r$ . If (in addition)  $f_r$  is irreducible,  $\deg f_r = d$  and  $\deg g = n$ , then  $f_r$  can have at most  $n/d$  cousins.

If  $p_3$  is reducible, then it has a factor of degree  $\leq 2$ . Suppose that  $p_3$  has a linear factor  $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ . Upon setting  $x_3 = 0$ , we see that

$$\alpha_1 x_1 + \alpha_2 x_2 \mid 4x_1^4 - 17x_1^2 x_2^2 + 4x_2^4 = (x_1 + 2x_2)(x_1 - 2x_2)(2x_1 + x_2)(2x_1 - x_2),$$

so  $\alpha_2/\alpha_1 \in \{\pm 1/2, \pm 2\}$ . Similarly,  $\alpha_3/\alpha_2 \in \{\pm 1/2, \pm 2\}$ , so  $\alpha_3/\alpha_1 \in \{\pm 1/4, \pm 1, \pm 4\}$ , which contradicts  $\alpha_3/\alpha_1 \in \{\pm 1/2, \pm 2\}$ . It follows that  $p_3$  has no linear factors.

Suppose  $p_3$  has a quadratic (irreducible) factor  $f = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \dots$ . If it is not true that  $\alpha_1 = \alpha_2 = \alpha_3$ , then by permuting variables,  $f$  has at least  $3 > 4/2$  cousins. Thus  $\alpha_1 = \alpha_2 = \alpha_3$ , and by scaling we may assume the common value is 2. The binary quartic  $4x_1^4 - 17x_1^2 x_2^2 + 4x_2^4$  has six quadratic factors, found by taking pairs of linear factors as above. Of these, the ones in which  $\alpha_1 = \alpha_2$  are  $2x_1^2 \pm 5x_1 x_2 + 2x_2^2$ . It follows that, more generally, the coefficient of  $x_i x_j$  is  $\pm 5$  and that

$$f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 + \pm_{12}(5x_1 x_2) + \pm_{13}(5x_1 x_3) + \pm_{23}(5x_2 x_3).$$

Regardless of the initial choice of signs, making the single sign changes  $x_i \mapsto -x_i$  for  $i = 1, 2, 3$  shows that  $f$  has 4 cousins, which again is too many. Therefore, we may conclude that  $p_3$  is irreducible.

We turn to  $q_3$  and first observe that

$$q_3(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^3 - 106x_1^2 x_2^2 x_3^2.$$

Suppose now that  $q_3$  is reducible, so it has at least one factor of degree  $\leq 3$ , and let  $f$  be such a factor of  $q_3$ . Once again, we set  $x_3 = 0$  and observe that

$$f(x_1, x_2, 0) \mid q_3(x_1, x_2, 0) = (x_1^2 + x_2^2)^3.$$

Since  $x_1^2 + x_2^2$  is irreducible over  $\mathbb{R}$ , we conclude that  $\deg f = 2$  and  $f(x_1, x_2) = \lambda(x_1^2 + x_2^2)$ . Writing

$$f(x_1, x_2, x_3) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \sum_{1 \leq i < j \leq 3} \beta_{ij} x_i x_j,$$

we see from the foregoing that  $\alpha_1 = \alpha_2$  and  $\beta_{12} = 0$ . By setting  $x_2 = 0$  and  $x_1 = 0$  in turn, we see that the  $\alpha_i$ 's are equal and  $\beta_{ij} = 0$ , so  $f$  is a multiple of  $x_1^2 + x_2^2 + x_3^2$ . But since  $x_1^2 x_2^2 x_3^2$  is not a multiple of  $x_1^2 + x_2^2 + x_3^2$ ,  $f$  cannot divide  $q_3$ , completing the proof.  $\square$

**Lemma 2.2.** Let  $f$  be a psd not sos  $n$ -ary  $2d$ -ic form and  $p$  an irreducible indefinite form of degree  $r$  in  $\mathbb{R}[x_1, \dots, x_n]$ . Then the  $n$ -ary  $(2d + 2r)$ -ic form  $p^2 f$  is also psd not sos.

*Proof.* See [6, Lemma 2.1].  $\square$

**Theorem 2.3. (Degree Jumping Principle)** Suppose  $f \in \Delta_{n,2d}$  for  $n \geq 3$ , then

1. for any integer  $r \geq 2$ , the form  $f p_n^{2a} q_n^{2b} \in \Delta_{n,2d+4r}$ , where  $r = 2a + 3b$ ;  $a, b \in \mathbb{Z}_+$ , and  $p_n, q_n$  are as defined in Lemma 2.1;
2.  $(x_1 \dots x_n)^2 f \in \Delta_{n,2d+2n}$ .

- Proof.* 1. For  $r \in \mathbb{Z}_+$ ,  $r \geq 2$ , there exists non-negative  $a, b \in \mathbb{Z}$  such that  $r = 2a + 3b$ . Since  $f p_n^{2a} q_n^{2b}$  is a product of even symmetric forms, it is even and symmetric; since it is a product of psd forms, it is psd. Thus we have  $f p_n^{2a} q_n^{2b} \in S\mathcal{P}_{n, 2d+4r}^e$ . Since  $p_n$  and  $q_n$  are indefinite and irreducible forms by Lemma 2.1, we get  $f p_n^2 \in \Delta_{n, 2d+8}$  and  $f q_n^2 \in \Delta_{n, 2d+12}$  by Lemma 2.2. Finally, by repeating this argument we get  $f p_n^{2a} q_n^{2b} \in \Delta_{n, 2d+4r}$ .
2. Taking  $p = x_i$  in turn for each  $1 \leq i \leq n$ , the assertion follows by Lemma 2.2.  $\square$

**Theorem 2.4. (Reduction to Basic Cases)** If  $\Delta_{n, 2d} \neq \emptyset$  for  $(n, 8)_{n \geq 4}$ ,  $(n, 10)_{n \geq 3}$  and  $(n, 12)_{n \geq 3}$ , then  $\Delta_{n, 2d} \neq \emptyset$  for  $(n, 2d)_{n \geq 3, d \geq 7}$ .

*Proof.* For  $n = 3$ , the basic examples are  $R(x, y, z) \in \Delta_{3, 6}$  (by Robinson [10]), several examples in  $\Delta_{3, 10}$  (by Harris [7]) and  $q_3 R(x, y, z) \in \Delta_{3, 14}$  (by Theorem 2.3 (1)). Every even integer  $\geq 12$  can be written as  $6 + 6k$ ,  $10 + 6k$  or  $14 + 6k$ ,  $k \geq 0$ , and so by Theorem 2.3 (2),  $\Delta_{3, 2d}$  is non-empty for  $2d \geq 6$ ,  $2d \neq 8$ .

For  $n \geq 4$ ,  $\Delta_{n, 6}^e \neq \emptyset$  (by Choi, Lam, Reznick [3]). We shall show in Sections 3 and 4 that  $\Delta_{n, 8}^e, \Delta_{n, 10}^e, \Delta_{n, 12}^e$  are non-empty. Every even integer  $\geq 14$  can be written as  $6 + 8k$ ,  $8 + 8k$ ,  $10 + 8k$  or  $12 + 8k$  and so, given our claimed examples, by Theorem 2.3,  $\Delta_{n, 2d}^e$  is non-empty for  $n \geq 4$ ,  $2d \geq 6$ .  $\square$

In order to find psd not sos even symmetric  $n$ -ary octics, psd not sos even symmetric  $n$ -ary decics and psd not sos even symmetric  $n$ -ary dodecics for  $n \geq 4$ , we first recall the following theorems which will be particularly useful in proving the main results of Sections 3 and 4.

**Theorem 2.5.** Suppose  $p = \sum_{i=1}^r h_i^2$  is an even sos form. Then we may write  $p = \sum_{j=1}^s q_j^2$ , where each form  $q_j^2$  is even. In particular,  $q_j(\underline{x}) = \sum c_j(\alpha) \underline{x}^\alpha$ , where the sum is taken over  $\alpha$ 's in one congruence class mod 2 component-wise.

*Proof.* See [3, Theorem 4.1].  $\square$

**Theorem 2.6.** A symmetric  $n$ -ary quartic  $f$  is psd if and only if  $f(\underline{x}) \geq 0$  for every  $\underline{x} \in \mathbb{R}^n$  with at most two distinct coordinates (if  $n \geq 4$ ).

*Proof.* This was originally proved in [2]; see [5, Corollary 3.11], [7, Section 2].  $\square$

**Theorem 2.7.** (i) For odd  $2m + 1 \geq 5$ , the symmetric  $2m + 1$ -ary quartic

$$L_{2m+1}(\underline{x}) := m(m+1) \sum_{i < j} (x_i - x_j)^4 - \left( \sum_{i < j} (x_i - x_j)^2 \right)^2$$

is psd not sos.

(ii) For  $2m \geq 4$ , the symmetric  $2m$ -ary quartic

$$C_{2m}(x_1, \dots, x_{2m}) := L_{2m+1}(x_1, \dots, x_{2m}, 0)$$

is psd not sos.

*Proof.* See [6, Theorems 2.8, 2.9].  $\square$

**Theorem 2.8.** For an integer  $r \geq 1$ , let  $M_r = M_r(x_1, \dots, x_n) := x_1^r + \dots + x_n^r$ . For reals  $a, b, c$ , the sextic  $p = aM_2^3 + bM_2M_4 + cM_6$  is psd if and only if  $at^2 + bt + c \geq 0$  for  $t \in \{1, 2, \dots, n\}$  and sos if and only if  $at^2 + bt + c \geq 0$  for  $t \in \{1\} \cup [2, n]$ .

*Proof.* See [3, Theorems 3.7, 4.25].  $\square$

**Observation 2.9.** Let  $v_t$  denote any  $n$ -tuple with  $t$  components equal to 1 and  $n - t$  components equal to zero. Then  $M_r(v_t) = t$ , so  $p(v_t) = t(at^2 + bt + c)$ . It will be helpful to let  $v_t(a_1, \dots, a_t)$  denote the particular  $v_t$  with 1's in positions  $a_1, \dots, a_t$ .

### 3. PSD NOT SOS EVEN SYMMETRIC $n$ -ARY OCTICS FOR $n \geq 4$

It follows from Theorem 2.7 that for  $m \geq 2$ ,

$$\begin{aligned} G_{2m+1}(x_1, \dots, x_{2m+1}) &:= L_{2m+1}(x_1^2, \dots, x_{2m+1}^2) \in \mathcal{SP}_{2m+1,8}^e; \\ D_{2m}(x_1, \dots, x_{2m}) &:= G_{2m+1}(x_1, \dots, x_{2m}, 0) \in \mathcal{SP}_{2m,8}^e. \end{aligned}$$

We showed in [6] that  $G_{2m+1}(\underline{x}) = 0$  precisely at those  $\underline{x} \in \mathbb{R}^{2m+1}$  which are a permutation of  $m + 1$   $r$ 's and  $m$   $s$ 's for any choice of reals  $r, s$ , so that  $D_{2m}(\underline{x}) = 0$ , projectively, precisely at points of the shape  $v_m$  or  $v_{m+1}$ .

**Theorem 3.1.** For  $m \geq 2$ ,  $D_{2m} \in \Delta_{2m,8}$  and  $G_{2m+1} \in \Delta_{2m+1,8}$ .

*Proof.* We observe that  $D_{2m}(v_1) > 0$ ; in fact, it is equal to  $m(m+1)(2m) - (2m)^2 = 2m^2(m-1)$ . Thus, the coefficient of  $x_i^8$  in  $D_{2m}$  is positive. Suppose  $D_{2m} = \sum h_t^2$ . Then  $x_i^4$  must appear with non-zero coefficient in at least one  $h_t$ . Since we may assume that  $h_t^2$  is even (using Theorem 2.5), we must have

$$h_t = \sum_{i=1}^n a_i x_i^4 + \sum_{1 \leq i < j \leq n} b_{i,j} x_i^2 x_j^2.$$

Since  $D_{2m}(v_m) = D_{2m}(v_{m+1}) = 0$ , it follows that  $h_t(v_m) = h_t(v_{m+1}) = 0$ , and this holds for all permutations of  $v_m$  and  $v_{m+1}$ . Our goal is to show that these equations imply that  $h_t = 0$ , which will contradict the assumption that  $D_{2m}$  is sos. By symmetry, it suffices to prove that  $a_i = 0$  for one choice of  $i$ .

To this end, let  $y^{(1)} = v_m(1, \dots, m-1, 2m-1)$ ,  $y^{(2)} = v_m(1, \dots, m-1, 2m)$  and  $y^{(3)} = v_{m+1}(1, \dots, m-1, 2m-1, 2m)$ . Then

$$\begin{aligned} 0 = h_t(y^{(1)}) &= \sum_{i=1}^{m-1} a_i + a_{2m-1} + \sum_{1 \leq i < j \leq m-1} b_{i,j} + \sum_{i=1}^{m-1} b_{i,2m-1}; \\ 0 = h_t(y^{(2)}) &= \sum_{i=1}^{m-1} a_i + a_{2m} + \sum_{1 \leq i < j \leq m-1} b_{i,j} + \sum_{i=1}^{m-1} b_{i,2m}; \\ 0 = h_t(y^{(3)}) &= \sum_{i=1}^{m-1} a_i + a_{2m-1} + a_{2m} + \sum_{1 \leq i < j \leq m-1} b_{i,j} + \sum_{i=1}^{m-1} b_{i,2m-1} + \sum_{i=1}^{m-1} b_{i,2m} + b_{2m-1,2m}. \end{aligned}$$

Taking the first equation plus the second minus the third yields

$$\sum_{i=1}^{m-1} a_i + \sum_{1 \leq i < j \leq m-1} b_{i,j} = b_{2m-1,2m}.$$

Since  $m \geq 2$ ,  $m - 1 < 2m - 2$ ; thus, the same argument implies that

$$\sum_{i=1}^{m-1} a_i + \sum_{1 \leq i < j \leq m-1} b_{i,j} = b_{2m-2,2m}.$$

That is, the coefficient of  $x_{2m-1}^2 x_{2m}^2$  in  $h_t$  equals the coefficient of  $x_{2m-2}^2 x_{2m}^2$ , and so by symmetry, for all distinct  $i, j, k, \ell$ , the coefficient of  $x_i^2 x_j^2$  equals the coefficient of  $x_i^2 x_k^2$ , which equals the coefficient of  $x_k^2 x_\ell^2$ . Thus, for all  $i \neq j$ ,  $b_{i,j} = u$  for some  $u$ .

Subtracting the first from the second equation gives now  $a_{2m-1} = a_{2m}$ , and so for all  $i$ ,  $a_i = v$  for some  $v$ . Finally, our previous equations imply that

$$\begin{aligned} 0 &= mv + \binom{m}{2}u = (m+1)v + \binom{m+1}{2}u = 0 \\ \implies -v &= \frac{m-1}{2} \cdot u = \frac{m}{2} \cdot u \implies u = 0 \implies v = 0. \end{aligned}$$

In other words,  $h_t = 0$ , establishing the contradiction.

Suppose now that  $G_{2m+1}$  were sos. Then

$$G_{2m+1} = \sum_{t=1}^r h_t^2 \implies D_{2m} = \sum_{t=1}^r h_t^2(x_1, \dots, x_{2m}, 0),$$

a contradiction. Thus  $G_{2m+1}$  is not sos.  $\square$

**Remark 3.2.** It was asserted in [3] that the psd even symmetric  $n$ -ary octic

$$M_2(M_2^3 - (2k+1)M_2M_4 + k(k+1)M_6)$$

is not sos, provided  $2 \leq k \leq n - 2$ . We prove this below for  $k = 2$  and  $n \geq 4$ .

**Theorem 3.3.** For  $n \geq 4$ ,

$$T_n(x_1, \dots, x_n) = M_2(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,8}.$$

*Proof.* Suppose

$$T_n(x_1, \dots, x_n) = \sum_{r=1}^m h_r^2(x_1, \dots, x_n).$$

Then,  $T_n(v_2) = T_n(v_3) = 0$  but  $T_n(v_1) > 0$ . In particular, the terms  $x_j^4$  must appear on the right hand side. As in the proof of Theorem 3.1, these terms must appear in

$$\sum_{k=1}^n a_k x_k^4 + \sum_{1 \leq j < k \leq n} b_{jk} x_j^2 x_k^2,$$

which must vanish at every  $v_2$  and every  $v_3$ . In particular, for  $i < j < k$ , we have

$$\begin{aligned} a_i + a_j + b_{ij} &= 0, \\ a_i + a_k + b_{ik} &= 0, \\ a_j + a_k + b_{jk} &= 0, \\ a_i + a_j + a_k + b_{ij} + b_{ik} + b_{jk} &= 0. \end{aligned}$$

It easily follows that  $a_i + a_j + a_k = 0$ . Now assume  $i, j, k$  are distinct, but not necessarily increasing. Since  $n \geq 4$ , there is an unused index  $\ell$  and we may conclude that  $a_i + a_j + a_\ell = 0$ . Hence  $a_k = a_\ell$ . Since these are arbitrary, we conclude that

$a_m$  is independent of  $m$ , and since  $a_i + a_j + a_k = 0$ , it follows that each  $a_m = 0$ , a contradiction.  $\square$

**Remark 3.4.** For  $n = 3$ ,  $M_2(M_2^3 - 5M_2M_4 + 6M_6) = 2M_2R$  is sos, see [10], or equation (7.4) in [3].

#### 4. PSD NOT SOS EVEN SYMMETRIC $n$ -ARY DECIS AND DODECIS FOR $n \geq 4$

**Theorem 4.1.** For  $n \geq 4$ ,

$$P_n(x_1, \dots, x_n) = (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,10}.$$

*Proof.* First recall that

$$nM_4 - M_2^2 = n \sum_{k=1}^n x_k^4 - \left( \sum_{k=1}^n x_k^2 \right)^2 = \sum_{i < j} (x_i^2 - x_j^2)^2$$

is psd by Cauchy-Schwarz. The zero set is  $(\pm 1, \dots, \pm 1)$ .

Second, recall from Theorem 2.8 that the quadratic  $t(at^2 + bt + c)$  gives the value of the sextic  $aM_2^3 + bM_2M_4 + cM_6$  at an  $n$ -tuple  $v_t$  with  $t$  1's and  $n - t$  0's. Since  $t(t - 2)(t - 3) \geq 0$ , this criterion is satisfied, and the second factor is also psd with zeros at  $v_2$  and  $v_3$ .

It follows that  $P_n$  is psd and its coefficient of  $x_1^{10}$  is  $(n - 1)(1 - 5 + 6) > 0$ . We show that  $P_n$  is not sos by showing that in any sos expression,  $x_1^{10}$  cannot occur.

Using Theorem 2.5, we see that if  $P_n = \sum h_r^2$  and  $x_1^5$  occurs in  $h_r$ , then

$$h_r = ax_1^5 + x_1^3 \left( \sum_{k=2}^n b_k x_k^2 \right) + x_1 \left( \sum_{k=2}^n c_k x_k^4 \right) + x_1 \left( \sum_{2 \leq j < k < n} d_{jk} x_j^2 x_k^2 \right).$$

Since  $P_n(v_2(1, j)) = P_n(v_3(1, j, k)) = 0$  for all  $j, k, 2 \leq j < k \leq n$ , it follows that  $0 = h_r(v_2(1, j)) = h_r(v_3(1, j, k)) = 0$ , and we have the equations

$$0 = a + b_j + c_j,$$

$$0 = a + b_j + b_k + c_j + c_k + d_{jk} = (a + b_j + c_j) + (a + b_k + c_k) + d_{jk} - a.$$

From these equations, we may conclude that for all  $2 \leq j < k \leq n$ ,

$$b_k + c_k = -a, \quad d_{jk} = a.$$

Finally,  $P_n(v_n) = 0$ , so  $h_r(v_n) = 0$ ; that is,

$$0 = a + \sum_{k=2}^n (b_k + c_k) + \sum_{2 \leq j < k < n} d_{jk} = a \left( 1 - (n - 1) + \binom{n - 1}{2} \right) = a \cdot \frac{(n - 2)(n - 3)}{2}.$$

Thus,  $a = 0$  and  $x_1^5$  occurs in no  $h_r$ . This gives the contradiction.  $\square$

**Remark 4.2.** When  $n = 3$ ,  $P_n$  is sos:

$$\begin{aligned} P_3 &= (3M_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6) \\ &= 4(x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2)R(x, y, z) \\ &= 4(x^2(x^2 - y^2)^2(x^2 - z^2)^2 + y^2(y^2 - x^2)^2(y^2 - z^2)^2 + z^2(z^2 - x^2)^2(z^2 - y^2)^2). \end{aligned}$$

**Remark 4.3.** We have also shown that for  $m \geq 2$ ,  $M_2G_{2m+1} \in \Delta_{2m+1,10}$ . We shall discuss  $M_2G_{2m+1}$  and  $M_2D_{2m}$  in a future publication.

**Theorem 4.4.** For  $n \geq 5$ ,

$$Q_n(x_1, \dots, x_n) = (M_2^3 - 5M_2M_4 + 6M_6)(M_2^3 - 7M_2M_4 + 12M_6) \in \Delta_{n,12}.$$

*Proof.* Since  $(t-2)(t-3) \geq 0$  and  $(t-3)(t-4) \geq 0$ , both factors in  $Q_n$  are psd by Theorem 2.8. The first has zeros at every  $v_2$  and  $v_3$  and the second has zeros at every  $v_3$  and  $v_4$ . But note that neither has a zero at  $v_1$ . In fact, the coefficient of  $x_1^6$  in  $Q_n$  is  $(1-5+6)(1-7+12) > 0$ ,

Suppose  $Q_n$  is sos and  $Q_n = \sum f_k^2$ . As before, assume the  $f_k^2$ 's are even (using Theorem 2.5). Then  $f_k(v_t) = 0$  for every  $v_t$  with  $t = 2, 3, 4$ . Since  $Q_n(v_1) > 0$ , there must be an  $f_k$  containing  $x_i^6$ , which will be itself even. To this end, suppose

$$f_k = \sum_{i=1}^n \alpha_i x_i^6 + \sum_{1 \leq i \neq j \leq n} \beta_{ij} x_i^4 x_j^2 + \sum_{1 \leq i < j < k \leq n} \gamma_{ijk} x_i^2 x_j^2 x_k^2.$$

For  $i < j$ , let  $\mu_{ij} = \beta_{ij} + \beta_{ji}$ . By evaluating at  $v_2(i, j)$ , we see that

$$0 = \alpha_i + \alpha_j + \beta_{ij} + \beta_{ji} = \alpha_i + \alpha_j + \mu_{ij} \implies \mu_{ij} = -\alpha_i - \alpha_j.$$

By evaluating at  $v_3(i, j, k)$ , we have

$$\begin{aligned} 0 &= \alpha_i + \alpha_j + \alpha_k + \mu_{ij} + \mu_{ik} + \mu_{jk} + \gamma_{ijk} = \\ &= (\alpha_i + \alpha_j + \alpha_k) - 2(\alpha_i + \alpha_j + \alpha_k) + \gamma_{ijk} \implies \gamma_{ijk} = (\alpha_i + \alpha_j + \alpha_k). \end{aligned}$$

Finally, by evaluating at  $v_4(i, j, k, \ell)$ , we have

$$\begin{aligned} 0 &= \alpha_i + \alpha_j + \alpha_k + \alpha_\ell + \mu_{ij} + \mu_{ik} + \mu_{jk} + \mu_{i\ell} + \mu_{j\ell} + \mu_{k\ell} + \gamma_{ijk} + \gamma_{ij\ell} + \gamma_{ik\ell} + \gamma_{jkl} \\ &= (\alpha_i + \alpha_j + \alpha_k + \alpha_\ell)(1 - 3 + 3) \implies \alpha_i + \alpha_j + \alpha_k + \alpha_\ell = 0. \end{aligned}$$

In other words, the sum of any four distinct  $\alpha_r$ 's is 0. Since  $n \geq 5$ , there exists  $m \in \{1, \dots, n\}$  different from  $i, j, k, \ell$  and we have  $\alpha_i + \alpha_j + \alpha_k + \alpha_m = 0$ . Thus  $\alpha_\ell = \alpha_m$ , and since the choice of  $\ell$  and  $m$  was arbitrary, we conclude that  $\alpha_1 = \dots = \alpha_n = \alpha$ , so that  $4\alpha = 0$  and thus the coefficient of  $x_i^6$  in  $f_k$  must be zero, a contradiction.  $\square$

**Remark 4.5.** We have been unable to determine whether  $Q_3$  and  $Q_4$  are sos.

**Theorem 4.6.** For  $n \geq 3$ ,

$$\begin{aligned} R_n(x_1, \dots, x_n) &= \frac{1}{12} \cdot (M_2^3 - 3M_2M_4 + 2M_6)(M_2^3 - 5M_2M_4 + 6M_6) \\ &= \left( \sum_{1 \leq i < j < k \leq n} x_i^2 x_j^2 x_k^2 \right) \left( \sum_{i=1}^n x_i^6 - \sum_{1 \leq i \neq j \leq n} x_i^4 x_j^2 + 3 \sum_{1 \leq i < j < k \leq n} x_i^2 x_j^2 x_k^2 \right) \in \Delta_{n,12}. \end{aligned}$$

*Proof.* Since  $(t-1)(t-2) \geq 0$  and  $(t-2)(t-3) \geq 0$ , both factors in  $R_n$  are psd by Theorem 2.8. Moreover, the first factor implies that

$$(4.1) \quad R_n(t, u, 0, \dots, 0) = 0$$

for all real  $t, u$ , and at all  $n$ -tuples which are permutations of  $(t, u, 0, \dots, 0)$ . We also have, for all  $t$ ,

$$(4.2) \quad R_n(t, 1, 1, 0, \dots, 0) = t^2 \cdot \frac{1}{2} \cdot ((2+t^2)^3 - 5(2+t^2)(2+t^4) + 6(2+t^6)) = t^4(t^2-1)^2;$$

this also holds by symmetry at any permutation of  $n-3$  0's, two 1's and one  $t$ .

We first remark that if  $n = 3$ ,  $R_n(x, y, z) = x^2 y^2 z^2 R(x, y, z)$ ; since  $R$  is not sos, the same holds for  $R$  multiplied by a product of squared linear factors. For  $n \geq 4$ , more work is necessary.

Suppose  $R_n = \sum_r h_r^2$ , so that  $\deg h_r = 6$ . Suppose as usual that each  $h_r^2$  is even (using Theorem 2.5). It follows from equation (4.1) that for any such  $h_r$ ,  $h_r(t, u, 0, \dots, 0) = 0$ , for all  $(t, u)$ . If, say, the terms in  $h_r$  involving only  $x_1^{6-k} x_2^k$  are  $\sum_{k=0}^6 a_k x_1^{6-k} x_2^k$ , then  $\sum a_k t^{6-k} u^k = 0$  for all  $t, u$ , which implies that  $a_k = 0$ .



Proceeding similarly for all pairs of variables, we see that no monomial involving one or two variables can appear in any  $h_r$ .

For equation (4.2), let  $\phi_r(t) = h_r(t, 1, 1, 0, \dots, 0)$ . We have

$$\sum_{r=1}^w \phi_r(t)^2 = t^4(t^2 - 1)^2.$$

Evaluation at  $t = 0, 1, -1$  shows that  $\phi_r(t) = t(t^2 - 1)\psi_r(t)$ , so that

$$\sum_{r=1}^w \psi_r(t)^2 = t^2,$$

which in turn implies that  $\psi_r(t) = \lambda_r t$  for some real  $\lambda_r$ . To recapitulate, we have

$$(4.3) \quad h_r(t, 1, 1, 0, \dots, 0) = \lambda_r t^2(t^2 - 1),$$

and similar equations hold for all permutations of the variables.

Since  $x_1^2 x_2^2 x_3^8$  appears in  $r_n$  with coefficient 1, it also appears in  $\sum h_r^2$  with coefficient 1. Since no monomial occurs in any  $h_r$  with only two variables, it follows that  $x_1 x_2 x_3^4$  must appear in at least one  $h_r$ , and since  $h_r^2$  is even, all terms in  $h_r$  must be  $x_1 x_2$  times an even quartic monomial. Further, we already know that  $x_1^5 x_2, x_1^3 x_2^3, x_1 x_2^5$  do not occur. Thus

$$h_r(x_1, \dots, x_n) = x_1 x_2 \left( \sum_{j=3}^n (a_j x_1^2 x_j^2 + b_j x_2^2 x_j^2 + c_j x_j^4) + \sum_{3 \leq j < k \leq n} d_{jk} x_j^2 x_k^2 \right).$$

Thus,

$$h_r(t, 1, 1, 0, \dots, 0) = t(a_3 t^2 + b_3 + c_3),$$

$$h_r(1, t, 1, 0, \dots, 0) = t(a_3 + b_3 t^2 + c_3).$$

In view of equation (4.3), both of these cubics must be identically zero, hence  $a_3 = b_3 + c_3 = b_3 = a_3 + c_3 = 0$ , and so, in particular,  $c_3 = 0$ . This means that  $x_1 x_2 x_3^4$  does *not* appear in any  $h_r$ , establishing the contradiction.  $\square$

### Proof of the Main Theorem.

Combine Theorems 2.4, 3.1, 4.1 and 4.6.  $\square$

## 5. A GLOSSARY OF FORMS

For easy reference, we list the examples discussed in this paper.

$$L_{2m+1}(x_1, \dots, x_{2m+1}) := m(m+1) \sum_{i < j} (x_i - x_j)^4 - \left( \sum_{i < j} (x_i - x_j)^2 \right)^2, \quad (\text{Theorem 2.7});$$

$$C_{2m}(x_1, \dots, x_{2m}) = L_{2m+1}(x_1, \dots, x_{2m}, 0), \quad (\text{Theorem 2.7});$$

$$M_r(x_1, \dots, x_n) = x_1^r + \dots + x_n^r, \quad (\text{Theorem 2.8});$$

$$G_{2m+1}(x_1, \dots, x_{2m+1}) = L_{2m+1}(x_1^2, \dots, x_{2m+1}^2), \quad (\text{Section 3});$$

$$D_{2m}(x_1, \dots, x_{2m}) = G_{2m+1}(x_1, \dots, x_{2m}, 0), \quad (\text{Section 3});$$

$$T_n(x_1, \dots, x_n) = M_2(M_2^3 - 5M_2M_4 + 6M_6), \quad (\text{Theorem 3.3});$$

$$P_n(x_1, \dots, x_n) = (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6), \quad (\text{Theorem 4.1});$$

$$Q_n(x_1, \dots, x_n) = (M_2^3 - 5M_2M_4 + 6M_6)(M_2^3 - 7M_2M_4 + 12M_6), \quad (\text{Theorem 4.4});$$

$$R_n(x_1, \dots, x_n) = \frac{1}{12} \cdot (M_2^3 - 3M_2M_4 + 2M_6)(M_2^3 - 5M_2M_4 + 6M_6), \quad (\text{Theorem 4.6}).$$

## 6. REDUCTION OF PSDNESS OF A FORM TO PREORDERING

Let  $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[\underline{x}]$ ,

$$T_S := \left\{ \sum_{e_1, \dots, e_s \in \{0,1\}} \sigma_{\underline{e}} g_1^{e_1} \dots g_s^{e_s} \mid \sigma_{\underline{e}} \in \Sigma \mathbb{R}[\underline{x}]^2, \underline{e} = (e_1, \dots, e_s) \right\}$$

be the finitely generated quadratic preordering, and

$$K_S := \{\underline{x} \in \mathbb{R}[\underline{x}] \mid g_1(\underline{x}) \geq 0, \dots, g_s(\underline{x}) \geq 0\}$$

be the basic closed semi-algebraic set.

We recall the following result which follows from [11, Proposition 6.1]:

**Proposition 6.1.** Let  $S$  be a finite subset of  $\mathbb{R}[\underline{x}]$ , such that  $\dim(K_S) \geq 3$ . Then  $\exists g \in \mathbb{R}[\underline{x}]$  s.t.  $g \geq 0$  on  $K_S$  but  $g \notin T_S$ .

For  $S' = \{x_1, \dots, x_n\}$ , the relation of  $T_{S'}$  and even sos forms is explained by the following result due to Frenkel and Horváth [4, Lemma 1].

**Lemma 6.2.** Let  $p \in \mathbb{R}[x_1, \dots, x_n]$ . Then  $p \in T_{S'}$  if and only if  $p(x_1^2, \dots, x_n^2) \in \Sigma \mathbb{R}[x_1, \dots, x_n]^2$ .

**Remark 6.3.** Let  $f(x_1, \dots, x_n)$  be an even symmetric form of degree  $2d$ , assume  $f$  is psd, and let  $g(y_1, \dots, y_n)$  be such that  $g(x_1^2, \dots, x_n^2) = f(x_1, \dots, x_n)$ . Then  $g$  is a  $n$ -ary symmetric form of degree  $d$  and  $g \geq 0$  on  $K_{S'}$ , where  $S' := \{y_1, \dots, y_n\}$ , i.e.  $g$  is nonnegative on the positive orthant  $\mathbb{R}_+^n$ . By Lemma 6.2,  $f \in \Sigma \mathbb{R}[\underline{x}]^2$  if and only if  $g \in T_{S'}$ . So, finding out pairs  $(n, 2d)$  for which an even symmetric  $n$ -ary  $2d$ -ic psd form  $f$  is sos is equivalent to finding the pairs  $(n, d)$  for which the corresponding symmetric  $n$ -ary  $d$ -ic form  $g$  nonnegative on  $\mathbb{R}_+^n$  belongs to  $T_{S'}$ .

In the following proposition we consider the case  $K_{S'} = \mathbb{R}_+^n$  and describe for which pairs  $(n, d)$  the form  $g$  of Proposition 6.1 above can be chosen to be symmetric.

**Proposition 6.4.** If  $n \geq 3$ , let  $S' = \{x_1, \dots, x_n\} \subseteq \mathbb{R}[\underline{x}]$ , then  $\dim(K_{S'}) \geq 3$  and we have:

- (1)  $\forall g \in S\mathcal{F}_{n,2}$ :  $g \geq 0$  on  $\mathbb{R}_+^n$  implies  $g \in T_{S'}$
- (2)  $\forall g \in S\mathcal{F}_{3,4}$ :  $g \geq 0$  on  $\mathbb{R}_+^3$  implies  $g \in T_{S'}$
- (3) for  $d \geq 5$ ,  $\exists g \in S\mathcal{F}_{3,d}$  s.t.  $g \geq 0$  on  $\mathbb{R}_+^3$  but  $g \notin T_{S'}$
- (4)  $\exists g \in S\mathcal{F}_{n,3}$  s.t.  $g \geq 0$  on  $\mathbb{R}_+^n$  but  $g \notin T_{S'}$
- (5) if  $n \geq 4$ , then  $\exists g \in S\mathcal{F}_{n,4}$  s.t.  $g \geq 0$  on  $\mathbb{R}_+^n$  but  $g \notin T_{S'}$
- (6) if  $n \geq 4$ , then  $\exists g \in S\mathcal{F}_{n,5}$  s.t.  $g \geq 0$  on  $\mathbb{R}_+^n$  but  $g \notin T_{S'}$
- (7) if  $n \geq 4$ , then  $\exists g \in S\mathcal{F}_{n,6}$  s.t.  $g \geq 0$  on  $\mathbb{R}_+^n$  but  $g \notin T_{S'}$
- (8) if  $n \geq 4$  and  $d \geq 7$ , then  $\exists g \in S\mathcal{F}_{n,d}$  s.t.  $g \geq 0$  on  $\mathbb{R}_+^n$  but  $g \notin T_{S'}$ .

*Proof.* Using Main Theorem and Remark 6.3 we are done.  $\square$

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