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On the Cohen–Macaulay Type of s-Lines in A^{n+1}

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Let A be the coordinate ring of s-lines through the origin in $A^{n+1}(k)$. We discuss what it means for these lines to be in "sufficiently general position" and then, with this restriction on the lines, we attempt to verify our belief that the Cohen–Macaulay type of A depends only on s and n. We characterize those s and n for which A is a Gorenstein ring (i.e. of C–M type 1) and explicitly calculate the Cohen–Macaulay type in several other cases. We then extend some of our results to the case of an arbitrary reduced curve whose tangent cone consists of lines in sufficiently general position. Finally, we calculate the Hilbert–Samuel polynomials of the curves we have been considering.

INTRODUCTION

Let $(A, \mathfrak{m})$ be the local ring at a point of a reduced curve in $A^{n+1}(k)$, k an algebraically closed field. Ideally, one would like to have a dictionary which would allow the passage from algebraic properties of the local ring to geometric properties of the point on the curve and vice versa.

In this paper we shall restrict ourselves to singular points for which the associated graded ring, $G(A)$, is reduced. This is very close to, but not precisely the same as, the notion of "ordinary multiple point." (See [7] for a

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discussion of the differences between these notions.) In any case, singularities for which the associated graded ring is reduced are ordinary multiple points, and our ultimate goal is to completely describe the Cohen–Macaulay type of the local ring $(A, \mathfrak{m})$ solely in terms of the tangent directions at the singularity.

A first step in that direction would be to calculate the Cohen–Macaulay type at the common point of intersection of s distinct lines in $A^{n+1}(k)$. Our investigations of this case have led us to believe that if the lines are in "sufficiently general position" then there is an expected Cohen–Macaulay type for the singularity at the point of intersection which depends only on s and n.

We have not, as yet, been able to completely verify that belief, but we have been able to decide, when the lines are in "sufficiently general position", precisely when the singularity in question is Gorenstein. We also give other indications that suport our belief.

There are various possible notions of "sufficiently general position" for a collection of lines through the origin in A^{n+1}. We describe what we want this to mean and verify that "most" collections of lines satisfy the condition.

We then try to apply our calculations for the case of lines to the case of an ordinary singularity whose tangent lines are in sufficiently general position. More precisely, we conjecture that if A is the local ring of a reduced curve at an ordinary singularity and if the tangent lines at that singularity are in sufficiently general position then the Cohen–Macaulay type of A is equal to the Cohen–Macaulay type of $G(A)$. We prove that this conjecture is valid when A or $G(A)$ is a Gorenstein ring.

We conclude the paper with some comments on the Hilbert polynomials of some of the rings we have discussed.

We are very grateful to our colleague L. G. Roberts for a useful discussion concerning the remarks which precede Theorem 5. His comments allowed us to make simplifications of our earlier proofs. We are also very grateful to Joseph Harris for his illuminating comments about "sufficiently general position." The remarks after Theorem 9 are entirely due to him.

All rings are assumed to be commutative noetherian rings with identity having finite integral closure in their total ring of fractions. The assumption that our base field, k, be algebraically closed could largely be ignored if one restricted the discussion to lines rational over k. For simplicity in the exposition we shall retain the assumption that k be algebraically closed throughout the paper.

1. Preliminaries

(a) *The Cohen–Macaulay Type*

If R is the coordinate ring of a reduced one-dimensional variety in $A^{n+1}(k)$, then there are non-zero-divisors in any maximal ideal of R and so R is a one-dimensional Cohen–Macaulay (C–M) ring. (We refer the reader to [4] for the definition and basic properties of C–M rings.)

If $(A, \mathfrak{m})$ is any one-dimensional local C–M ring and $x \in \mathfrak{m}$ is a non-zero-divisor in $\mathfrak{m}$, then $(x) = I_1 \cap \cdots \cap I_m$, where each I_j $(1 \leqslant j \leqslant m)$ is an irreducible ideal and the representation is irredundant $(I_j \not\subset \bigcap_{l \neq j} I_l)$. It is a well-known theorem of Northcott and Rees (see e.g. [4, Theorem 27.3, p. 290]) that the number m is independent of x and is thus an invariant of A. We call m the *Cohen–Macaulay type* of A and say that A is C–M (m). If $m = 1$, A is called a *Gorenstein* ring. We extend the definitions to a non-local Cohen–Macaulay ring R by taking the supremum over the localizations at the maximal ideals of R, of the Cohen–Macaulay types.

The main tool we shall use to determine the type of a C–M ring is a theorem due to Gröbner (see e.g. [4, Theorem 27.5, p. 291]).

THEOREM 1. *Let $(A, \mathfrak{m})$ be a local ring and let q be an $\mathfrak{m}$-primary ideal. The number of irreducible components in an irredundant irreducible decomposition of q is*

$$l_{A/q}((q : m)/q) = l_{A/q}(\operatorname{ann}_{A/q}(\mathfrak{m}/q))$$

(where "$l_{A/q}$" denotes length as A/q-module.)

(b) *The Total Ring of Quotients of a One-dimensional Reduced Ring*

The remarks here are all well known and may be found, for example in ([2, Chap. V, Sect. 1, No. 5]). We repeat them here in order to establish some notation that we shall use later.

Let R be a reduced noetherian ring. Then $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$, where the $\mathfrak{p}_i$ are the minimal prime ideals of R. Set $S = \prod_{i=1}^{s} R/\mathfrak{p}_i$ and let $\pi_i : R \to R/\mathfrak{p}_i$ be the canonical surjections. If $\pi = (\pi_1, \ldots, \pi_s) : R \to S$ then the following are easily established.

Theorem 2. (1) *π: $R \to S$ is an injection.*

(2) *S is (naturally) contained in the total ring of fractions of R.*

(3) *S is integral over R.*

(4) *$\bar{R}$ = integral closure of R in its total ring of fractions is $= \prod_{i=1}^{s} (\overline{R/\mathfrak{p}_i})$, where $(\overline{R/\mathfrak{p}_i})$ = integral closure of $R/\mathfrak{p}_i$ in its field of fractions.*

(c) *The Position of Points in* $\mathbb{P}^n(k)$

One of us has already pointed out the importance of the concepts of this section to the discussion of ordinary singularities [6, Sect. 3]. Since then, the method of exposition of the ideas has undergone a series of revisions and changes in terminology. For our purposes in this paper the method of exposition we now present is more suitable for the calculations we want to make.

If k is a field then the polynomial ring, $k[X_0,..., X_n]$, is a graded ring (with its usual grading) and the dth homogeneous piece, denoted $V(d, n)$, is a k-vector space with basis given by all the monomials of degree $= d$. It is an easy exercise to show that $\dim_k V(d,n) = \binom{d+n}{n}$.

Let $P_1,..., P_s$ be points of $\mathbb{P}^n$, $P_i = [a_{i0} : a_{i1} : \cdots : a_{in}]$, and let $F_{d,1}, F_{d,2},..., F_{d,\binom{d+n}{n}}$ be all the monomials of degree d in $k[X_0,..., X_n]$. We form the $s \times \binom{d+n}{n}$ matrix

$$G_d(P_1,..., P_s) = (g_{ij}), \qquad \text{where} \quad g_{ij} = F_{d,j}(P_i).$$

Notice that if we consider the homogeneous system of linear equations with coefficient matrix $G_d(P_1,..., P_s)$ then any non-trivial solution to that system provides us with the coefficients to construct a form, of degree d, in $k[X_0,..., X_n]$, which vanishes at the points $P_1,..., P_s$. Conversely, any such form will give a non-trivial solution to the system of linear equations. Consequently, if $rk(G_d(P_1,..., P_s)) = t$ there are precisely $\binom{d+n}{n} - t$ linearly independent forms of degree d passing through $P_1,..., P_s$.

DEFINITIONS (1) The set of points $\{P_1,..., P_s\}$ is said to be in *generic s-position* if the matrices $G_d(P_1,..., P_s)$ have maximum rank for every $d \geqslant 1$; i.e.,

$$\text{if} \quad \binom{d+n}{n} > s \qquad \text{then} \quad \mathrm{rk}(G_d(P_1,..., P_s)) = s,$$

and

$$\text{if} \quad \binom{d+n}{n} \leqslant s \qquad \text{then} \quad \mathrm{rk}(G_d(P_1,..., P_s)) = \binom{d+n}{n}.$$

(2) The set of points $\{P_1,..., P_s\}$ is said to be in *generic t-position*, $0 < t \leqslant s$, if every *t-subset* of $\{P_1,..., P_s\}$ is in generic t-position.

(3) The set of points $\{P_1,..., P_s\}$ is said to be in *uniform position* if it is in generic t-position for every $0 < t \leqslant s$.

Remarks. (1) The notion of generic t-position is a projective invariant for a set of points $\{P_1,..., P_s\}$ in $\mathbb{P}^n$.

(2) The set of points $\{P_1,..., P_s\}$ is in generic t-position if and only if every $t \times \binom{d \times n}{n}$ submatrix of $G_d(P_1,..., P_s)$ has maximum rank, for every d.

(3) If the points $\{P_1,..., P_s\}$ are in generic s-position then the least degree, d_0, of a hypersurface passing through $P_1,..., P_s$ is the least integer d_0 such that $\binom{d_0+n}{n} > s$ (see also [6, Lemma 3.3]).

EXAMPLES. (1) Any set of points of $\mathbb{P}^n$ are in generic 1-position.

(2) Any two points of $\mathbb{P}^n$ are in generic 2-position hence in uniform position.

(3) Any three non-collinear points of $\mathbb{P}^2$ are in uniform position.

(4) Any set of points of $\mathbb{P}^1$ are in uniform position. (The matrices in question all have Vandermonde sub-matrices.)

(5) Four points of $\mathbb{P}^2$, no three collinear, are in uniform position.

(6) Four non-collinear points of $\mathbb{P}^2$ are in generic 4-position.

(7) $N = \binom{d+n}{n}$ points lying on a hypersurface of degree d are not in generic N-position, but if every $N-1$ of these points are contained in only one hypersurface of degree d then the N points are in generic $N-1$ position. This applies specifically to any six points on an irreducible conic.

(8) The nine distinct points of intersection of two irreducible cubics in $\mathbb{P}^2$ are not in generic 9-position. Notice that in this case, the least degree of a homogeneous polynomial containing the nine points is 3 and that $d=3$ is the least integer such that $\binom{d+2}{2} > s = 9$. But, rk $G_3(P_1,..., P_9) < 9$ since there are at least two independent cubics containing these nine points.

Notice further that if we have any cubic passing through these nine points then Noether's conditions are satisfied at these points [see [3, p. 120] and thus, by Noether's $AF + BG$ theorem, this cubic must be in the pencil determined by the original two cubics. This says that rk $G_3(P_1,..., P_9) = 8$.

These nine points are, however, in generic 8-position. To see this first recall that any cubic passing through any eight of these nine points also passes through the ninth. Thus rk $G_3(P_1,..., \hat{P}_i,..., P_9) = 8$ also, $1 \leqslant i \leqslant 9$ and every 8×10 submatrix of $G_3(P_1,..., P_9)$ has the same "solutions" as $G_3(P_1,..., P_9)$ itself. The conclusion that the points $P_1,..., P_9$ are then in generic 8-position will follow from the next proposition.

PROPOSITION 3. *Let $P_1,..., P_s$ be points of $\mathbb{P}^n$. The set of points is in generic s-position if and only if the least degree, d_0, of a non-zero form vanishing at $P_1,..., P_s$ is the least integer d_0 such that* $\binom{d_0+1}{n} > s$ and *the subspace of $V(d_0, n)$ of forms vanishing at $P_1,..., P_s$ has dimension exactly* $\binom{d_0+n}{n} - s$.

Proof. $\Rightarrow$ Obvious from the definition.

$\Leftarrow$ We need to show that the matrices $G_d(P_1,..., P_s)$ have maximum rank for every d.

Case 1. $d < d_0$.

In this case $G_d(P_1,..., P_s)$ is an $s \times \binom{d+n}{n}$ matrix and $s \geqslant \binom{d+n}{n}$. Since, by assumption, there is no form of degree d vanishing at the P_i's, we must have $\operatorname{rk} G_d = \binom{d+n}{n}$, which is the maximum possible.

Case 2. $d = d_0$.

Our assumption on the dimension of the space of solutions implies $\operatorname{rk} G_{d_0} = s$, the maximum possible.

Case 3. $d > d_0$.

To handle this case we prove the following:

LEMMA. *Let $P_1,..., P_s$ be points of $\mathbb{P}^n$ and let d be such that $s \leqslant \binom{d+n}{n}$ and* rk $G_d(P_1,..., P_s) = s$. *Then* rk $G_{d+1}(P_1,..., P_s) = s$ *also.*

Proof (*of Lemma*). First notice that $\operatorname{rk} G_i(P_1,..., P_s)$ is unchanged by a linear change of coordinates in $\mathbb{P}^n$. Thus, we may assume that the P_i all have X_0-coordinate $= 1$. Hence $G_d(P_1,..., P_s)$ is a sub-matrix of $G_{d+1}(P_1,..., P_s)$. Since $\operatorname{rk} G_d(P_1,..., P_s) = s$ and both matrices have no more rows than columns, we obtain $\operatorname{rk} G_{d+1}(P_1,..., P_s) = s$ also.

Now Case 3 of the Proposition follows by induction using Case 2 and the Lemma.

We finish off this section by showing that the notion of generic s-position (or t-position, or uniform position) is not a vacuous notion. In fact, we show more.

THEOREM 4. *The s-tuples of points of $\mathbb{P}^n$, $(P_1,..., P_s)$, considered as points of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (s-times), which are in generic s-position* (*respectively, generic t-position for any $t < s$; respectively uniform position*) *form a non-empty open subset of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (s-times).*

Proof. We first observe that it is enough to prove that for any s, the elements of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (s-times) which correspond to points in generic s-position form a non-empty open set.

Suppose this is the case and we choose $t < s$. Then, for any t-element subset of $\{1, 2,..., s\}$ we have a projection $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (s-times) onto $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (t-times). We label these projections $\pi_1, \pi_2,..., \pi_{\binom{s}{t}}$. By assumption, the elements of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (t – times) which correspond to t points in generic t-position form a non-empty open set V_t. Thus,

$$\bigcap_{i=1}^{\binom{s}{t}} \pi_i^{-1}(V_t) = U_t$$

gives a non-empty open subset of $\mathbb{P}^n \times \cdots \mathbb{P}^n$ (s-times) which corresponds precisely to those sets of s points in generic t-position. Further, $\bigcap_{t=1}^{s} U_t$ is also a non-empty open set and describes those sets of s points in uniform position.

It remains to show that the elements of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (s-times) corresponding to points in generic s-position form a non-empty open set.

We first show the set is open. Recall that for any d we have the Veronese embedding

$$v_d\colon \mathbb{P}^n \to \mathbb{P}^{N-1}, \qquad N = \binom{d+n}{n},$$

defined by $v_d(x_0 : \cdots : x_n) = (\cdots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \cdots)$, $\sum_{j=0}^{n} i_j = d$ (i.e., evaluation at $(x_0 : \cdots : x_n)$ of the basis monomials of $V(d, n)$). This then gives us a morphism

$$\begin{array}{c} \phi_d = (v_d : \cdots : v_d)\colon \mathbb{P}^n \times \cdots \times \mathbb{P}^n \text{ (s-times)} \to \mathbb{P}^{N-1} \times \cdots \times \mathbb{P}^{N-1} \text{ (s-times)} \\ \hphantom{\phi_d = (v_d : \cdots : v_d)\colon \mathbb{P}^n \times \cdots \times \mathbb{P}^n \text{ (s-times)} \to \mathbb{P}^{N-1} \times} \| \hphantom{\mathbb{P}^{N-1} \text{ (s-times)}} \\ \hphantom{\phi_d = (v_d : \cdots : v_d)\colon \mathbb{P}^n \times \cdots \times \mathbb{P}^n \text{ (s-times)} \to \mathbb{P}^{N-1} \times} V \hphantom{\mathbb{P}^{N-1} \text{ (s-times)}} \end{array}$$

We may loosely consider the elements of V as $s \times N$ matrices. (Notice that $\phi_d(P_1, \ldots, P_s)$ "is" $G_d(P_1, \ldots, P_s)$.) Now, write the equations describing the situation that every maximal minor of an $s \times N$ matrix is zero. These equations are homogeneous in each set of variables corresponding to a factor of V and so they describe a closed subset C of V. Thus, $\phi_d^{-1}(V - C)$ is the open set consisting of those s-tuples, $(P_1, \ldots, P_s)$ for which $G_d(P_1, \ldots, P_s)$ has maximal rank. Thus, $\bigcap_{j=1}^{d_0} \phi_j^{-1}(V - C)$, where d_0 is the least integer for which $\binom{d_0+n}{n} > s$, will correspond to the set of points in generic s-position and is now, clearly, an open set. (Note that we are using here the fact, implicit from the proof of Proposition 3, that $P_1, \ldots, P_s \in P^n$ are in generic position iff rk $G_j(P_1, \ldots, P_s)$ is maximal $\forall j$, $1 \leqslant j \leqslant d_0$, where d_0 is the least integer for which $\binom{d_0+n}{n} > s$.

The proof will now be finished once we can show that $\phi_d^{-1}(V - C) \neq \varnothing$ for any d.

Clearly, if we can show that for any s there is at least one set of points in $\mathbb{P}^n$ in generic s-position, that will finish off the theorem. We prove this by induction on s; the case $s = 1$ being trivial.

So, let $P_1, \ldots, P_{s-1}$ be $s - 1$ points of $\mathbb{P}^n$ in generic $(s-1)$-position. Let d_0 be the least integer such that $N = \binom{d_0+n}{n} \geqslant s$ and consider the Veronese embedding $v_{d_0}\colon \mathbb{P}^n \to \mathbb{P}^{N-1}$. It is well known that $v_{d_0}(\mathbb{P}^n)$ is not contained in any linear subvariety of $\mathbb{P}^{N-1}$.

Let L_{s-1} be the proper linear subvariety of $\mathbb{P}^{N-1}$ spanned by $v_{d_0}(P_1), \ldots, v_{d_0}(P_{s-1})$. Then $X = v_{d_0}(\mathbb{P}^n) \backslash \{v_{d_0}(\mathbb{P}^n) \cap L_{s-1}\}$ is non-empty, so we may choose $P_s \in v_{d_0}^{-1}(X)$. We now claim that $P_1, \ldots, P_s$ are in generic s-

position. To see this we must show $G_d(P_1,\ldots,P_s)$ has maximum rank for every d.

Case 1. *d is an integer for which* $\binom{d+n}{n} \leqslant s-1$.

In this case we have $\operatorname{rk} G_d(P_1,\ldots,P_{s-1}) = \binom{d+n}{n}$, since $P_1,\ldots,P_{s-1}$ are in generic $(s-1)$-position and so $\operatorname{rk} G_d(P_1,\ldots,P_s) = \binom{d+n}{n}$ also, which is the maximum it could be.

Case 2. *d is an integer for which* $\binom{d+n}{n} \geqslant s$.

By the lemma in the proof of Proposition 3 it is enough to show that $\operatorname{rk} G_{d_0}(P_1,\ldots,P_s) = s$ for the least d_0 with $\binom{d_0+n}{n} \geqslant s$. But, this is precisely what it means to say that $v_{d_0}(P_s)$ is not in the linear span of $v_{d_0}(P_1),\ldots, v_{d_0}(P_{s-1})$.

2. THE COORDINATE RING OF S-LINES IN A^{n+1} AND ITS INTEGRAL CLOSURE

If we look at the affine variety, $\mathscr{L}$, consisting of s distinct lines in $A^{n+1}(k)$, all passing through the origin, then this variety has coordinate ring $R = k[x_0,\ldots,x_n]/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$, where the $\mathfrak{p}_i$ are homogeneous prime ideals of height n. In our case $(k=\bar{k})$, each $\mathfrak{p}_i$ is then a complete intersection generated by n linearly independent linear forms. The ideal $\mathfrak{p}_i$ thus corresponds to a line $\mathscr{L}_i$. If $(a_{i0},\ldots,a_{in})$ is a point on $\mathscr{L}_i$ then we can identify that line with the point $P_i = [a_{i0} : a_{i1} : \cdots : a_{in}] \in \mathbb{P}^n(k)$.

Now, let $k[T_i]$ be the affine coordinate ring of the affine line $A^1(k)$. We map $A^1(k) \to \mathscr{L}_i$ by sending $t \to (a_{i0}t, a_{i1}t,\ldots, a_{in}t)$. This mapping induces an isomorphism of rings $k[x_0,\ldots,x_n]/\mathfrak{p}_i \to k[T_i]$, and the composite mapping $R \to k[x_0,\ldots,x_n]/\mathfrak{p}_i \simeq k[T_i]$ given by $\bar{X}_j \to a_{ij}T_i$ is precisely the map π_i we described just before Theorem 2. So, we have $R \hookrightarrow \prod_{i=1}^{s} k[T_i] = S$ and S is the integral closure of R in its total ring of quotients. Furthermore, this embedding is explicitly given by $\bar{X}_j \to (a_{1j}T_1, a_{2j}T_2,\ldots, a_{sj}T_s) = \varepsilon_j$.

We view S as a graded ring, in the obvious way, and observe that the inclusion of the graded ring R into S is a graded homomorphism of degree zero. Thus, viewed as a subring of S, we see that R is generated over k, by $\varepsilon_0, \varepsilon_1, \varepsilon_2,\ldots, \varepsilon_n$.

If we write

$$R = R_0 \oplus R_1 \oplus \cdots \qquad (R_0 = k),$$

then each R_i is a subspace of an s-dimensional k vector space. The following crucial observation tells us the dimension of each graded piece of R.

THEOREM 5. *With all notation as above,*

$$\dim_k R_d = \operatorname{rk} G_d(P_1, \ldots, P_s), \qquad \forall d \geqslant 1.$$

Proof. Since R is generated over $R_0 = k$ by R_1 and R_1 is generated as a k-space by $\varepsilon_0, \ldots, \varepsilon_n$, we know that R_d is generated, as a k-space, by all the monomials of degree $= d$ in $\varepsilon_0, \ldots, \varepsilon_n$. If we write

$$\begin{aligned} \varepsilon_0 &= (a_{10}, a_{20}, \ldots, a_{s0}) \\ \varepsilon_1 &= (a_{11}, a_{21}, \ldots, a_{s1}) \\ \vdots\ & \quad \vdots \\ \varepsilon_n &= (a_{1n}, a_{2n}, \ldots, a_{sn}) \end{aligned}$$

and if we let $F \in k[Y_0, \ldots, Y_n]$ represent a monomial expression of degree d, $F = Y_0^{\alpha_0} \cdots Y_n^{\alpha_n}$, $\sum \alpha_i = d$, then

$$\begin{aligned} \varepsilon_0^{\alpha_0} \varepsilon_1^{\alpha_1} \cdots \varepsilon_n^{\alpha_n} &= a_{10}^{\alpha_0} a_{11}^{\alpha_1} \cdots a_{1n}^{\alpha_n}, a_{20}^{\alpha_0} a_{21}^{\alpha_1} \cdots a_{2n}^{\alpha_n}, \ldots, a_{s0}^{\alpha_0} a_{s1}^{\alpha_1} \cdots a_{sn}^{\alpha_n} \\ &= (F(P_1), F(P_2), \ldots, F(P_s)). \end{aligned}$$

But, this is precisely how we obtain the columns of $G_d(P_1, \ldots, P_s)$.

3. GORENSTEIN SINGULARITIES

In this section we would like to begin an investigation of when the coordinate ring of s-lines through the origin of A^{n+1} is a Gorenstein ring. It is well known, (see [1 Cor. 6.5]) that if R is a one-dimensional reduced ring with finite integral closure $\bar{R}$ and if $\mathscr{C}$ is the conductor of R in $\bar{R}$ then R is Gorenstein iff $l_R(\bar{R}/\mathscr{C}) = 2 l_R(R/\mathscr{C})$.

Let $R = k[x_0, \ldots, x_n]/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$, $\quad \operatorname{ht} \mathfrak{p}_i = n$, $\mathfrak{p}_i \leftrightarrow P_i \in \mathbb{P}^n$ and we suppose the points $\{P_1, \ldots, P_s\}$ are in generic $(s-1)$ and s position. (From now on, we shall say that the lines in A^{n+1}, corresponding to the primes $\mathfrak{p}_i$, are in uniform position (or generic t-position, etc.) if the points $P_i \in \mathbb{P}^n$ are in uniform position (or generic t-position, etc.).)

We have seen how to obtain $R \hookrightarrow S = \prod_{i=1}^{s} k[T_i] = \bar{R}$, and if we let M denote the homogeneous maximal ideal of R generated by $\varepsilon_0, \ldots, \varepsilon_n$, then in [6, Theorem 5.1] it was shown that, with the assumption of generic $(s-1)$ and s-position, the conductor, $\mathscr{C}$, of R in $\bar{R}$ is M^{d_0}, where d_0 is the least integer for which $s \leqslant \binom{d_0+n}{n}$. So, if

$$\begin{aligned} \bar{R} &= k^s \oplus S_1 \oplus S_2 \oplus \cdots, \\ R &= k \ \oplus R_1 \oplus R_2 \oplus \cdots, \end{aligned}$$

and $\mathscr{C} = \oplus_{t \geq d_0} R_t$; then, since $\dim_k R_{d_0} = rk\, G_{d_0}(P_1,\ldots, P_s)$ and $\{P_1,\ldots, P_s\}$ are in generic s-position we have that $\dim_k R_{d_0} = s = \dim_k S_{d_0}$. Also, $\dim_k S_t = \dim_k R_t$, $\forall t > d_0$. Thus, considering $\mathscr{C}$ as an $\bar{R}$ ideal, we have that $\mathscr{C} = \oplus_{t \geq d_0} S_t$.

We now calculate $\dim_k(R/\mathscr{C})$ and $\dim_k(\bar{R}/\mathscr{C})$. As we have seen what $\mathscr{C}$ is in both R and $\bar{R}$, we find that

$$R/\mathscr{C} \simeq k \oplus R_1 \oplus \cdots \oplus R_{d_0-1},$$

and

$$\bar{R}/\mathscr{C} \simeq k^s \oplus S_1 \oplus \cdots \oplus S_{d_0-1},$$

We thus have that $\dim_k \bar{R}/\mathscr{C} = sd_0$ and $\dim_k R/\mathscr{C} = 1 + \mathrm{rk}\, G_1(P_1,\ldots, P_s) + \cdots + \mathrm{rk}\, G_{d_0-1}(P_1,\ldots, P_s)$. But, since $\{P_1,\ldots, P_s\}$ are in generic s-position and d_0 was the least integer $\ni: s \leqslant \binom{d_0+n}{n}$, we have

$$\mathrm{rk}\, G_i(P_1,\ldots, P_s) = \binom{i+n}{n}, \qquad 1 \leqslant i \leqslant d_0 - 1,$$

$$\therefore \dim_k R/\mathscr{C} = \binom{n}{n} + \binom{1+n}{n} + \cdots + \binom{d_0-1+n}{n}$$
$$= \binom{d_0+n}{n+1}$$

We summarize these calculations here.

PROPOSITION 6. *Let R be the affine coordinate ring of s-lines of A^{n+1} in generic $(s-1)$ and s position. Let $\bar{R}$ = integral closure of R in its total ring of quotients and let $\mathscr{C}$ be the conductor of R in $\bar{R}$.*

If d_0 is the least integer for which $\binom{d_0+n}{n} \geqslant s$ then $\dim_k \bar{R}/\mathscr{C} = sd_0$ and $\dim_k R/\mathscr{C} = \binom{d_0+n}{n+1}$.

In view of the characterization of Gorenstein rings we gave in the beginning of this section, we obtain:

COROLLARY. *Let R be as in Proposition 6. Then R is a Gorenstein ring iff $sd_0 = \binom{d_0+1}{n+1}$.*

It remains only to find the s's and n's for which this equality holds.

First notice that for $n = 1$ we have $d_0 = s - 1$ and it is true that $s(s-1) = 2\binom{s}{2}$, i.e., the conditions are always satisfied. This conforms with the classical result that the coordinate ring of a plane curve is always Gorenstein.

Also note that if $d_0 = 1$ (i.e., $s \leqslant n+1$) then the ring can be Gorenstein only for $s = 2$.

So, we can suppose $d_0 > 1$ and $n > 1$, and we wish to find the integers s which satisfy:

$$sd_0 = 2\binom{d_0+n}{n+1}$$

and

$$\binom{d_0+n-1}{n} < s \leqslant \binom{d_0+n}{n}$$

These imply

$$\binom{d_0+n-1}{n} < \frac{2}{d_0}\binom{d_0+n}{n+1};$$

i.e.,

$$d_0 < \frac{2n}{n-1}.$$

Thus, if $n \geqslant 3$, $d_0 = 2$ and $s = n+2$.

If $n = 2$ then $d_0 = 2$ or 3 appear possible. Only $d_0 = 2$ yields a solution, however, and it is $s = 4$. We summarize all these calculations.

THEOREM 7. *Let R be the coordinate ring of s lines through the origin of A^{n+1} that are in generic $(s-1)$ and s position. If $n = 1$, R is always a Gorenstein ring. If $n > 1$, R is a Gorenstein ring if and only if $s = 2$ or $s = n+2$.*

From the discussion preceeding Theorem 7 it follows that:

COROLLARY. *With the hypothesis of Theorem 7 and $n > 1$, if R is Gorenstein then the conductor of R in $\bar{R}$ is always M^2.*

4. SOME OTHER GORENSTEIN SINGULARITIES

If we examine the ideas that were crucial to the calculations in the preceeding section we find that the essential ingredients were:

(1) knowing that the conductor of R in $\bar{R}$ was a power of the homogeneous maximal ideal of R.

(2) knowing precisely the dimensions of the graded pieces of R.

In [6] the exact structure of the conductor of $R = k[x_0,\dots,x_n]/\mathfrak{p}_1 \cap \cdots \mathfrak{p}_s$, ht $\mathfrak{p}_i = n$, in $\bar{R}$ was determined. It follows from [6, Theorem 4.4] that if $\mathfrak{p}_i \leftrightarrow P_i \in \mathbb{P}^n$ and if the points $\{P_1,\dots,P_s\}$ are in generic $(s-1)$-position but *not* generic s-position then the conductor is again a power of the homogeneous maximal ideal of R. In this case if d_0 is the least integer d for which $\binom{d+n}{n} \geqslant s$ and M is the homogeneous maximal ideal of R then the conductor is M^{d_0+1}.

Thus, our goal in this section will be to completely describe the occurrence of Gorenstein singularities when we consider the reduced varieties consisting of s-lines through the origin in A^{n+1} which are in generic $(s-1)$-position but not in generic s-position (see Examples 7 and 8 of Section 1c).).

We continue using the notation introduced in Section 2; $R = R_0 \oplus R_1 \oplus \cdots$ $(R_0 = k)$. Then $R/\ell = \oplus \sum_{i=0}^{d_0} R_i$ and $\dim_k R_i =$ rk $G_i(P_1,\dots,P_s)$.

PROPOSITION 8. *Let $\{P_1,\dots,P_s\}$ be points of $\mathbb{P}^n$ in generic $(s-1)$-position but not in generic s-position and let d_0 be the least integer d for which $\binom{d+n}{n} \geqslant s$. Then*

(i) *for* $0 \leqslant i \leqslant d_0 - 1$, rk $G_i(P_1,\dots,P_n) = \binom{i+n}{n}$;

(ii) *for* $\binom{d_0-1+n}{n} < s < \binom{d_0+n}{n}$, rk $G_{d_0}(P_1,\dots,P_s) = s-1$;

(iii) *for* $s = \binom{d_0+n}{n}$, *rk* $G_{d_0}(P_1,\dots,P_s) = \binom{d_0+n}{n} - 1 = s-1$.

Proof. (i) $G_i(P_1,\dots,P_s)$ is an $s \times \binom{i+n}{n}$-matrix. Since $P_1,\dots,P_s$ are in generic $(s-1)$-position, every $(s-1) \times \binom{i+n}{n}$-submatrix has maximal rank.

Suppose $\binom{i+n}{n} > (s-1)$, then $s \leqslant \binom{i+n}{n}$. But d_0 was chosen as the least integer, d, for which $s \leqslant \binom{d+n}{n}$ and $i \leqslant d_0 - 1$. Thus, we must conclude that $\binom{i+n}{n} \leqslant (s-1)$ for $0 \leqslant i \leqslant d_0 - 1$ and so every $(s-1) \times \binom{i+n}{n}$ submatrix has rank $= \binom{i+n}{n}$. But then $G_i(P_1,\dots,P_s)$ must also have rank $= \binom{i+n}{n}$.

(ii) In any case we know that rk $G_{d_0} \geqslant s-1$. If rk $G_{d_0} = s$ then from Proposition 3 it follows that $\{P_1,\dots,P_s\}$ are in generic s-position, contrary to our assumption. Thus, rk $G_{d_0} = s-1$.

(iii) As in (ii) we know that rk $G_{d_0} \geqslant s-1$. If rk $G_{d_0} = s$ then rk $G_{d_0+l} = s$, $\forall l \geqslant 0$ and so, using (i), rk G_i is maximal for every i. But, this would imply that $\{P_1,\dots,P_s\}$ are in generic s-position, contrary to our assumption. Thus, we conclude rk $G_{d_0} = s-1$ in this case also.

To find the Gorenstein singularities we proceed as in Section 3.

Case 1. $\binom{d_0-1+n}{n} < s < \binom{d_0+n}{n}$.

In this case (which corresponds to (ii) of Proposition 8 we have $\dim_k \bar{R}/\ell = (d_0+1)s$ and $\dim_k R/\ell = \binom{d_0+n}{n+1} + (s-1)$.

As before, to find the Gorenstein singularities we will want to know when

$\dim_k \bar{R}/\ell = 2 \dim_k R/\ell$ and so we first seek to find out when this equality does not hold.

Notice that as s increases by 1 over the interval $\binom{d_0-1+n}{n}$ to $\binom{d_0+n}{n}$ the $\dim_k \bar{R}/\ell$ increases by d_0+1, while $\dim_k R/\ell$ increases only by 1. Hence, if we can show that for the first s in this interval we have $\dim_k \bar{R}/\ell > 2 \dim_k R/\ell$ then there are no Gorenstein rings for s's in that open interval.

So, let $s = \binom{d-1+n}{n} + 1$; we want to know when

$$(d+1)\left[\binom{d-1+n}{n}+1\right] > 2\left[\binom{d+n}{n+1}+\binom{d-1+n}{n}\right]. \qquad (\dagger)$$

After some calculation we find that ($\dagger$) is valid iff

$$\begin{aligned}(d+n-1)!\,[(n+1)(d+1)-2(n+1)-2(d+n)]\\ > -[(n+1)!][(d-1)!](d+1).\end{aligned}$$

This last inequality will certainly be valid if

$$(n+1)(d+1)-2(n+1)-2(d+n) \geqslant 0. \qquad (*)$$

Since we may as well assume $n \geqslant 2$ (there are no points of $\mathbb{P}^1$ that are in generic $(s-1)$-position but not generic s-position), one sees immediately that ($*$) holds iff $d \geqslant (3+1/n)/(1-1/n)$.

Thus, ($\dagger$) holds for

$$\begin{array}{lll} n=2 & \text{and} & d\geqslant 7,\\ n=3 & \text{and} & d\geqslant 5,\\ n=4 & \text{and} & d\geqslant 5,\\ n\geqslant 5 & \text{and} & d\geqslant 4.\end{array}$$

Hence, the only possibilities for Gorenstein singularities in Case 1 can occur for

$$\begin{array}{lll} n=2 & \text{and} & d\leqslant 6,\\ n=3 & \text{and} & d\leqslant 4,\\ n=4 & \text{and} & d\leqslant 4,\\ n\geqslant 5 & \text{and} & d\leqslant 3.\end{array}$$

Notice that for $d=1$, $s<n+1$. We ignore this case since then all the lines really lie in an A^n.

Some simple calculations reveal that the only possibilities for Gorenstein singularities are:

In A^3 when $s = 9,17,22$.

In $A^{n+1} (n>2)$, when $s = 2(n+1)$ or $s = [(n+3)(n+2)/2]-1$.

Case 2. $s = \binom{d_0+n}{n}$.

In this case (which corresponds to (iii) of Proposition 8) we have $\dim_k \bar{R}/\mathscr{C} = s(d_0+1) = (d_0+1)\binom{d_0+n}{n}$ and $\dim_k R/\mathscr{C} = \binom{d_0+n}{n+1} + \binom{d_0+n}{n} - 1$.

For such an s we have $\dim_k \bar{R}/\mathscr{C} = s(d+1) = (d+1)\binom{d+n}{n}$ and $\dim_k R/\mathscr{C} = \binom{d+n}{n+1} + \binom{d+n}{n} - 1$.

So, the question now is for which d, n we have

$$(d+1)\binom{d+n}{n} = 2\left[\binom{d+n}{n+1} + \binom{d+n}{n} - 1\right];$$

i.e.,

$$(d-1)\binom{d+n}{n} = 2\binom{d+n}{n+1} - 2.$$

This happens iff

$$(d+n)!\,[(n+1)(d-1) - 2d] = -2[(n+1)!][d!]. \tag{†}$$

For (†) to hold we must have

$$(n+1)(d-1) - 2d < 0 \tag{*}$$

Since $d > 1$ is the only relevant d, (*) is true iff

$$n + 1 < \frac{2d}{d-1};$$

i.e.,

$$n < \frac{d+1}{d-1} = \frac{1+1/d}{1-1/d}$$

But, since $n < 2$ is not interesting for us, we are reduced to considering only the case $n = 2$, $d = 2$. (We again ignore the case $d = 1$ for then $s = n+1$ and these $n+1$ lines of A^{n+1} really lie in A^n.)

So, in this case $s = \binom{4}{2} = 6$ and we have

$$6 \cdot (3) = \dim \bar{R}/\mathscr{C} = 18$$

and

$$\binom{4}{3} + \binom{4}{2} - 1 = \dim R/\mathscr{C} = 9;$$

i.e.,

we have a Gorenstein singularity in this case.

Thus, for Case 2 we obtain a Gorenstein singularity only in A^3 for six points in generic 5-position, but not in generic 6-position.

Putting all these possibilities together we find that the only times there may be Gorenstein singularities in A^{n+1} corresponding to lines in generic $(s-1)$-position but not generic s-position are:

$$A^{n+1}(n \geqslant 2),\ s = 2(n+1), \qquad \text{or} \quad s = \frac{(n+3)(n+2)}{2} - 1,$$

$$\text{and } A^3 \text{ for } s = 17, 22.$$

Obviously, these last two possibilities in A^3 seem anomalous. In fact, we now show that there do not exist 17 (respectively 22) points of $\mathbb{P}^2$ in generic 16-position (respectively 21-position) not in generic 17-position (respectively 22-position). These arguments are due to Joseph Harris.

17 *points*. Suppose there are 17 points of $\mathbb{P}_2$ in generic 16-position but not in generic 17-position. Then, there is no quartic through any 16 of these points and every quintic containing 16 of the points contains all 17.

Now, suppose there is a quartic containing 15 of the points, then, the remaining two points cannot lie on this quartic. We may choose a line containing one of these two points and missing the other. This line and the quartic give a quintic which contains 16, but not 17, of the points and this is impossible.

So, we are forced to assume that there is no quartic through any 15 of the points, i,e., the 17 points are in generic 15-position.

Pick any two of the 17 points and consider the line through them, and then pick any 14 of the remaining 15 points. There is a quartic through these 14 points not containing any of the other three points. The line and the quartic then give a quintic containing 16 of the points, hence all 17. Since this 17th point is not on the quartic, it must be on the line, i.e., all 17 points must lie on the line. This is a contradiction.

22 *points*. Suppose that there are 22 points of $\mathbb{P}^2$ in generic 21-position but not in generic 22-position.

This means that there is no quintic through any 21 of these points and any sextic containing 21 of the points contains all 22.

But, there is always a quintic through 20 points and we choose one through 20 of the points. We can then pick a line through one of the remaining two points which misses the other. This gives us a sextic through 21 points not containing all 22, which is a contradiction.

These two situations raise the possibility that for $n \geqslant 3$ there may not exist points in generic $(s-1)$-position which are not in generic s-position, for $s = 2(n+1)$ and $s = [(n+3)(n+2)/2] - 1$.

$s = 2(n+1)$ $(n \geqslant 3)$: Choose t_i, $1 \leqslant i \leqslant 2n+2$ distinct elements of k and consider the points $P_i = [1 : t_i : t_i^2 : \cdots : t_i^n] \in \mathbb{P}^n$.

CLAIM. *The points $\{P_1, \ldots, P_{2n+2}\}$ are in generic $2n+1$-position in $\mathbb{P}^n$ but not in generic $2(n+1)$-position.*

Proof. First note that when $n \geqslant 3$ then $\binom{2+n}{n} > 2(n+1)$, so we need only consider the matrices $G_1(P_1, \ldots, P_{2(n+1)})$ and $G_2(P_1, \ldots, P_{2(n+2)})$.

Now

$$G_1(P_1, \ldots, P_{2(n+2)}) = \begin{bmatrix} 1 & t_1 & \cdots & t_1^n \\ 1 & t_2 & \cdots & t_2^n \\ \vdots & & & \\ 1 & t_{2(n+1)} & \cdots & t_{2(n+1)}^n \end{bmatrix}$$

and it is clear that any $(2n+1) \times (n+1)$ submatrix has rank $= n+1$; since such a matrix has a Vandermonde submatrix.

Now

$$G_2(P_1, \ldots, P_{2(n+1)}) = \begin{bmatrix} 1 & t_1^2 & \cdots & t_1^{2n} & \\ 1 & t_2^2 & \cdots & t_2^{2n} & * \\ \vdots & & & & \\ 1 & t_{2(n+1)} & \cdots & t_{2(n+1)}^{2n} & \end{bmatrix}$$

where the columns of $(*)$ will just be equal to some of the earlier columns. These first $2n+1$ columns are linearly independent so

$$\operatorname{rk} G_2(P_1, \ldots, P_{2(n+1)}) = 2n+1$$

and so the points $\{P_1, \ldots, P_{2(n+1)}\}$ are *not* in generic $2(n+1)$-position. But, for any $2n+1$ rows, the resulting submatrix contains a $(2n+1) \times (2n+1)$ Vandermonde matrix and so has $\operatorname{rk} = 2n+1$; i.e., $\{P_1, \ldots, P_{2(n+1)}\}$ are in generic $(2n+1)$-position, as was to be shown.

Notice that by analogy with the case of six points on an irreducible conic in $\mathbb{P}^2$, these $2(n+2)$ points lie on the curve in $\mathbb{P}^n$ which is the intersection of the $(n-1)$ irreducible quadric hypersurfaces

$$x_1^2 = x_2 x_0, \qquad x_2^2 = x_1 x_3, \ldots, x_{n-1}^2 = x_{n-2} x_n.$$

For $n \geqslant 3$ we have not been able to show that there are points in generic $(s-1)$-position but not in generic s-position when $s = [(n+3)(n+2)/2] - 1$. Presumably these points should exist on an appropriate intersection of cubic hypersurfaces in $\mathbb{P}^n$.

We shall now summarize the results of the calculations in this section.

THEOREM 9. *Let R be the coordinate ring of $s > n+1$ lines through the origin in A^{n+1} $(n \geq 2)$. Suppose that these lines are in generic $(s-1)$-position but not in generic s-position.*

Then R is a Gorenstein ring iff $s = 2(n+1)$ or $s = [(n+3)(n+2)/2] - 1$.

5. ORDINARY SINGULARITIES IN GENERAL

So far we have only discussed the coordinate ring of s distinct lines through the origin in A^{n+1}. We would now like to turn our attention to a general reduced curve in A^{n+1} with an ordinary singularity at the origin (i.e., distinct tangents with linear braches, see [6]) and compare the nature of this singularity with the singularity described by the intersection of the lines of the tangent cone. From what we have already seen, we would do well to restrict ourselves to the situation where the tangent lines to the singularity are in uniform position. This restriction does not seem unnatural in view of the difficulties one encounters in the study simply of lines not in uniform position.

In fact, under these circumstances it was shown in [6] that if $(A, \mathfrak{m})$ is the local ring at the ordinary singularity and $G(A)$ is the associated graded ring then $G(A)$ is precisely the coordinate ring of the lines of the tangent cone. We believe that the C–M type of A is the same as the C–M type of $G(A)$ in this case; but we can prove this only for Gorenstein singularities, which we now proceed to do.

We recall that if B is a semi-local ring then $G(B)$, the associated graded ring, is $\oplus J^n/J^{n+1}$, where J is the Jacobson radical of B. If I is any ideal of B then $G(I)$ denotes the homogeneous ideal of $G(B)$ generated by the initial forms of the elements of I. Then, if $G(I)_n$ denotes the homogeneous elements of degree n in $G(I)$ we have

$$G(I)_n = (I \cap J^n) + J^{n+1}/J^{n+1}.$$

Also, if $f(n) = l_{B/J^n}(B/J^n)$, then f is called the Hilbert–Samuel function of B.

Now, notice that $G(B)/G(J^n) \simeq \oplus_{i=0}^{n-1} J^i/J^{i+1}$ has a natural structure as a graded B/J-module. We have,

LEMMA 1. $l_{B/J^n}(B/J^n) = l_{B/J}(G(B)/G(J^n))$.

Proof. B/J^n has a filtration induced by $B \supset J \cdots \supset J^{n-1} \supset J^n$, and $l_{B/J^n}(B/J^n) = \sum_{i=0}^{n-1} l_{B/J^n}(J^i/J^{i+1})$: Since J^i/J^{i+1} is annihilated by J, $l_{B/J^n}(J^i/J^{i+1}) = l_{B/J}(J^i/J^{i+1})$; and the result follows.

LEMMA 2 [6]. *If A is a reduced local ring of dimension 1 for which $G(A)$ is reduced and if $\bar{A}$ is the integral closure of A (in its total ring of* *quotients) then $G(\bar{A})$ is the integral closure of $G(A)$ (in its total ring of quotients).*

THEOREM 10. *Let $(A, \mathfrak{m})$ be a reduced local ring of dimension 1 for which $G(A)$ is also reduced. Suppose further that the conductor of $G(A)$ in $G(\bar{A})$ is a power of the homogeneous maximal ideal of $G(A)$. Then A is a Gorenstein ring if and only if $G(A)$ is a Gorenstein ring.*

Proof. Let $\mathscr{C}$ be the conductor of A in $\bar{A}$. Now A is a Gorenstein ring if and only if $l(\bar{A}/\mathscr{C}) = 2l(A/\mathscr{C})$. Let $\mathscr{G}$ be the conductor of $G(A)$ in $G(\bar{A})$. It will be enough to prove that $l(G(A)/\mathscr{G}) = l(A/\mathscr{C})$ and that $l(G(\bar{A})/\mathscr{G}) = l(\bar{A}/\mathscr{C})$.

Now in [6, Theorem 2.4] it was proved under the hypothesis of this theorem that if $\mathscr{G} = G(\mathfrak{m})^n$ then $\mathscr{G} = G(J)^n$, J the Jacobson radical of B, and $\mathscr{C} = \mathfrak{m}^n = J^n$.

Now the result follows from Lemma 1.

Let $(A, \mathfrak{m})$ be the local ring of a point on a curve and suppose the embedding dimension of A (i.e., $\dim_{A/\mathfrak{m}} (\mathfrak{m}/\mathfrak{m}^2)$) is $n + 1$. Then the tangent cone at that point is a closed subscheme of A^{n+1} which, if we simply consider it set-theoretically as an algebraic set, consists of s-lines in A^{n+1}.

COROLLARY 1. *Let (A, m) be the local ring at an ordinary singularity of a reduced curve which has embedding dimension $n + 1$ and suppose the tangent cone, as algebraic set, consists of s lines through the origin of A^{n+1} in generic $(s - 1)$-position and generic s-position.*

Then, A is a Gorenstein ring if and only if either $n = 1$ or $n > 1$ and $s = 2$ or $n + 2$.

If the lines of the tangent cone are assumed to be in generic $(s - 1)$-position but not generic s-position and if $G(A)$ is reduced then A is Gorenstein if and only if $s = 2(n + 1)$ or $s = [(n + 3)(n + 2)/2] - 1$ $(n > 1)$.

Proof. In the first instance, the assumption of generic s-position on the lines of the tangent cone implies that $G(A)$ is reduced [6] and consequently $G(A)$ is the coordinate ring of s-lines in generic s-position in A^{n+1}. We have noted (see Section 3) that under the assumption of generic $(s - 1)$-position and generic s-position the conductor of $G(A)$ in $G(\bar{A})$ is a power of the homogeneous maximal ideal of $G(A)$.

We may thus use Theorem 10 and Theorem 7 to complete the proof.

The second conclusion follows similarly from Theorems 10 and 9 and the discussion at the beginning of Section 4.

COROLLARY 2. *Let $(A, \mathfrak{m})$ be the local ring of a non-plane curve at an ordinary singularity with embedding dimension $= n + 1$ and suppose the lines of the tangent cone* (Spec$(G(A))$) *are in generic $(s - 1)$- and generic s-*

position in A^{n+1}. If A is a Gorenstein ring then the conductor of A in $\bar{A}$ is m^2.

Proof. See Corollary to Theorem 7.

6. The Cohen–Macaulay Type of Lines in A^{n+1}

We return to the consideration of s-lines in uniform position passing through the origin in A^{n+1}. We have already characterized those s for which the coordinate ring of the variety is Gorenstein (See Theorem 7). We now attempt to consider other Cohen–Macaulay types.

As earlier we let $R = k[x_0,..., x_n]/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ denote the coordinate ring of s-lines in uniform position passing through the origin in A^{n+1} and let M denote the homogeneous maximal ideal of R. As we mentioned earlier (see Section 3) the conductor of R in $\bar{R}$ (the integral closure of R in the total quotient ring) is M^{d_0},, where d_0 is the least integer for which $s \leqslant \binom{d_0+n}{n}$. We continue with the notation we established in Section 2 to describe the embedding of R in $\bar{R}$.

Now $\mathfrak{p}_i \leftrightarrow P_i \in P^n$ and without loss of generality we may assume the x_0-coordinate of each $P_i = 1$. Then, under the embedding $R \to \bar{R}$, $x_0 \to (T_1,..., T_s)$ so, x_0 is not a zero-divisor in R. Thus, to discover the C–M type of R it will be sufficient to find the number of irreducible ideals in an irredundant irreducible decomposition of $(x_0) \subset R$. (See Section 1(a)) and recall that R_N is regular for any maximal ideal $N \neq M$.)

If we use $\bar{R}_m$ to denote the mth homogeneous piece of $\bar{R}$ and R_m to denote the mth homogeneous piece of R, then our observation about the conductor of R in $\bar{R}$ gives that

$$R = R_0 \oplus R_1 \oplus \cdots \oplus R_{d_0-1} \oplus \bar{R}_{d_0} \oplus \bar{R}_{d_0+1} \oplus \cdots \quad (R_0 = k)$$

Thus, to find the C–M type of R we need only calculate (see Theorem 1) $l_{R/x_0R}(\operatorname{ann}(M/x_0R))$. Since $\operatorname{ann}(M/x_0R) \cdot M/x_0R = 0$ we have that $\operatorname{ann}(M/x_0R)$ is an $R/M = k$-module and so its length as R/x_0R-module is equal to its dimension as a k-vector space. Thus, we are reduced to finding the k-space dimension of $\operatorname{ann}(M/x_0R)$. But, $\operatorname{ann}(M/x_0R)$ is a homogeneous ideal so it will be enough to find the dimensions of its homogeneous pieces. Now observe that the mapping $R_t \to R_{t+1}$ $(t \geqslant 0)$ given by $y \to x_0 y$ is injective since x_0 is not a zero-divisor. Thus, $\dim_k(x_0R_t) = \dim_k R_t$. Now if $t \geqslant d_0$ then $\dim_k R_t = \dim_k \bar{R}_t = s$ and so $\dim_k R_t = s$ for $t \geqslant d_0$ and $x_0R_t = R_{t+1}$ for $t \geqslant d_0$. Thus,

$$R/x_0R = R_0 \oplus R_1/x_0R_0 \oplus R_2/x_0R_1 \oplus \cdots \oplus R_{d_0}/x_0R_{d_0-1}.$$

Hence, in computing $[\mathrm{ann}_{R/x_0R}(M/x_0R)]$ we need only consider $y \in (R/x_0)$ of degree $\leqslant d_0$.

To facilitate the discussion we recall the matrices $G_d(P_1,..., P_s)$ we discussed earlier (see Section 1 (c) and Section 2, Theorem 5). The matrix $G_d(P_1,..., P_s)$ is an $s \times \binom{d+n}{n}$ matrix and its columns (which are indexed by the monomials of degree d in $k[x_0,..., x_n]$) give us representatives for a set of generators of R_d. The mapping $x_0: R_d \to R_{d+1}$ can be easily described in terms of these columns, namely if $\mathscr{C}$ is the column of G_d corresponding to the monomial F then $x_0\mathscr{C}$ is the column of R_{d+1} corresponding to the monomial x_0F.

We now claim that if $t < d_0 - 1$ then no element of R_t/x_0R_{t-1} is in $\mathrm{ann}\,(M/x_0R)$. (This fact was observed independently by L. Roberts using different ideas.) To see this, observe that we thus have $t + 1 < d_0$ and so $G_{t+1}(P_1,..., P_s)$, which must have maximal rank since $P_1,..., P_s$ are in uniform position (see Section 1 (c)) must therefore have all of its columns independent.

Now, choose $y \in R_t - x_0R_{t-1}$. Since the columns of G_t are linearly independent and span R_t, we must have that y is a linear combination of the columns of G_t corresponding to those monomials of degree t which do not involve x_0. In order for y to belong to the $\mathrm{ann}\,(M/x_0R)$ we must have $yx_j \in x_0R_t$ for $1 \leqslant j \leqslant n$. But yx_j is a linear combination of the columns of G_t which do not involve x_0. We thus must have a linear relation among the columns of G_t which is absurd since they are independent. Thus, the annihilator can only consist of elements of degree d_0 and $d_0 - 1$ in R/x_0.

On the other hand, if $y \in R_{d_0}$ then $yM \subset \sum_{i=1}^{\infty} R_{d_0+i}$. Since $R_{d_0+i} = x_0R_{d_0+(i-1)}$ (as we observed earlier) we have that $yM \subset (x_0)$; i.e., every homogeneous element of degree d_0 in R/x_0 is in $\mathrm{ann}\,(M/x_0R)$. Since the degree d_0 part of R/x_0 is $R_{d_0}/x_0R_{d_0-1}$ we have that

$$\begin{aligned}\dim_k \mathrm{ann}\,(M/x_0R) = {} & (\dim_k R_{d_0}) - \dim_k(x_0R_{d_0-1}) \\ & + \dim_k (\text{degree } (d_0 - 1)\text{-part of ann } (M/x_0R)).\end{aligned}$$

We shall denote this last summand by h.

Since $\dim_k R_{k_0} = s$ and $\dim_k(x_0R_{d_0-1}) = \dim_k R_{d_0-1} = \binom{d_0+n-1}{n}$ we have that $\dim_k \mathrm{ann}\, M/x_0R = s - \binom{d_0+n-1}{n} + h$. Clearly, the only difficulty is to find h.

We first observe that the preceeding discussion which showed that no homogeneous element of R/x_0R of degree $< d_0 - 1$ was in $\mathrm{ann}\,(M/x_0R)$ would also show that no homogeneous element of degree $(d_0 - 1)$ was in the annihilator *if* we know that the columns of G_{d_0} where also linearly independent. This occurs when $s = \binom{d_0+n}{n}$. We have thus shown:

PROPOSITION 11. *If R is the coordinate ring of $s = \binom{d+n}{n}$ lines through* *the origin of* A^{n+1}, *which are in uniform position, then the C–M type of R is* $\binom{d+n}{n} - \binom{d+n-1}{n} = \binom{d+n-1}{n-1}$.

Notice that when $d = 1$ in this proposition, i.e., when the conductor of R in $\bar{R}$ is M, we are in the case where R is seminormal (see [6, Sect. 4, Remark]); i.e., we are considering $n+1$ lines in A^{n+1} passing through the origin, which are in uniform position (i.e., linearly independent).

We can make another general observation about the number h.

CLAIM. $\operatorname{ann}(M/(x_0))_{d-1} \subsetneq (R/(x_0))_{d-1}$.

Proof. We may as well assume $s < \binom{d_0+n}{n}$ and so $G_{d_0}(P_1,\dots,P_s)$ has $\mathrm{rk} = s < \binom{d_0+n}{n} =$ number of columns of G_{d_0}. Now the columns of $G_{d_0}(P_1,\dots,P_s)$ corresponding to monomials involving x_0 are linearly independent, since they are x_0-multiples of the linearly independent columns of G_{d_0-1}. These are $\binom{d_0+n-1}{n} < s$ columns of $G_{d_0}(P_1,\dots,P_s)$, hence there is a column of G_{d_0} independent of the columns corresponding to the monomials involving x_0. Let this column correspond to the monomial F. Then $F = x_i F'$ where $i \neq 0$ and F' is a monomial of degree $d_0 - 1$ not involving x_0. Now consider the column of G_{d_0-1} corresponding to F'. This gives a non-zero element of $(R/x_0)_{d_0-1}$ (since the columns of G_{d_0-1} are independent and F' does not involve x_0) and this element does not belong to the annihilator since it does not annihilate x_i. We summarize these observations.

PROPOSITION 12. *If R is the coordinate ring of s lines through the origin of* A^{n+1}, *in uniform position, then the C–M type of R is*

$$s - \binom{d_0+n-1}{n} + h,$$

where d_0 *is the least integer for which* $\binom{d_0+n}{n} \geqslant s$ *and* $0 \leqslant h \leqslant \binom{d_0+n-1}{n} - \binom{d_0+n-2}{n} - 1 = \binom{d_0+n-2}{n-1} - 1$.

Proposition 11 gives us an instance where $h = 0$. The next proposition gives another such case.

PROPOSITION 13. *If R is the cordinate ringe of* $s = \binom{d+n}{n} - 1$ *lines in uniform position in* A^{n+1} *then the C–M type of R is* $\binom{d+n}{n} - 1 - \binom{d+n-1}{n} = \binom{d+n-1}{n-1} - 1$.

Proof. It will be sufficient to show that no non-zero element of $(R/x_0)_{d-1}$ is in $\operatorname{ann}(M/x_0)$.

Consider the matrix G_d, it is an $s \times (s+1)$-matrix of $\mathrm{rk} = s$. The columns of G_d are labeled by the monomials of degree d in $k[x_0,\dots,x_n]$ and the set of columns corresponding to monomials involving x_0 are linearly independent.

Thus, these columns can be extended to a basis for the column space of G_d, which has dimension s. Then, there is a basis for the column space of G_d obtained by omitting one of the columns of G_d which corresponds to a monomial in $k[x_0,..., x_n]$ not involving x_0.

Case 1. We can omit a column of G_d corresponding to a monomial in $k[x_0,..., x_n]$ not involving x_0 *and* x_i $(i > 0)$ and obtain a basis for the column space of G_d.

We will show in this case that no non-zero element of $(R/x_0)_{d-1}$ is in $\operatorname{ann}(M/x_0)$.

Since the columns of G_{d-1} (independently) generate R_{d-1}, a non-zero element of $(R/x_0)_{d-1}$ can be represented by a linear combination of the columns of G_{d-1} corresponding to monomials of degree $(d-1)$ in $k[x_0,..., x_n]$. We abusively write such an element as a linear combination of the monomials which denote those columns; i.e., $\bar{f} \in (R/x_0)_{d-1}$ is represented by $f = \sum c_{i_1, i_2,..., i_n} \; x_1^{i_1} \cdots x_n^{i_n}$, $\sum_{j=1}^n i_j = d-1$. If $f \in \operatorname{ann}(M/x_0)$ then $fx_i \in x_0 R_{d-1}$; i.e., in terms of the columns of G_d, $\sum c_{i_1, i_2,..., i_n} x_1^{i_1} \cdots x_n^{i_n} x_i$ is in the space spanned by the columns of G_d corresponding to all monomials involving x_0. This contradicts our assumption that these columns were linearly independent, unless $f = 0$.

Case 2. We can omit a column of G_d corresponding to a monomial in $k[x_0,..., x_n]$ not involving x_0 yet involves all x_i, $1 \leqslant i \leqslant n$, and obtain a basis for the column space of G_d.

We shall also show in this case that no non-zero element of $(R/x_0)_{d-1}$ is in $\operatorname{ann}(M/x_0)$.

Let us suppose that the column we have omitted corresponds to $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\sum_{i=1}^n \alpha_i = d$ and $\alpha_i \geqslant 1$, $1 \leqslant i \leqslant n$. Let $\bar{f} \neq 0 \in (R/x_0)_{d-1}$, be in $\operatorname{ann}(M/x_0)$ where, as earlier, we abusively write $f = \sum c_{i_1,..., i_n} x_1^{i_1} \cdots x_n^{i_n}$, $\sum_{j=1}^n i_j = d-1$. We must have $fx_j \in x_0 R_{d-1}$ for each j, $1 \leqslant j \leqslant n$; i.e., in terms of the columns of G_d, $\sum c_{i_1, i_2,..., i_n} x_1^{i_1} \cdots x_1^{i_1} \cdots x_j^{i_j+1} \cdots x_n^{i_n}$ is in the space spanned by the columns of G_d corresponding to the monomials involving x_0. Thus, in order not to contradict our assumption that the columns of G_d, with the $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ column omitted, are linearly independent, we must have that

$$c_{\alpha_1, \alpha_2,..., \alpha_j - 1,..., \alpha_n} \neq 0 \qquad \forall j, 1 \leqslant j \leqslant n.$$

In particular we have that $c_{\alpha_1, \alpha_2 - 1, \alpha_3,..., \alpha_n} \neq 0$; i.e., $f = c_{\alpha_1, \alpha_2 - 1,..., \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n} + f'$.

But, since $fx_1 = 0$ in $(R/x_0)_d$ we obtain that the column of G_d corresponding to $x_1^{\alpha_1+1} x_2^{\alpha_2-1} x_3^{\alpha_3} \cdots x_n^{\alpha_n}$ is a linear combination of the remaining columns. By a simple dimension count this implies that the remaining columns of G_d are a basis for the column space of G_d.

Thus, we have shown that when we can omit the column of G_d corresponding to the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ $(\alpha_i \geqslant 1)$ and the remaining columns are independent then the existence of a non-zero element of $\operatorname{ann}(M/x_0)$ in $(R/x_0)_{d-1}$ implies that we can omit the column of G_d corresponding to $x_1^{\alpha_1+1} x_2^{\alpha_2-1} x_3^{\alpha_3} \cdots x_n^{\alpha_n}$ and have the remaining columns be independent. By iterating this procedure we can thus show we can omit the column of G_d corresponding to $x_1^{\alpha_1+\alpha_2} x_3^{\alpha_3} \cdots x_n^{\alpha_n}$ and have the remaining columns be independent. But this brings us back to Case 1 and so $\operatorname{ann}(M/x_0)_{d-1} = 0$.

Herzog has given a very general theorem concerning the Cohen–Macaulay type which is sometimes useful for us.

PROPOSITION 14. (See [5, p. 28, Satz 3.6]). *If R is a local C–M ring with finite integral closure $\bar{R}$ and conductor $\mathscr{C}$ then the*

$$\text{C–M type of } R \leqslant l_R(\bar{R}/\mathscr{C}) - 2l_R(R/\mathscr{C}) + 1.$$

If we apply this proposition to the coordinate ring of seven lines of A^3 in uniform position, R, we find that the C–M type is $\leqslant 2$. By Theorem 7 we know that R cannot be Gorenstein, so the C–M type is exactly 2. Notice that Proposition 12 would only give that the C–M type of R is $1 + h$ where $0 \leqslant h \leqslant 2$. This also provides an example to show that h need not always be 0.

7. THE HILBERT POLYNOMIAL OF CURVES WITH REDUCED TANGENT CONE

We have shown (§ 5, Lemma 1) that for any local ring $(A, \mathscr{m})$, $\dim_{A/\mathscr{m}^n}(A/\mathscr{m}^n) = \dim_k(G(A)/G(\mathscr{m})^n)$ and so to compute the Hilbert–Samuel function (and so the Hilbert–Samuel polynomial) of curves with reduced tangent cone it is enough to compute the corresponding function (polynomial) for the local ring $R_m = G(A)_{G(m)}$ of the lines of the tangent cone.

PROPOSITION 15. *Let R be the coordinate ring of s lines through the origin in uniform position in A^{n+1}. Let M be the homogeneous maximal ideal of R and suppose the embedding dimension of R_M is $n+1$. The Hilbert–Samuel polynomial of R_M is*

$$f(x) = sx + \left[\binom{d_0 + n}{n+1} - d_0 s \right],$$

where d_0 is the least integer for which $s \leqslant \binom{d_0+n}{n}$.

Proof. The multiplicity of R_M is clearly s and so $f(x) = sx + a$ is the Hilbert–Samuel polynomial. It remains only to compute a. Now $f(t) = \dim_k R/M^t$ for $t \gg 0$. Thus, for $t \gg 0$, $f(t) = \dim_k (\sum_{i=0}^{t-1} R_i)$. But, since the lines are in uniform position we can compute $\dim_k R_i$ as $rk\, G_i$. Thus, for $t > d_0$ we have

$$f(t) = \left[\sum_{i=0}^{d_0-1} \binom{i+n}{n} \right] + (t - d_0)\, s$$

$$= \binom{d_0+n}{n+1} + (t-d_0)\, s = ts + a.$$

$$\therefore\ a = \binom{d_0+n}{n+1} - d_0 s,$$

as was to be shown.

COROLLARY. *If R is as in Proposition* 15 *and in addition R is a Gorenstein ring then*

$$f(x) = sx - \binom{s}{2} \qquad \textit{if} \quad n = 1,$$

and

$$f(x) = sx - s, \qquad \textit{if} \quad n > 1.$$

Proof. If we are considering plane curves then $n = 1$ and $d = s - 1$ and the result follows.

If $n > 1$ then from Theorem 7 and its Corollary we have $d = 2$ and $s = n + 2$.

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