A GENERALIZED SUPER FERMAT PROBLEM FOR BINARY FORMS

GLEB NENASHEV, BORIS SHAPIRO, AND MICHAEL SHAPIRO

To James Joseph (Sylvester),

Abstract. The space $Pol_d \simeq \mathbb{C}P^d$ of all complex-valued binary forms of degree d has a standard stratification, each stratum of which contains all forms with the multiplicities of their distinct roots given by a fixed partition $\mu \vdash d$. For each such stratum S_μ, we introduce its index ℓ_μ which is the minimal number of projectively dependent pairwise distinct points on S_μ, i.e., points whose projective span has dimension smaller than $\ell_\mu - 1$. We obtain several results about ℓ_μ and apply them to the super Fermat problem for binary forms.

1. Introduction

In what follows by a form we will always mean a binary form. The standard stratification of the d-dimensional projective space Pol_d of all complex-valued binary forms of degree d (considered up to a non-vanishing constant factor) according to the multiplicities of their distinct roots is a well-known and widely used construction in mathematics, see e.g. [Ar, Va, KhSh]. Its strata denoted by S_μ are enumerated by all partitions $\mu \vdash d$. Cohomology of S_μ with different coefficients appear in different contexts and were intensively studied over the years, see [Va]. It seems however that the question about the dimensions of the secant varieties to S_μ, as well as Problem 1 below having an immediate application to the so-called super Fermat problem for binary forms which can be traced back to J. J. Sylvester[1], have not been earlier discussed. (When working with binary forms of degree d, we will consider their zero loci as positive divisors of degree d on $\mathbb{C}P^1$.)

Definition 1. Given a positive-dimensional quasiprojective variety $V \subset \mathbb{C}P^d$, we define its *secant degeneracy index* ℓ_V as the minimal positive integer ℓ such that there exists ℓ distinct points on V which are projectively dependent, i.e. whose projective span has dimension at most $\ell - 2$.

Obviously, $3 \leq \ell_V \leq d+2$. The upper bound is attained, for example, for a parabola in the plane or, more generally, for a rational normal curve $V \subset \mathbb{C}P^d$. On the other hand, if V contains $\mathbb{C}P^1 \setminus \{\text{finite set}\}$, then $\ell_V = 3$. To the best of our knowledge, the invariant ℓ_V has not been previously studied. However a related question about the uniqueness of representation of generic forms of subgeneric rank

2010 *Mathematics Subject Classification.* Primary 14M99, Secondary 11E25, 11P05.

Key words and phrases. strata of the classical discriminant, secant varieties, super Fermat problem.

[1] James Joseph Sylvester was born as James Joseph in a wealthy Jewish family in London in 1814. He took an extra name Sylvester due to US immigration requirements and in order to avoid problems related to his ethnic origin which however has not helped him that much.

as sums of powers of linear forms has been studied in e.g. [COV]. The principal question considered in the present paper is as follows.

Problem 1. *For a given partition $\mu \vdash d$, calculate/estimate its secant degeneracy index $\ell_\mu := \ell_{S_\mu}$.*

For a given partition μ, the equation

$$f_1 + f_2 + \cdots + f_{\ell_\mu} = 0, \tag{1}$$

is called the *minimal secant degeneracy relation* for μ. A solution of the latter equation is a collection of pairwise non-proportional forms from S_μ satisfying (1).

We will now explain that Problem 1 is a natural generalization of the so-called super Fermat problem for binary forms. Recall that a super Fermat equation

$$x_1^k + x_2^k + \cdots + x_l^k = x^k, \tag{2}$$

is an analog of the Fermat equation with l terms in the left-hand side where $l > 2$. It has been studied already by L. Euler [Eu] who conjectured that, for any positive integer k, the number $G_0(k)$ defined as the minimal value of l for which (2) has a solution in positive integers, satisfies the inequality $G_0(k) \geq k$.

We will refer to the question of determining the number $G_0(k)$ as the *super Fermat problem for integers*. Euler's conjecture was disproved two centuries later by R. Frye who found the counterexample $95800^4 + 217519^4 + 414560^4 = 422481^4$, for $k = 4$.

N. Elkies found that $2682440^4 + 15365634^4 + 18796760^4 = 20161567 3^4$ which also shows that $G_0(4) = 3$ and disproves Euler's conjecture, see [El]. A little earlier Lander and Parkin [LaPa] found the counterexample to Euler's conjecture for $k = 5$, namely $27^5 + 84^5 + 110^5 + 133^5 = 144^5$. Thus, $G_0(5) = 3$ or 4.

For higher values of k, Euler's conjecture is still open. For $k = 6$, an interesting example $74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6 = 1141^6$ is contained in [LaPaSe]; it shows that $3 \leq G_0(6) \leq 7$. There are no corresponding examples for $k > 6$.

An analog of the super Fermat problem for binary forms (and other classes of functions) has also been studied at least since the late 19-th century, see [Li]. The main question under consideration in this area is as follows.

Problem 2. *Given positive integers k, d, find/estimate the minimal number $\ell(d, k)$ such that there exist pairwise non-proportional binary forms $f_1, f_2, \ldots, f_{\ell(d,k)}$ of degree d satisfying the super Fermat equation*

$$f_1^k + f_2^k + \cdots + f_{\ell(d,k)}^k = 0. \tag{3}$$

Obviously, $\ell(d, 1) = 3$ and $\ell(1, k) = k + 2$. Substituting $x \mapsto x^d$ and $y \mapsto y^d$, in a solution for $d = 1$, we conclude that $\ell(d, k) \leq k + 2$. A known result of Liouville implies that $\ell(d, k) > 3$ for $k \geq 3$, see [Li] and [Ri].

As an illustration, let us briefly discuss the case $d = 2$. Pythagoras' theorem inspires the example

$$(x^2 + y^2)^2 + (i(x^2 - y^2))^2 + (2ixy)^2 = 0,$$

giving $\ell(2, 2) = 3$ which implies $\ell(d, 2) = 3$ for $d \geq 2$. An identity

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

FIGURE 1. Two main heroes of this story: Professor J. J. Sylvester and Professor B. Reznick. They look rather alike, don't they?

discovered by S. Ramanujan in 1913 implies that $\ell(2,3)=4$ which, in its turn, gives $\ell(d,3)=4$ for $d\geq 2$. Another example of this kind which can be found in [Re] is as follows

$$(x^2+xy-y^2)^3+(x^2-xy-y^2)^3=2(x^2)^3-2(-y^2)^3.$$

Next $\ell(2,4)=\ell(2,5)=4$ (implying $\ell(d,4)=4$ and $\ell(d,5)=4$, for $d\geq 2$) which is proven by

$$(x^2+y^2)^4+(\omega x^2+\omega^2y^2)^4+(\omega^2x^2+\omega y^2)^4=18(xy)^4$$

(WHAT IS ω HERE) and

$$\sum_{j=0}^{3}(-1)^j(i^jx^2+\sqrt{-2}xy+i^{-j}y^2)^5=0.$$

Additionally in [Re], B. Reznick shows that $\ell(2,6)=\ell(2,7)=5$ which is proven by

$$\sum_{j=0}^{3}\left(i^{-j}x^2+\sqrt{-\frac{2}{5}}xy+i^ky^2\right)^6=-\frac{5632}{125}x^6y^6$$

and

$$\sum_{j=0}^{3}\left(i^{-j}x^2+\sqrt{-\frac{6}{5}}xy+i^ky^2\right)^7=-\frac{2^{23/2}3^{1/2}13}{5^{7/2}}ix^7y^7.$$

[Re] contains the equalities showing that $\ell(2,9)\leq 6$ and $\ell(2,14)=6$. The latter fact follows from the miraculous identity $\sum_{j=0}^{5}q_j^{14}(x,y)=0$ where $q_k(x,y)=\zeta_5^kx^2+ixy+\zeta_5^{-k}y^2$ for $0\leq k\leq 4$ and $q_5(x,y)=\sqrt{-5}xy$. B. Reznick has also shown that $\ell(2,2k)\leq k+2$ and $\ell(2,2k+1)\leq k+3$ (unpublished).

In connection with $\ell(d,k)$, the following question looks very natural.

Problem 3. *Determine* $L(k)=\lim_{d\to\infty}\ell(d,k)$.

According to [Ha],

$$\frac{1}{2}+\sqrt{\left(k+\frac{1}{4}\right)}<L(k)\leq\sqrt{4k+1}. \tag{4}$$

Below we slightly generalize Problem 2 and discuss its relations with Problem 1. Motivated by Problem 2, we sugggest its natural generalization as follows.

Problem 4. *For a given partition $\mu \vdash d$, find/estimate the minimal number ℓ_μ such that there exist pairwise non-proportional binary forms $q_1, q_2, \dots, q_{\ell_\mu}$ of degree d satisfying the generalized super Fermat equation*

$$q_1 + q_2 + \cdots + q_{\ell_\mu} = 0. \tag{5}$$

Obviously, ℓ_μ is exactly the secant degeneracy index of μ introduced above and Problem 2 corresponds to the case of rectangular partitions $(k^d) \vdash kd$.

The structure of the paper is as follows. In § 2 we present our results related to the secant degeneracy index ℓ_μ. In § 3 we apply these results to the (generalized) super Fermat problem for binary forms. Finally, in § 4 we state a number of open problems and conjectures.

Acknowledgements. The second author is grateful to Professor B. Reznick of UIUC for numerous discussions of the topic. The third author wants to acknowledge the hospitality of the Mathematics Department, Stockholm University in September-October 2015.

2. On secant degeneracy index

2.1. **General results on ℓ_μ.** For a given positive-dimensional quasiprojective $V \subset \mathbb{C}P^d$, let us additionally introduce the number $\ell_{\bar V}$ which is the secant degeneracy index of the projective closure $\bar V$. Recall the notion of the refinement partial order "$\succ$" on the set of all partitions of a given positive integer d. Namely, $\mu' \succ \mu$ in this order if μ' is obtained from μ by merging of some parts of μ. The unique minimal element of this partial order is $(1)^d$, while its unique maximal element is (d).

Proposition 1. *The number $\ell_{\bar\mu}$ is monotone non-decreasing in the refinement partial order. In other words, if $\mu' \succ \mu''$, then $\ell_{\bar\mu'} \geq \ell_{\bar\mu''}$.*

Proof. Obvious, since $\bar S_{\mu'} \subset \bar S_{\mu''}$. □

The following observation is in place here. There are partitions μ for which $\ell_\mu \neq \ell_{\bar\mu}$. For example, for $\mu_d = (2d+1, 2d, d, d)$, $\ell_{\bar\mu_d} = 3$, but ℓ_{μ_d} grows to infinity when $d \to \infty$. However we conjecture the following.

Conjecture 1. *For any partition $\mu \vdash d$ there exists $\mu' \succeq \mu$ such that $\ell_{\bar\mu} = \ell_{\mu'}$.*

Problem 5. *For which quasiprojective varieties $V \subset \mathbb{C}P^d$, $\ell_V = \ell_{\bar V}$?*

Another important observation is as follows. Given a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, we call $\widetilde\mu = (\mu_{i_1} \geq \mu_{i_2} \geq \cdots \geq \mu_{i_s})$ where $1 \leq i_1 < i_2 < \cdots < i_s \leq r$, a *subpartition* of μ.

Proposition 2. *For a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$ and any its subpartition $\widetilde\mu$, $\ell_\mu \leq \ell_{\widetilde\mu}$. In particular, $\ell_\mu \leq \mu_r + 2$.*

Proof. Given a subpartition $\widetilde\mu = (\mu_{i_1} \geq \mu_{i_2} \geq \cdots \geq \mu_{i_s})$ of a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, let

$$f_1 + f_2 + \cdots + f_{\ell_{\widetilde\mu}} = 0, \tag{6}$$

be a linear dependence of pairwise non-proportional binary forms from $S_{\widetilde\mu}$ realizing its secant degeneracy index. Take the partition $\widehat\mu = \mu \backslash \widetilde\mu = (\widehat\mu_1 \geq \widehat\mu_1 \geq \cdots \geq \widehat\mu_{r-s})$. Multiplying the latter equality by $\prod_{j=1}^{r-s}(x - a_j y)^{\widehat\mu_j}$, where a_j are generic complex number, we get a linear dependence between polynomials in S_μ. The inequality $\ell_\mu \le \mu_r + 2$ is a special case of the general inequality, if one chooses $\widetilde{\mu} = (\mu_r)$. Observe that for the partition $(d) \vdash d$, $\ell_{(d)} = d + 2$, since the set of binary forms of degree d with a root of multiplicity d is a rational normal curve in $Pol_d \simeq \mathbb{C}P^d$. □

The latter upper bound $\ell_\mu \le \mu_r + 2$ is sharp, for example, for any partition with $\mu_r = 1$, but not in general. Namely, already in the case $\mu = (2^2) \vdash 4$, $\ell_\mu = 3 < 4 = \mu_2 + 2$. For $\mu = (3^2) \vdash 6$, $\ell_\mu = 4 < 5$; $\mu = (4^2) \vdash 8$, $\ell_\mu = 4 < 6$.

Before formulating general results about ℓ_μ, let us provide more concrete examples and information about ℓ_μ.

Proposition 3. *Let $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_r)$ be a partition with two different indices i_1 and i_2 such that $\mu_{i_1} - \mu_{i_1+1} = \mu_{i_2} - \mu_{i_2+1} = 1$. Then, $\ell_\mu \le 4$.*

Proof. Wlog assume that $i_1 < i_2$, and consider two different cases.
Case 1. $i_2 = i_1 + 1$. Consider a subpartition $\widetilde{\mu} = (\mu_{i_1}, \mu_{i_1+1}, \mu_{i_1+2}) = (\mu_{i_1+2} + 2, \mu_{i_1+2} + 1, \mu_{i_1+2})$; set $k = \mu_{i_1+2}$. We know that $\ell_\mu \le \ell_{\widetilde{\mu}}$. So it is enough to prove that $\ell_{\widetilde{\mu}} \le 4$. Consider three distinct complex numbers p, q and r, and four polynomials $g_1 = (x-p)^{k+2}(x-q)^{k+1}(x-r)^k$, $g_2 = (x-p)^{k+2}(x-r)^{k+1}(x-q)^k$, $g_3 = (x-q)^{k+2}(x-p)^{k+1}(x-r)^k$ and $g_3 = (x-r)^{k+2}(x-p)^{k+1}(x-q)^k$. Take a linear combination $ag_1 + bg_2 + cg_3 + dg_4$; it is given by

$$Q(x)(a(x-p)(x-q) + b(x-p)(x-r) + c(x-q)^2 + d(x-r)^2),$$

where $Q(x) = (x-p)^{k+1}(x-q)^k(x-r)^k$. Polynomials $(x-p)(x-q)$, $(x-p)(x-r)$, $(x-q)^2$ and $(x-r)^2$ are linearly dependent. Thus there exist a, b, c, d such that $ag_1 + bg_2 + cg_3 + dg_4 = 0$. Hence $\ell_{\widetilde{\mu}} \le 4$.
Case 2. $i_2 > i_1 + 1$. Consider a subpartition $\widetilde{\mu} = (\mu_{i_1}, \mu_{i_1+1}, \mu_{i_2}, \mu_{i_2+1}) = (\mu_{i_1+1} + 1, \mu_{i_1+1}, \mu_{i_2+1} + 1, \mu_{i_2+1})$; set $k_1 = \mu_{i_1+1}$ and $k_1 = \mu_{i_2+1}$. We know that $\ell_\mu \le \ell_{\widetilde{\mu}}$. So it is enough to prove that $\ell_{\widetilde{\mu}} \le 4$. Consider four different complex numbers p, q, r and t, and four polynomials $g_1 = (x-p)^{k_1+1}(x-q)^{k-1}(x-r)^{k_2+1}(x-s)^{k_2}$, $g_2 = (x-q)^{k_1+1}(x-p)^{k-1}(x-r)^{k_2+1}(x-s)^{k_2}$, $g_3 = (x-p)^{k_1+1}(x-q)^{k-1}(x-s)^{k_2+1}(x-r)^{k_2}$ and $g_4 = (x-q)^{k_1+1}(x-p)^{k-1}(x-s)^{k_2+1}(x-r)^{k_2}$

Consider a linear combination $ag_1 + bg_2 + cg_3 + dg_4$, which is equal to

$$R(x)(a(x-p)(x-r) + b(x-q)(x-r) + c(x-p)(x-s) + d(x-q)(x-s)),$$

where $R(x) = (x-p)^{k_1}(x-q)^{k_1}(x-r)^{k_2}(x-r)^{k_2}$. Polynomials $(x-p)(x-r)$, $(x-q)(x-r)$, $(x-p)(x-s)$ and $(x-q)(x-s)$ are linearly dependent. Thus there exist a, b, c, d such that $ag_1 + bg_2 + cg_3 + dg_4 = 0$, and hence $\ell_{\widetilde{\mu}} \le 4$. □

Proposition 4. *Take any partition $\hat{\mu} = (\hat{\mu}_1 \ge \hat{\mu}_2 \ge \cdots \ge \hat{\mu}_n)$. Given an arbitrary positive integer i, form a partition $\mu' = (\hat{\mu}_1 + i \ge \hat{\mu}_2 + i \ge \cdots \ge \hat{\mu}_n + i \ge i, i, \ldots, i)$, where the entry i is repeated $n(\ell_{\hat{\mu}} - 1)$ times at the end of μ', $\ell_{\hat{\mu}}$ being the index of $\hat{\mu}$. Then any partition μ containing the subpartition μ' satisfies the inequality $\ell_{\mu'} \le \ell_{\hat{\mu}}$. In particular, the secant degeneracy index ℓ_μ equals 3, for any partition μ containing the subpartitions: $\widetilde{\mu} = (t+1, t, t)$, where t is any positive integer. More generally, the secant degeneracy index ℓ_μ is at most $i + 2$, for any partition μ containing the subpartition: $\widetilde{\mu} = (t+i, t, t, \ldots, t)$, where t is any positive integer repeated $i + 1$ times.*

Proof. BLA □

Proposition 5. *The secant degeneracy index ℓ_μ equals μ_r+2, for any $\mu=(\mu_1\geq \mu_2\geq\dots\geq\mu_r)$ satisfying the condition:*

$$\mu_1+\mu_2+\dots+\mu_{r-1}>(r-1)\mu_r(\mu_r+2).$$

Proof. BLA ☐

Corollary 1. *For $\mu=(p,q)$, $\ell_\mu=q+2$ if $p\geq q^2-1$.*

For a partition $\mu=(\mu_1\geq\mu_2\geq\dots\geq\mu_r)$, we define its jump multiset J_μ as the multiset of all positive numbers in the set $\{\mu_1-\mu_2,\dots,\mu_{r-1}-\mu_r,\mu_r\}$; we denote by h_μ the minimal (positive) jump of μ, i.e. the element of J_μ with minimal length. Our first general result is as follows.

Theorem 6. *For any $\mu=(\mu_1\geq\mu_2\geq\dots\geq\mu_r)$, $\ell_\mu\geq\sqrt{h_\mu+1}+1$.*

Proof. Given μ, let $\{f_1,\dots,f_{\ell_\mu}\}$ be a collection of forms satisfying (1). Assume that $\{f_1,\dots,f_{\ell_\mu}\}$ gives a counterexample to the statement. Denote by g the GCD of $\{f_1,\dots,f_{\ell_\mu}\}$ and consider the relation

$$\frac{f_1}{g}+\ldots+\frac{f_{\ell_\mu}}{g}=0.$$

For any i, every root of the polynomial $\frac{f_i}{g}$ has multiplicity not smaller than h_μ, where $h_\mu>\ell_\mu$, since it is not a counterexample otherwise.

Consider the sequence of Wronskians $w_i=W\left(\frac{f_1}{g},\dots,\frac{f_{i-1}}{g},\frac{f_{i+1}}{g},\dots,\frac{f_{\ell_\mu}}{g}\right)$, $i=1,\dots,\ell_\mu$. All these Wronskians are proportional because of the latter relation.

Let α be a root of some f_i. There exists s such that $\frac{f_s}{g}$ is not divisible by $(x-\alpha)$, since otherwise g is not the GCD.

For any t consider the multiplicity of the root of w_t at α. It satisfies the inequality:

$$ord_\alpha(w_t)\geq\sum\left(ord_\alpha\left(\frac{f_j}{g}\right)\right)-(\ell_\mu-2)\#\left\{i:(x-\alpha)|\frac{f_i}{g}\right\},$$

because any column of the Wronski matrix corresponding to $(x-\alpha)|\frac{f_j}{g}$ is divisible by $(x-\alpha)^{ord_\alpha\left(\frac{f_j}{g}\right)-l+2}$.

Hence,

$$\deg w_1\geq\sum_{i=1}^{\ell_\mu}\left(\deg\left(\frac{f_i}{g}\right)-(\ell_\mu-2)\#_{roots}\left(\frac{f_i}{g}\right)\right)=$$

$$=\ell_\mu(|\mu|-\deg g)-(\ell_\mu-2)\sum_{i=1}^{\ell_\mu}\#_{roots}\left(\frac{f_i}{g}\right).$$

On the other hand,

$$\deg w_1\leq(\ell_\mu-1)(\deg\left(\frac{f_i}{g}\right)-\ell_\mu+2)=$$

$$=(\ell_\mu-1)(|\mu|-\deg g)-(\ell_\mu-1)(\ell_\mu-2).$$

We obtain

$$(\ell_\mu-1)(|\mu|-\deg g)-(\ell_\mu-1)(\ell_\mu-2)\geq\ell_\mu(|\mu|-\deg g)-(\ell_\mu-2)\sum_{i=1}^{\ell_\mu}\#_{roots}\left(\frac{f_i}{g}\right),$$

i.e.,

$$(\ell_\mu - 2)\sum_{i=1}^{\ell_\mu} \#_{roots}\left(\frac{f_i}{g}\right) - (\ell_\mu - 1)(\ell_\mu - 2) \geq |\mu| - \deg g.$$

The number $\#_{roots}\left(\frac{f_i}{g}\right)$ of distinct roots is at most $\frac{|\mu| - \deg g}{h_\mu}$, because each root has multiplicity at least h_μ. Thus

$$(\ell_\mu - 2)(\ell_\mu - 1)\frac{|\mu| - \deg g}{h_\mu} - (\ell_\mu - 1)(\ell_\mu - 2) \geq |\mu| - \deg g.$$

Hence,

$$(\ell_\mu - 2)(\ell_\mu - 1) > h_\mu.$$

□

2.2. **Partitions with growing and with stabilizing secant degeneracy index.** By a *radical* of a given form we mean the binary form obtained as the product of all distinct linear factors of the original form.

Definition 2. *Given a partition μ and a positive integer m, a solution of*

$$f_1 + f_2 + \cdots + f_m = 0, \tag{7}$$

with pairwise non-proportional $f_i \in S_\mu$ is called a common radical solution if all f_i's have the same radical, i.e. the same set of distinct linear factors.

Theorem 7. *For $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, either $\ell_\mu \geq \sqrt{\frac{\mu_r}{r-1}} + 1$ or any solution of* (1) *is a common radical solution.*

Proof. Assume the opposite. Let $\{f_1, \ldots, f_{\ell_\mu}\}$ be a solution of (1), which is not a common radical solution. Let g be the GCD of $\{f_1, \ldots, f_{\ell_\mu}\}$.

For the term $f_i = c_i(x - a_{i,1})^{\mu_1} \cdots (x - a_{i,r})^{\mu_r}$, define

$$g_i := (x - a_{i,1})^{\mu_1 - \ell_\mu + 2} \cdots (x - a_{i,r})^{\mu_r - \ell_\mu + 2}.$$

Observe that g_i is a polynomial, because any root of f_i has multiplicity at least $\mu_r > \ell_\mu$.

Consider the sequence of Wronskians

$$w_i = W(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{\ell_\mu}),\ i = 1, \ldots, \ell_\mu.$$

They are proportional because of (1). Notice that the column in the Wronski matrix for w_t corresponding to f_i ($i \neq t$) is divisible by g_i. Hence w_t is divisible by $\frac{\prod_{i=1}^{\ell_\mu} g_i}{g_t}$.

Since $\{f_1, \ldots, f_{\ell_\mu}\}$ is not a common radical solution, there exists $\alpha \in \mathbb{C}$, such that α is a root of f_p but it is not a root of f_q for some $p \neq q$.

Since the Wronskians w_p and w_q are proportional, they are divisible by

$$LCM\left(\frac{\prod_{i=1}^{\ell_\mu} g_i}{g_p}, \frac{\prod_{i=1}^{\ell_\mu} g_i}{g_q}\right) = \frac{\prod_{i=1}^{\ell_\mu} g_i}{GCD(g_p, g_q)} = \frac{\prod_{i=1}^{\ell_\mu} g_i}{g_p}\frac{g_p}{GCD(g_p, g_q)}.$$

Then these Wronskians are divisible by $\frac{\prod_{i=1}^{\ell_\mu} g_i}{g_p}(x - \alpha)^{\mu_r - \ell_\mu + 2}$. Therefore their degrees are greater than or equal to

$$(\ell_\mu - 1)(|\mu| - r(\ell - 2)) + \mu_r - \ell_\mu + 2.$$

On the other hand, the degrees of the Wronskians are at most $(\ell_\mu - 1)(|\mu| - \ell_m u + 2)$. Thus,

$$(\ell_\mu - 1)(|\mu| - \ell_\mu + 2) \geq (\ell_\mu - 1)(|\mu| - r(\ell - 2)) + \mu_r - \ell_\mu + 2,$$

which implies $-(\ell_\mu - 1)(\ell_\mu - 2) \geq -r(\ell_\mu - 1)(\ell - 2) + \mu_r - \ell_\mu + 2$. The latter inequality after straightforward simplifications gives

$$\ell_\mu - 1 \geq \sqrt{\frac{\mu_r}{r-1}}.$$

Contradiction. □

Notation. For a given partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, define $\mu^{\langle d \rangle} := (\mu_1 + d \geq \mu_2 + d \geq \cdots \geq \mu_r + d)$.

Definition 3. *We say that a partition μ has a growing secant degeneracy index if $\lim_{d\to\infty} \ell_{\mu^{\langle d \rangle}} = +\infty$ and that μ has a stabilizing secant degeneracy index otherwise.*

Definition 4. *Given a partition $\mu = (i_1^{m_1}, i_2^{m_2}, \ldots, i_s^{m_s})$, $\sum_{j=1}^{s} m_j = r$, define the stabilizing subgroup $Sym_\mu \subseteq Sym_r$ as.... Observe that the number of elements in Sym_μ equals the multinomial ccoefficient $\frac{d!}{m_1!m_2!\ldots m_s!}$.*

Corollary 2. *A partition μ has a stabilizing secant degeneracy index if and only if there is a common radical solution of* (6)*. A partition μ has a growing secant degeneracy index if and only if the linear span of the S_k-orbit of any form $f \in S_\mu$ has dimension $k! - 1$.*

For any partition μ with a growing secant degeneracy index, i.e., for $\ell_{\mu^d} \to \infty$, we know that

$$\sqrt{\frac{\mu_r + d}{r-1}} + 1 \leq \ell_{\mu^d} \leq \mu_r + d + 2,$$

see Proposition 2 and Theorem 7.

Problem 6. *For any partition μ with a growing secant degeneracy index, what is the leading term of the asymptotic of ℓ_{μ^d}, when $d \to +\infty$? Does it depend on a particular choice of μ?*

2.3. More on partitions with stabilizing secant degeneracy index.

Proposition 8. *Let multiset $\tau_\mu = \{\mu_1, \ldots, \mu_r\}$ and sequence $\{a_1, \ldots, a_r\}$ with following properties (i) for any i, $a_i \in [1, r]$; (ii) number of different permutation π of τ_μ such that $\pi_i \geq \mu_{a_i}$ at least $|\mu| - \sum_{i=1}^{r} \mu_{a_i} + 2$. Then there is solution of 7 with common radical, furthermore there is such solution with any set of distinct r roots.*

2.4. More on partitions with growing secant degeneracy index.

Corollary 3. *Any partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, such that every its jump is at least $(r!)^2$, has a growing secant degeneracy index.*

Proof. Assume that ℓ_{μ^d} does not grow to infinity. Then by Theorem 6

$$\ell_{\mu^d} \geq \sqrt{h_\mu^d + 1} + 1 \geq \sqrt{h_\mu^0 + 1} + 1 \geq \sqrt{(r!)^2 + 1} + 1 > r!.$$

However the number of different polynomials (up to a constant factor) with fixed r roots and their multiplicities μ^d is at most $r!$. Hence no common radical solution might exist. Contradiction. □

Proposition 9. *If, for partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$, number of jumps which at most d at least $2(log_2(d) + log_2(log_2(d)))$, then $\ell_\mu \leq d(log_2(d) + log_2(log_2(d)))$.*

Proof. BLA ☐

2.5. **Partitions with two parts.** In this section motivated by a number of interesting observations of B. Reznick about relations among powers of binary quadratic forms, we concentrate on the special case of partitions with exactly two parts.

Proposition 10. *For partitions $\mu = (p, 2)$, $\ell_\mu = 3$ when $p = 2$, and $\ell_\mu = 4$ when $p > 2$.*

For partitions $\mu = (p, 3)$, $\ell_\mu = 4$ when $p = 3, 4, 5$ and $\ell_\mu = 5$ when $p > 7$. Cases $\mu = (6, 3)$ and $\mu = (7, 3)$ are still open.

Proof. In case $\mu = (3, 4)$ the first author found the example:

$$y^3(x+y)^4 - y^3x^4 = L(x+ay)^3y^4 + (1-L)(x+by)^3y^4,$$

where $a = 3 - \sqrt{3}$, $b = 3 + \sqrt{3}$ and $L = \frac{9-5\sqrt{3}}{18}$;

In case $\mu = (5, 3)$ the first author suggests the example:

$$f_1 + f_2 - f_3 - f_4 = 0,$$

where $f_1(x) = (x + c_1^5 y)^3(x + c_1^{-3} y)^5$; $f_2(x) = (x + c_2^5 y)^3(x + c_2^{-3} y)^5$; $f_3(x) = (x + c_1^{-5} y)^3(x + c_1^3 y)^5$; $f_4(x) = (x + c_2^{-5} y)^3(x + c_2^3 y)^5$. Additionally, $c_1 = -c_2 =$???. ☐

3. Application to the classical super Fermat problem for binary forms

Additionally,

Lemma 11. *If $\ell(d, k) = l$ then $\ell(d', k) \leq l$ for $d' \geq d$. In other words, for any fixed k, the sequence $\ell(d, k)$ is monotone non-increasing w.r.t. d.*

Proof. HOW??? ☐

Proposition 12. *For any positive integers k and d, $\ell := \ell(d, k)$ satisfies the inequality*

$$\ell(\ell - 2) - \frac{(\ell - 2)(\ell - 1)}{d} \geq k. \tag{8}$$

Proof. Observe that we can always assume that a) $q_1, q_2, \ldots q_l$ have no non-trivial common factor; and b) the dimension of the linear space spanned by $q_1^k, q_2^k, \ldots, q_\ell^k$ equals $\ell - 1$ which implies that the Wronskian of any $\ell - 1$ of these functions is not vanishing identically. The latter implies that all such Wronskians coincide up to a constant factor. By a linear change of variables we can achieve that $\deg_x q_i = d$ for all i. Consider the Wronskian $W := W(q_1^k, q_2^k, \ldots, q_{\ell-1}^k)$. Observe that $\deg_x W \leq (\ell - 1)(kd - \ell + 2)$. Indeed,

Lemma 13. *For any m-tuple of linearly independent polynomials $f_1, f_2, \ldots, f_m$,*

$$\deg_x W(f_1, f_2, \ldots, f_m) \leq m \max_i \deg_x(f_i - m + 1).$$

Proof. BLA ☐

On the other hand, W is divisible by $\prod_{j=1}^{\ell-1} q_j^{k-\ell+2}$. By the above W is divisible by $\prod_{j=1}^{\ell} q_j^{k-\ell+2}/q_i^{k-\ell+2}$, for any $i=1,2,\dots,\ell$ implying by the absence of a common factor that W is divisible by $\prod_{j=1}^{\ell} q_j^{k-\ell+2}$. We obtain $\deg_x W \geq d\ell(k-\ell+2)$. Thus

$$(\ell-1)(kd-\ell+2) \geq d\ell(k-\ell+2) \Leftrightarrow \ell(\ell-2) - \frac{(\ell-2)(\ell-1)}{d} \geq k.$$

□

The following conjecture might be true.

Conjecture 2. *For any fixed positive integers k and d, $\ell(k,d)$ equals the minimal value of ℓ such that (8) holds.*

In particular, for any positive integer $j \geq 3$, $\ell\left(2, \frac{(j+1)(j-2)}{2}\right) = j$. More generally, for any d and any j such that $(j-2)(j-1)$ is divisible by d,

$$\ell\left(d, j(j-2) - \frac{(j-1)(j-2)}{d}\right) = j.$$

4. Related Problems

1. Let us introduce the maximum rank $r_{max}(V)$ of a positive-dimensional quasiprojective subvariety $V \subset \mathbb{C}P^d$. Given such a V, denite by $\sigma_k^o(V)$ its open k-th secant variety, which is the union of all the linear spans $\langle p_1, \dots, p_k\rangle$ where $p_i \in V$ Obviously $r_{max}(V) \geq r_{gen}(V)$. Notice that according to [BT], under very weak assumptions on V, its maximum rank does not exceed twice its generic rank. Our interest in the maximum rank is motivated by the following fact.

Proposition 14. *For any irreducible positive-dimensional quasiprojective variety $V \subset \mathbb{C}P^d$, $\ell_V \leq r_{max}(V) + 2$.*

Proof. BLA □

We hope that one can calculate the maximum rank of S_μ for any partition μ. In case of $\mu = (d)$, Theorem 5.4 of [Re2] (which was initially proven by J. J. Sylvester) by B. Reznick claims that for $d \geq 3$, the maximal Waring rank of binary polynomials of degree k equals k and this maximal value is attained exactly on binary forms representable as $l_1 l_2^{d1}$, where l_1 and l_2 are distinct linear forms.

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Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
E-mail address: nenashev@math.su.se

Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
E-mail address: shapiro@math.su.se

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027
E-mail address: mshapiro@math.msu.edu