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Classifying Hilbert functions of fat point subschemes in $\mathbb{P}^2$

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Abstract

The paper [10] raised the question of what the possible Hilbert functions are for fat point subschemes of the form $2p_1 + \cdots + 2p_r$, for all possible choices of r distinct points in $\mathbb{P}^2$. We study this problem for r points in $\mathbb{P}^2$ over an algebraically closed field k of arbitrary characteristic in case either $r \leq 8$ or the points lie on a (possibly reducible) conic. In either case, it follows from [17, 18] that there are only finitely many configuration types of points, where our notion of configuration type is a generalization of the notion of a representable combinatorial geometry, also known as a representable simple matroid. (We say $p_1, \dots, p_r$ and $p'_1, \dots, p'_r$ have the same *configuration type* if for all choices of nonnegative integers m_i, $Z = m_1p_1 + \cdots + m_rp_r$ and $Z' = m_1p'_1 + \cdots + m_rp'_r$ have

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the same Hilbert function.) Assuming either that $7 \leq r \leq 8$ (see [12] for the cases $r \leq 6$) or that the points p_i lie on a conic, we explicitly determine all the configuration types, and show how the configuration type and the coefficients m_i determine (in an explicitly computable way) the Hilbert function (and sometimes the graded Betti numbers) of $Z = m_1p_1 + \cdots + m_rp_r$. We demonstrate our results by explicitly listing all Hilbert functions for schemes of $r \leq 8$ double points, and for each Hilbert function we state precisely how the points must be arranged (in terms of the configuration type) to obtain that Hilbert function.

1. Introduction

Macaulay completely solved the problem of describing what can be the Hilbert function of a homogeneous ideal I in a polynomial ring R. His solution, however, left open the question of which of these functions are the Hilbert functions of ideals in specific classes, such as prime homogeneous ideals (i.e., ideals of irreducible projective varieties) or, what will be a focus in this paper, symbolic powers of ideals of reduced zero-dimensional subschemes of projective 2-space.

Various versions of these questions have been studied before. A complete determination of the Hilbert functions of reduced 0-dimensional subschemes of projective space is given in [9]. But it is not, for example, known which functions arise as Hilbert functions of points taken with higher multiplicity, even if the induced reduced subscheme consists of generic points; see, for example, [5] and its bibliography, [11, 20, 21, 24]. The paper [10] asks what can be said about Hilbert functions and graded Betti numbers of symbolic powers $I^{(m)}$ of the ideal I of a finite set of reduced points in $\mathbb{P}^2$, particularly when $m = 2$, while the papers [13, 14] study $I^{(m)}$ for larger m, but only when I defines a reduced subscheme of projective space whose support is close to a complete intersection. Other work has focused on what one knows as a consequence of knowing the Hilbert function; e.g., [1] shows how the growth of the Hilbert function of a set of points influences its geometry, [3] studies how the Hilbert function constrains the graded Betti numbers in case I has height 2, and [8] studies graded Betti numbers but more generally for graded modules M over R.

In this paper we will determine precisely what functions occur as Hilbert functions of ideals $I(Z)$ of fat point subschemes Z of $\mathbb{P}^2$ over an algebraically closed field k of arbitrary characteristic, in case either $I(Z)$ contains some power of a form of degree 2 (i.e., the support of Z lies on a conic), or in case the support of Z consists of 8 or fewer points. If either the support of Z lies on a conic or consists of 6 or fewer points, by results of [4, 17, 12], one can also determine the graded Betti numbers of $I(Z)$. In the case of ideals of the form $I(Z)$ when the support of Z is small or lies on a conic, our results allow us to give explicit answers for many of the questions raised in the papers cited above. For example, given the Hilbert function h of $I(Z')$ for a reduced subscheme $Z' = p_1 + \cdots + p_r$, we can explicitly determine the Hilbert functions of all symbolic powers $I(Z)^{(m)}$ for all Z whose ideal $I = I(Z)$ has Hilbert function h, as long as Z' has support on a conic or at 8 or fewer points, and we can in addition explicitly determine the graded Betti numbers of $I(Z)^{(m)}$ as long as Z' has support either on a conic (by [4, 17]) or at 6 or fewer points (by [12]).

We now discuss the context of our work in more detail. Let Z be a subscheme of $\mathbb{P}^2$. Let $I(Z)$ be the corresponding saturated homogeneous ideal in the polynomial ring $R = k[x, y, z]$ (over an algebraically closed field k of arbitrary characteristic) which defines Z. Then $R(Z) = R/I(Z) = \oplus_{j\geq 0} R(Z)_j$ is a graded ring whose Hilbert function h_Z is defined by $h_Z(t) = \dim_k R(Z)_t$.

If Z is a fat point subscheme (defined below), then h_Z is a nondecreasing function with $h_Z(0) = 1$ and such that $h_Z(t) = \deg(Z)$ for all $t \geq \deg(Z) - 1$. Thus there are only finitely many possible Hilbert functions for a fat point subscheme Z of given degree. In case Z is smooth (i.e. a finite set of distinct points) it is known what these finitely many Hilbert functions are [9]. The paper [10] raises the problem of determining the Hilbert functions, and even the graded Betti numbers for a minimal free resolution of $R(Z)$ over R, for fat point subschemes of $\mathbb{P}^2$ more generally.

If Z is a reduced 0-dimensional subscheme of $\mathbb{P}^2$, let mZ denote the subscheme defined by the mth symbolic power $I(mZ) = (I(Z))^{(m)}$ of the ideal $I(Z)$. If Z consists of the points $p_1, \ldots, p_r$, we write $Z = p_1 + \cdots + p_r$, and we have $I(Z) = I(p_1) \cap \cdots \cap I(p_r)$, where $I(p_i)$ is the ideal generated by all forms of R that vanish at p_i. We also write $mZ = mp_1 + \cdots + mp_r$. The ideal $I(mZ)$ is just $I(p_1)^m \cap \cdots \cap I(p_r)^m$. More generally, given any nonnegative integers m_i, the *fat point* subscheme denoted $m_1p_1 + \cdots + m_rp_r$ is the subscheme defined by the ideal $I(m_1p_1 + \cdots + m_rp_r) = I(p_1)^{m_1} \cap \cdots \cap I(p_r)^{m_r}$. Given a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r$, we refer to the sum of the points for which $m_i > 0$ as the *support* of Z and we define the *degree* of Z, denoted $\deg Z$, to be the integer $\sum_{i=1}^{r} \binom{m_i+1}{2}$.

Given the Hilbert function h of a reduced fat point subscheme, the focus of the paper [10] is to determine what Hilbert functions occur for $2Z$, among all reduced Z whose Hilbert function is h. The paper [10] represents a step in the direction of answering this problem: For each h which occurs for a reduced fat point subscheme of $\mathbb{P}^2$, [10] determines the Hilbert function of $2Z$ for an explicitly constructed subscheme $2Z$ whose support Z has Hilbert function h. This leaves open the problem of giving a complete determination of the Hilbert functions which occur among all fat points subschemes $2Z$ whose support Z has Hilbert function h, but [10] proves that there is a maximal such Hilbert function. This proof is nonconstructive; what this maximal Hilbert function is and how it can be found is an open problem. The paper [10] also raises the question of whether there is a minimal Hilbert function (among Hilbert functions for all $2Z$ such that Z is reduced with given Hilbert function).

In this paper, we give a complete answer to all of the questions raised in [10] about Hilbert functions if the degree of the support is 8 or less or the support is contained in a conic. We classify all possible arrangements of r points p_i in case either $r \leq 8$ or the points are contained in a conic, and determine the Hilbert functions for all fat point subschemes $Z = m_1p_1 + \cdots + m_rp_r$, regardless of the multiplicities. This is possible because Bezout considerations (see [17, 18] and Example 3.3) completely determine the Hilbert functions for any Z if either $r \leq 8$ or the points p_i lie on a conic (and these same considerations even determine the graded Betti numbers of $I(Z)$ if either $r \leq 6$ or the points lie on a conic).

Without a bound on the degree of Z, there are clearly infinitely many Hilbert functions. However, results of [17] (in case the points $p_1, \ldots, p_r$ lie on a conic) or of [18] (in case $r \leq 8$) imply that to each set of points $p_1, \ldots, p_r$ we can attach one of a finite set of combinatorial structures we call *formal configuration types*. Those formal configuration types which are representable (meaning they arise from an actual set of points) turn out to be the same thing as what we define to be a configuration type. In case $r \leq 8$ or the points $p_1, \ldots, p_r$ lie on a conic, configuration types have the property that if we know the integers m_i and the configuration type of the points p_i, we can explicitly write down the Hilbert function of $m_1p_1 + \cdots + m_rp_r$ (and, if either $r \leq 6$ or the points lie on a conic, we can even write down the graded Betti numbers).

The ultimate goal of the program started in [10] is to determine all Hilbert functions (and the graded Betti numbers) for fat point subschemes Z whose support has given Hilbert function h. Note that $h(2)$ is less than 6 if and only if the points of the support lie on a conic. Thus in case $h(2) < 6$, the results of [17] give a complete answer to the program of [10], once one writes down the configuration types of points on a conic (which is easy). The results of [18] similarly give a complete answer to the Hilbert function part of the program of [10] when h is less than or equal to the constant function 8, modulo writing down the configuration types. But unlike the case of points on a conic, determining the configuration types for 7 or 8 distinct points takes a fair amount of effort. That determination is given here for the first time.

Giving the list of types for 8 points subsumes doing so for any fewer number of points, but it is of interest to consider the case of 7 points explicitly. The case of 6 points is worked out in [12]. (In fact, the main point of [12] is to determine graded Betti numbers. Thus [12] entirely resolves the program of [10] in case h is less than or equal to the constant function 6. The case of fewer than 6 points can easily be recovered from the case of 6 points.) It turns out that there are 11 different configuration types 6 distinct points of $\mathbb{P}^2$. We find 29 types for 7 points and 143 types for 8 points.

However, a new feature arises for the case of 7 or 8 points, compared with 6 points. Formal configuration types, as we define them, are matroid-like combinatorial objects which can be written down without regard to whether or not some set of points exists having that type. A combinatorial geometry (or, equivalently, a simple matroid) whose points have span of dimension $s \leq 2$ is a matroid, of rank $s + 1 \leq 3$, without loops or parallel elements. Our classifications of formal configuration types of sets of 7 (respectively 8) distinct points includes the classification of all combinatorial geometries on 7 (respectively 8) points whose span has dimension at most 2 [2] (alternatively, see [23]). A key question for combinatorial geometries is that of representability; that is: does there exist a field for which there occurs an actual set of points having the given combinatorial geometry? This is also an issue for our configuration types. In the case of 6 points, every formal configuration type is representable for every algebraically closed field, regardless of the characteristic. For 7 points, every formal configuration type is representable for some algebraically closed field, but sometimes representability depends on the characteristic. For 8 points, there are three formal configuration types which are not representable at all.

Given only the configuration type and the multiplicities m_i, we also describe an explicit procedure (which, if so desired, can be carried out by hand) for computing the Hilbert function of any $Z = m_1p_1 + \cdots + m_rp_r$ when either $r \leq 8$ or the points lie on a conic. If the points lie on a conic (using [4, 17]) or if $r \leq 6$ (using [12]), one can also determine the graded Betti numbers of the minimal free resolution of $R(Z)$. Example 3.3 demonstrates the procedure in detail in case the points lie on a conic. In case $6 \leq r \leq 8$, the procedure is implemented as a web form that can be run from any browser. To do so, visit http://www.math.unl.edu/ bharbourne1/FatPointAlgorithms.html.

2. Background

In this section we give some definitions and recall well known facts that we will need later. First, the minimal free resolution of $R(Z) = R/I(Z)$ for a fat point subscheme $Z \subset \mathbb{P}^2$ is

$$0 \to F_1 \to F_0 \to R \to R(Z) \to 0$$

where F_0 and F_1 are free graded R-modules of the form $F_0 = \oplus_{i\geq 0} R[-i]^{t_i}$ and $F_1 = \oplus_{i\geq 0} R[-i]^{s_i}$. The indexed values t_i and s_i are the graded Betti numbers of Z. Consider the map

$$\mu_i : I(Z)_i \otimes R_1 \to I(Z)_{i+1}$$

given by multiplication of forms of degree i in $I(Z)$ by linear forms in R. Then $t_{i+1} = \dim \operatorname{cok}(\mu_i)$, and $s_i = t_i + \Delta^3 h_Z(i)$ for $i \geq 1$, where Δ is the difference operator (so $\Delta h_Z(i+1) = h_Z(i+1) - h_Z(i)$). Note that knowing t_i and the Hilbert function of Z suffice to ensure that we know s_i. Thus once h_Z is known, if we want to know the graded Betti numbers for Z, it is enough to determine t_i for i.

We now introduce the notion of Hilbert function equivalence for ordered sets of points in $\mathbb{P}^2$.

DEFINITION 2.1 Let $p_1, \ldots, p_r$ and $p'_1, \ldots, p'_r$ be ordered sets of distinct points of $\mathbb{P}^2$. We say these sets are *Hilbert function equivalent* if $h_Z = h_{Z'}$ for every choice of nonnegative integers m_i, where $Z = m_1 p_1 + \cdots + m_r p_r$ and $Z' = m_1 p'_1 + \cdots + m_r p'_r$. We refer to a Hilbert function equivalence class as a *configuration type.*

Remark 2.2 Given r and $m_1, \ldots, m_r$ it is clear that there are only finitely many possible Hilbert functions for fat point schemes $Z = m_1 p_1 + \cdots + m_r p_r$. However, for any given r there is absolutely no guarantee that there are finitely many configuration types for sets of r points. Indeed, there are infinitely many types for $r = 9$ points (see Example 5.9). Thus it is interesting that by our results below there are only finitely many configuration types among ordered sets of r distinct points if either the points lie on a conic or $r \leq 8$ points. This allows us to answer many of the questions raised in [10] (and more) for these special sets of points.

The methods used in this paper are very different from those used in [10], partly because the focus in this paper is on special sets of points, in particular points on a conic and sets of $r \leq 8$ points in $\mathbb{P}^2$. For these cases, the second author has demonstrated, in a series of papers, the efficacy of using the theory of rational surfaces, in particular the blow-ups of $\mathbb{P}^2$ at such points.

The tools of this technique are not so familiar in the commutative algebra community (where the original questions were raised) and so we thought to take this opportunity not only to resolve the problems of [10] for such sets of points, but also to lay out the special features of the theory of rational surfaces which have proved to be so useful in these Hilbert function investigations.

We begin with basic terminology and notation. Let $p_1, \dots, p_r$ be distinct points of $\mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^2$ be the morphism obtained by blowing up the points. Then X is a smooth projective rational surface. Its divisor class group $\mathrm{Cl}(X)$ is a free Abelian group with basis $L, E_1, \dots, E_r$, where L is the class of the pullback via π^* of the class of a line, and E_i is the class of the fiber $\pi^{-1}(p_i)$. We call such a basis an *exceptional configuration* for X. It is an orthogonal basis for $\mathrm{Cl}(X)$ with respect to the bilinear form given by intersecting divisor classes. In particular, $-L^2 = E_i^2 = -1$ for all i, and $L \cdot E_i = 0 = E_i \cdot E_j$ for all i and all $j \neq i$. An important class is the *canonical class* K_X. In terms of the basis above it is $K_X = -3L + E_1 + \cdots + E_r$. Given a divisor F, we write $h^0(X, \mathcal{O}_X(F))$ for the dimension of the space $H^0(X, \mathcal{O}_X(F))$ of global sections of the line bundle $\mathcal{O}_X(F)$ associated to F. To simplify notation, we will hereafter write $h^0(X, F)$ in place of $h^0(X, \mathcal{O}_X(F))$, etc. Since $h^0(X, F)$ is the same for all divisors F in the same divisor class, we will abuse notation and use $h^0(X, F)$ even when F is just a divisor class. The projective space $\mathbb{P}(H^0(X, \mathcal{O}_X(F))$ will be denoted $|F|$.

Recall that a *prime divisor* is the class of a reduced irreducible curve on X, and an *effective divisor* is a nonnegative integer combination of prime divisors. For simplicity, we will refer to a divisor class as being *effective* if it is the class of an effective divisor. We will denote the collection of those effective divisor classes by $\mathrm{EFF}(X)$. Notice that for the cases we are considering in this paper (i.e. r points on a conic or $r \leq 8$ points) the *anticanonical divisor class* $-K_X$ is effective. We say that a divisor F is *ample* if $F \cdot D > 0$ for every effective divisor D.

Our interest in this paper is to decide when a divisor class F on X is effective and then to calculate $h^0(X, F)$. To see why this is relevant to the problem of computing Hilbert functions of ideals of fat points, let $Z = m_1p_1 + \cdots + m_rp_r$. Then under the identification of R_i with $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(i)) = H^0(X, iL)$, we have $I(Z)_i = H^0(X, F(Z, i))$, where $F(Z, i) = iL - m_1E_1 - \cdots - m_rE_r$. We also have a canonical map

$$\mu_F : H^0(X, F) \otimes H^0(X, L) \to H^0(X, F + L)$$

for any class F. In case $F = F(Z, i)$, then $t_{i+1} = \dim \mathrm{cok}(\mu_{F(Z,i)})$. As a result, we can use facts about divisors on X, in particular about $h^0(X, F)$, to obtain results about the Hilbert function and graded Betti numbers for Z.

Since F is a divisor on a rational surface, the Riemann-Roch Theorem gives

$$h^0(X, F) - h^1(X, F) + h^2(X, F) = \frac{F \cdot F - K.F}{2} + 1.$$

So, if for some reason we were able to prove that the divisor F we are interested in has $h^1(X, F) = h^2(X, F) = 0$ then the Riemann-Roch formula would calculate $h^0(X, F)$ for us.

A divisor whose intersection product with **every** effective divisor is ≥ 0 is called *numerically effective (nef)*. The collection of nef divisor classes forms a cone which is denoted $\mathrm{NEF}(X)$. It is always true for a nef divisor F that $h^2(X, F) = 0$ (since $(K - F) \cdot L < 0$, we see that $K - F$ is not effective; now use Serre duality). So, if we could somehow connect the divisor we are interested in to a nef divisor, then at least we might be able to get rid of the need to calculate $h^2(X, F)$. Unfortunately, it is not true, in general, that $h^1(X, F) = 0$ for F a nef divisor.

However, one of the fundamental results in this area asserts that for rational surfaces obtained by blowing up either any number of points on a conic, or any $r \leq 8$ points, nef divisors do have $h^1(X, F) = 0$. More precisely.

Theorem 2.3

Let X be obtained by blowing up r distinct points of $\mathbb{P}^2$. Suppose F is a nef divisor on X.

(a) *If either $2L - E_1 - \cdots - E_r$ is effective (i.e., the points p_i lie on a conic), or $r \leq 8$, then $h^1(X, F) = 0 = h^2(X, F)$ and so $h^0(X, F) = \frac{(F^2 - K_X \cdot F)}{2} + 1$.*
(b) *If $2L - E_1 - \cdots - E_r$ is effective, then μ_F is surjective.*
(c) *If $r \leq 6$, then μ_F has maximal rank (i.e., is either surjective or injective).*

Proof. As indicated above, $h^2(X, F) = 0$ for a nef divisor. When $2L - E_1 - \cdots - E_r$ is the class of a prime divisor, the fact that

$$h^0(X, F) = \frac{(F^2 - K_X \cdot F)}{2} + 1$$

was proved in [20], and surjectivity for μ_F was proved in [4]. Both results in the general case (assuming only that the points p_i lie on a conic, not that the conic is necessarily reduced and irreducible, hence subsuming the case that the points are collinear) were proved in [17]. The fact that $h^0(X, F) = (F^2 - K_X \cdot F)/2 + 1$ if $r \leq 8$ was proved in [18]. The fact that μ_F has maximal rank for $r \leq 6$ is proved in [12]. (We note that if $r \geq 7$, it need not be true that μ_F always has maximal rank if F is nef [16, 15], and if $r = 9$, it need not be true that $h^1(X, F) = 0$ if F is nef, as can be seen by taking $F = -K_X$, when X is the blow-up of the points of intersection of a pencil of cubics.) □

Remark 2.4 Let $Z = m_1p_1 + \cdots + m_rp_r$ and let $F = tL - m_1E_1 - \cdots - m_rE_r$ be nef. In Theorem 2.3, (a) says if either the points lie on a conic or $r \leq 8$ then (by means of a standard argument) $h_Z(t) = \deg Z$, i.e. $t + 1 \geq$ the Castelnuovo-Mumford regularity of $I(Z)$; (*b*) says that $I(Z)$ has no generator in degree $t + 1$ when the points lie on a conic and (*c*) says if $r \leq 6$ that the number of generators of $I(Z)$ in degree $t + 1$ is $\max\{\dim I(Z)_{t+1} - 3 \dim I_Z(t), 0\}$. Of course, (*b*) and (*c*) are interesting only if $t + 1$ is equal to the Castelnuovo-Mumford regularity of $I(Z)$, since μ_t is always surjective for t at or past the regularity.

In general, we will want to compute $h^0(X, F)$ and the rank of μ_F even if F is not nef. The first problem is to decide whether or not F is effective. If F is not effective, then $h^0(X, F) = 0$ by definition, and it is clear in that case that μ_F is injective, and hence that $\dim \operatorname{cok}(\mu_F) = h^0(X, F + L)$. So, if we can decide that F is not effective then we are done. What if F is effective?

Suppose F is effective *and* we can find all the fixed components N of F (so $h^0(X, N) = 1$)). We can then write $F = H + N$, and clearly H is nef with $h^0(X, F) = h^0(X, H)$. When Theorem 2.3(a) applies, this finishes our task of finding $h^0(X, F)$. The decomposition $F = H + N$ is called the *Zariski decomposition* of F.

Moreover, the restriction exact sequence

$$0 \to \mathcal{O}_X(-N) \to \mathcal{O}_X \to \mathcal{O}_N \to 0$$

tensored with $\mathcal{O}_X(F+L)$, gives an injection $|H+L| \to |F+L|$ which is defined by $D \mapsto D+N$. Now, $\mathrm{Im}(\mu_H)$ defines a linear subsystem of $|H+L|$, which we'll denote $|\mathrm{Im}(\mu_H)|$, and the image of μ_F is precisely the inclusion of $|\mathrm{Im}(\mu_H)|$ in $|F+L|$. Hence

$$\dim \mathrm{cok}(\mu_F) = \dim \mathrm{cok}(\mu_H) + (h^0(X, F+L) - h^0(X, H+L)).$$

So, if it were possible to find the Zariski decomposition of a divisor on X, then it would be possible to find $h^0(X, F)$ for any divisor class F when Theorem 2.3 applies, since it furnishes us with most of what we need to find Hilbert functions and graded Betti numbers.

To sum up, what we need is an explicit way to decide if a divisor F is effective or not, and when it is effective, a way to find its Zariski decomposition. These are now our two main tasks.

It turns out that in order to explicitly carry out these two tasks, we have to know the *prime* divisors C with $C^2 < 0$. We thus define $\mathrm{Neg}(X)$ to be the classes of those prime divisors C with $C^2 < 0$ and further define $\mathrm{neg}(X)$ to be the subset of $\mathrm{Neg}(X)$ of classes of those C with $C^2 < -1$.

We now give the definition of the *negative curve type* of a set of points $p_1, \dots, p_r \in \mathbb{P}^2$.

DEFINITION 2.5 Let $p_1, \dots, p_r$ and $p'_1, \dots, p'_r$ be ordered sets of distinct points of $\mathbb{P}^2$. Let X (X', respectively) be the surfaces obtained by blowing the points up. Let $L, E_1, \dots, E_r$ and $L', E'_1, \dots, E'_r$ be the respective exceptional configurations, and let $\phi : \mathrm{Cl}(X) \to \mathrm{Cl}(X')$ be the homomorphism induced by mapping $L \mapsto L'$ and $E_i \mapsto E'_i$ for all $i > 0$. We say $p_1, \dots, p_r$ and $p'_1, \dots, p'_r$ have the same *negative curve type* if ϕ maps $\mathrm{Neg}(X)$ bijectively to $\mathrm{Neg}(X')$.

What will turn out to be the case is that if either $r \leq 8$ or the points $p_1, \dots, p_r$ lie on a conic, then the configuration types are precisely the negative curve types.

In Section 3 we show how to carry out the two tasks mentioned above and, in the process, demonstrate that configuration type and negative curve type are the same, for the case that X is obtained by blowing up r points on a conic. The case of points on a line was handled in [17] and is, in any case, subsumed by the case of points on a conic, which also subsumes the case of $r \leq 5$ points. In Section 4 we treat the case of $7 \leq r \leq 8$ points.

3. Points on a conic

As mentioned above, in this section we show how to carry out our two tasks for the case of any r points on a conic. This case is technically simpler than the cases of $r = 6,\ 7$, or 8 points, but has the advantage that all the major ingredients for handling the cases of $6, 7$ or 8 points are already present in the simpler argument for points on a conic. What should become clear from the discussion of this special case is the reason why there are only finitely many configuration types and why they are the same as the negative curve types.

In order to state the next result efficiently, we introduce some finite families of special divisors on surfaces obtained by blowing up $\mathbb{P}^2$ at r distinct points lying on a conic. Let $\mathcal{B}$ be the classes $\{E_1, \cdots, E_r\}$ on X, obtained from the blow-ups of the points $p_1, \ldots, p_r$. These are always classes in $\mathrm{Neg}(X)$ but not in $\mathrm{neg}(X)$. Let $\mathcal{L}$ be the set of classes

$$\mathcal{L} := \{L - E_{i_1} - \cdots - E_{i_j} \mid 2 \leq j, \quad 0 < i_1 < \cdots < i_j \leq r\}.$$

Note that these classes are all potentially in $\mathrm{Neg}(X)$ (such a class may not be a prime divisor; e.g., if p_1, p_2, p_3 are three points on a line, then $L - E_1 - E_2 = (L - E_1 - E_2 - E_3) + E_3$ is a sum of two classes, where E_3 is clearly prime and so is $L - E_1 - E_2 - E_3$ if there are no additional points p_i on the line). If $r > 4$ then there is another effective divisor which is potentially in $\mathrm{Neg}(X)$, namely $Q = 2L - E_1 - \cdots - E_r$. Notice also that the divisor $A_r = (r-2)L - K_X$ meets all the divisors in the set $\mathcal{B} \cup \mathcal{L} \cup \{Q\}$ positively.

There is no loss in assuming $r \geq 2$, since the case $r < 2$ is easy to handle by ad hoc methods. (For example, if $r = 0$ and $F = tL$, then F is effective if and only if $t \geq 0$, and F is nef if and only if $t \geq 0$. If $r = 1$ and $F = tL - mE_1$, then F is effective if and only if $t \geq 0$ and $t \geq m$, and F is nef if and only if $t \geq m$ and $m \geq 0$. However, if $r \leq 1$, then it is not true that every element of $\mathrm{EFF}(X)$ is a nonnegative integer linear combination of elements of $\mathrm{Neg}(X)$.)

Proposition 3.1

Let X *be obtained by blowing up* $r \geq 2$ *distinct points of* $\mathbb{P}^2$*, and assume that* $Q = 2L - E_1 - \cdots - E_r$ *is effective.*

(a) *Then* $\mathrm{NEF}(X) \subset \mathrm{EFF}(X)$*, and every element of* $\mathrm{EFF}(X)$ *is a nonnegative integer linear combination of elements of* $\mathrm{Neg}(X)$*;*
(b) $\mathrm{Neg}(X) \subset \mathcal{L} \cup \mathcal{B} \cup \{Q\}$*;*
(c) $\mathrm{Neg}(X) = \mathrm{neg}(X) \cup \{C \in \mathcal{L} \cup \mathcal{B} \cup \{Q\} \mid C^2 = -1, C \cdot D \geq 0$ *for all* $D \in \mathrm{neg}(X)\}$*;*
(d) *for every nef divisor* F*,* $|F|$ *is base point (hence fixed component) free; and*
(e) *the divisor* A_r *is ample.*

Proof. Parts (a, b, d) follow from Lemma 3.1.1 [17]. For (c), we must check that if $C \in \mathcal{L} \cup \mathcal{B} \cup \{Q\}$, $C^2 = -1$ and $C \cdot D \geq 0$ for all $D \in \mathrm{neg}(X)$, then C is the class of a prime divisor.

First note that any $C \in \mathcal{L} \cup \mathcal{B} \cup \{Q\}$, with $C^2 = -1$ is effective. So, write $C = f_1F_1 + \cdots + f_sF_s$ with the F_i effective and prime and the f_i nonnegative integers. Now

$$-1 = C^2 = f_1 C \cdot F_1 + \cdots + f_s C \cdot F_s$$

and so $C \cdot F_i < 0$ for some i with $f_i > 0$. But

$$C \cdot F_i = f_1 F_1 \cdot F_i + \cdots + f_i F_i^2 + \cdots + f_s F_s \cdot F_i$$

and $F_j \cdot F_i \geq 0$ for all $j \neq i$. Thus $F_i^2 < 0$ as well.

Now suppose that $C \cdot D \geq 0$ for all $D \in \mathrm{neg}(X)$. If $F_i^2 \leq -2$ then $F_i \in \mathrm{neg}(X)$ and so $C \cdot F_i \geq 0$ which is a contradiction. So, we must have $F_i^2 = -1$. Hence, by (b), $F_i \in \mathcal{L} \cup \mathcal{B} \cup \{Q\}$. But we can write down every element of $\mathcal{L} \cup \mathcal{B} \cup \{Q\}$ with self-intersection -1 and explicitly check that no two of them meet negatively. Thus

$C \cdot F_i < 0$ implies $C = mF_i$ for some positive multiple m of F_i, but by explicit check, if mF_i is in $\mathcal{L} \cup \mathcal{B} \cup \{Q\}$, then $m = 1$, and hence $C = F_i$ is a prime divisor.

As for part (e), we already observed that A_r meets every element in $\mathcal{L} \cup \mathcal{B} \cup \{Q\}$ positively, and from (b) we have $\mathrm{Neg}(X) \subset \mathcal{L} \cup \mathcal{B} \cup \{Q\}$. Since $\mathrm{Neg}(X)$ generates $\mathrm{EFF}(X)$, that is enough to finish the proof. □

We now show how to use Proposition 3.1 to tell if F is effective or not, and to calculate $h^0(X, F)$ when F is effective — *assuming that we know the finite set* $\mathrm{neg}(X)$ *and hence* $\mathrm{Neg}(X)$.

Let $H = F$, $N = 0$. If $H \cdot C < 0$ for some $C \in \mathrm{Neg}(X)$, replace H by $H - C$ (note that this reduces $H \cdot A_r$ since $C \cdot A_r > 0$) and replace N by $N = N + C$.

Eventually either $H \cdot A_r < 0$ (hence H and thus F is not effective) or $H \cdot C \geq 0$ for all $C \in \mathrm{Neg}(X)$, hence H is nef and effective by Proposition 3.1 (a), and we have a Zariski decomposition $F = H + N$ with $h^0(X, F) = h^0(X, H)$ and so $h^0(X, F) = (H^2 - H.K_X)/2 + 1$ by Theorem 2.3.

What becomes clear from this procedure is that if we start with two sets of r points on a conic and the isomorphism ϕ of groups given in Definition 2.5 takes $\mathrm{neg}(X)$ bijectively to $\mathrm{neg}(X')$ then the calculations of $h^0(X, F)$ and $h^0(X', \phi(F))$ are formally the same and give $h^0(X, F) = h^0(X', F)$.

From the definition we see that the classes in $\mathrm{neg}(X)$ are simply the classes of self-intersection < -1 in $\mathrm{Neg}(X)$, so $\mathrm{Neg}(X)$ determines $\mathrm{neg}(X)$. Part (c) of Proposition 3.1 tells us that $\mathrm{neg}(X)$ determines $\mathrm{Neg}(X)$. Thus, it is clear that if we have two configurations of r points on a conic, then ϕ takes $\mathrm{Neg}(X)$ bijectively onto $\mathrm{Neg}(X')$ if and only if it takes $\mathrm{neg}(X)$ bijectively onto $\mathrm{neg}(X')$.

It follows that two configurations of r points on a conic have the same configuration type if and only if they have the same negative curve type. That type is also completely determined by the subset of $\mathcal{L} \cup \mathcal{B} \cup \{Q\}$ that turns out to be $\mathrm{neg}(X)$.

Thus, to enumerate all configuration types of r points on a conic, it is enough to enumerate all the subsets of $\mathcal{L} \cup \mathcal{B} \cup \{Q\}$ which can be $\mathrm{neg}(X)$.

This classification turns out to be rather simple in the case of r points on a conic. It is this fact that distinguishes this case from that $6, 7$ or 8 points (which we will treat in the following sections).

We now work out the classification, up to labeling of the points. We distinguish between two possibilities for r points on a conic: either the r points lie on an irreducible conic, or they lie on a reducible conic and no irreducible conic passes through the points.

If the r points are on an irreducible conic, then since no line meets an irreducible conic in more than 2 points, no elements of $\mathcal{L}$ are ever in $\mathrm{neg}(X)$. Elements of $\mathcal{B}$ are never in $\mathrm{neg}(X)$, so we need only decide if Q is in $\mathrm{neg}(X)$. Thus there are two cases:

I. $\mathrm{neg}(X)$ empty; i.e. $Q \notin \mathrm{neg}(X)$. In this case we must have $r \leq 5$.

II. $\mathrm{neg}(X) = \{Q\}$; in this case $r > 5$.

Now assume that the r points lie only on a reducible conic. In this case, Q is never a prime divisor and so we must decide only which of the elements of $\mathcal{L}$ are in $\mathrm{neg}(X)$. Note also that in this case we have $r \geq 3$ (and if $r = 3$, the 3 points must be collinear).

So, suppose that the r points consist of a on one line and b on another line (where we may assume that $0 \leq a \leq b$). We have to keep in mind the possibility that one of the points could be on both lines, i.e. it could be the point of intersection of the two lines. Thus a and b satisfy the formula $a + b = r + \epsilon$, where $\epsilon = 0$ if the point of intersection of the two lines is not among our r points and $\epsilon = 1$ if it is. We thus obtain two additional cases (depending on whether or not $a \geq 3$), with two subcases (depending on whether ϵ is 0 or 1) when $a \geq 3$:

III. $a \geq 3$ with $0 \leq \epsilon \leq 1$: in this case $\text{neg}(X)$ contains exactly two classes. Up to relabeling, these classes are $L - E_1 - \cdots - E_a$ and $L - E_{a+1} - \cdots - E_r$ if $\epsilon = 0$, and $L - E_1 - \cdots - E_a$ and $L - E_1 - E_{a+1} - \cdots - E_r$ if $\epsilon = 1$.

IV. $a < 3$, $b \geq 3$ with $\epsilon = 0$: in this case, up to relabeling, $\text{neg}(X)$ contains exactly one class, namely $L - E_1 - \cdots - E_b$. (Note that $a = 2$ with $\epsilon = 1$ gives the same configuration type as does $a = 1$ with $\epsilon = 0$, and that $a = 1$ with $\epsilon = 1$ gives the same configuration type as does $a = 0$ with $\epsilon = 0$. Thus when $a < 3$ we may assume $\epsilon = 0$.)

Remark 3.2 It is worth observing that here there were very few possibilities for $\text{neg}(X)$. The fact is that by Bezout considerations, certain prime divisors cannot '*coexist*' in $\text{neg}(X)$. For example, since a line cannot intersect an irreducible conic in more than two points we couldn't have Q and $L - E_{i_1} - E_{i_2} - E_{i_3}$ both in $\text{neg}(X)$ at the same time. The fact that it is easy to decide what can and cannot coexist is what makes the case of points on a conic relatively simple. In the next section we will see that the analysis of which divisors can coexist in $\text{neg}(X)$ is much more subtle and even depends on the characteristic of our algebraically closed field.

There are thus only finitely many configuration types for r points on a conic. Given any such points p_i, we also see by Theorem 2.3 and Proposition 3.1 that the coefficients m_i and the configuration type of the points completely determine h_Z and the graded Betti numbers of $I(Z)$ for $Z = m_1p_1 + \cdots + m_rp_r$.

EXAMPLE 3.3 We now work out a specific example, showing how Bezout considerations completely determine the Hilbert function and graded Betti numbers in the case of any Z with support in a conic. (By [12], the same considerations apply for any Z with support at any $r \leq 6$ points. If $7 \leq r \leq 8$, Bezout considerations still determine the Hilbert function, but the Betti numbers are currently not always known.)

By Bezout considerations we mean this: To determine Hilbert functions and graded Betti numbers it is enough to determine fixed components, and to determine fixed components, it is enough to apply Bezout's theorem (which says that two curves whose intersection is more than the product of their degrees must have a common component).

Consider distinct points $p_1, \ldots, p_5$ on a line L_1, and distinct points $p_6, \ldots, p_9$ on a different line L_2 such that none of the points is the point $L_1 \cap L_2$ which we take to be p_{10}. Let $Z = 4p_1 + 2(p_2 + \cdots + p_5) + 3p_6 + 3p_7 + 2p_8 + 2p_9 + 3p_{10}$. Then Z has degree 46 and $h_Z(t)$ for $0 \leq t \leq 15$ is 1, 3, 6, 10, 15, 21, 28, 31, 35, 38, 40, 42, 44, 45, 46, 46. The complete set of graded Betti numbers for F_0 is 7^5, 9^3, 10^1, 13^1, 15^1, and for F_1 they are 8^5, 9^1, 10^1, 11^1, 14^1, 16^1, where, for example, 7^5 for F_0 signifies that $t_7 = 5$, and 8^5 for F_1 signifies that $s_8 = 5$.

We first demonstrate how to compute $h_Z(t)$. First, $\binom{t+2}{2} - h_Z(t) = h^0(X, F(Z,t))$, where X is obtained by blowing up the points p_i, and

$$F(Z,t) = tL - 4E_1 - 2(E_2 + \cdots + E_5) - 3E_6 - 3E_7 - 2E_8 - 2E_9 - 3E_{10}.$$

Clearly the proper transform $L'_1 = L - E_1 - \cdots - E_5 - E_{10}$ of L_1 is in $\mathrm{Neg}(X)$, as are $L'_2 = L - E_6 - \cdots - E_9 - E_{10}$, $L'_{ij} = L - E_i - E_j$ for $1 \leq i \leq 5$ and $6 \leq j \leq 9$, and E_i for $1 \leq i \leq 10$. Since the proper transform of no other line has negative self-intersection, we see by Proposition 3.1 that $\mathrm{Neg}(X)$ consists exactly of L'_1, L'_2, the L'_{ij} and the E_i.

Consider $h(6)$. Then $F(X,6) \cdot L'_1 = -9$; i.e., any curve of degree 6 containing Z has an excess intersection with L_1 of 9: we expect a curve of degree 6 to meet a line only 6 times, but containing Z forces an intersection of 4 at p_1, 2 at p_2, etc, for a total of 15, giving an excess of 9. Thus any such curve must contain L_1 as a component. Said differently, if $F(Z,6)$ is effective, then so is

$$F(Z,6) - L'_1 = 5L - 3E_1 - 1(E_2 + \cdots + E_5) - 3E_6 - 3E_7 - 2E_8 - 2E_9 - 2E_{10}.$$

But $(F(Z,6) - L'_1) \cdot L'_2 < 0$, hence if $F(Z,6) - L'_1$ is effective , so is $F(Z,6) - L'_1 - L'_2$. Continuing in this way, we eventually conclude that if $F(Z,6)$ is effective, so is $F(Z,6) - 3L'_1 - 2L'_2 - (E_2 + \cdots + E_5) - 2E_{10} - L_{1,6} = -E_7$. But $A_{10} \cdot (-E_7) < 0$, so we know $F(Z,6) - (3L'_1 + 2L'_2 + (E_2 + \cdots + E_5) + 2E_{10} - L_{1,6})$ is not effective, so $h_Z(6) = 28$.

As another example, we compute $h_Z(12)$. As before, if $h^0(X, F(Z,12)) = h^0(X,H)$ where

$$H = F(Z,12) - L'_1 - L'_2 = 10L - 3E_1 - 1(E_2 + \cdots + E_5) - 2E_6 - 2E_7 - E_8 - E_9 - E_{10}.$$

But H is nef, since H meets every element of $\mathrm{Neg}(X)$ nonnegatively, so $h^0(X,H) = \binom{10+2}{2} - 19 = 47$ by Theorem 2.3(a). Thus $h_Z(12) = 91 - 47 = 44$. Also, the linear system $F(Z,12)$ decomposes as $H + (L'_1 + L'_2)$, where H is nef and $L'_1 + L'_2$ is fixed. A similar calculation shows $F(Z,13)$ has fixed part L'_1. Since the complete linear system of curves of degree 12 through Z has a degree 2 base locus (corresponding to $L'_1 + L'_2$), while the complete linear system of curves of degree 13 through Z has only a degree 1 base locus (i.e., L'_1), the map $\mu_{12} : I(Z)_{12} \otimes R_1 \to I(Z)_{13}$ cannot be surjective, so we know at least 1 homogeneous generator of $I(Z)$ is required in degree 13. But μ_H is surjective, by Theorem 2.3. Thus

$$\dim \mathrm{cok}(\mu_{12}) = h^0(X, F(Z,13)) - h^0(X, H+L) = 60 - 59 = 1.$$

Thus the graded Betti number for F_0 in degree 13 is exactly 1. We compute the rest of the graded Betti numbers for F_0 the same way. The graded Betti numbers for F_1 can be found using the well known formula $s_i = t_i + \Delta^3 h_Z(i)$ for $i > 0$ and the fact that $s_0 = 0$.

Remark 3.4 One problem raised in [10] is the existence and determination of maximal and minimal Hilbert functions. For example, [10] shows that there must be some Z' such that $h_{2Z'}$ is at least as big in every degree as h_{2Z} for every Z with $h_Z = h_{Z'}$; this $h_{2Z'}$ is referred to as $\underline{h}^{\max}$. The proof in [10] is nonconstructive, and [10] determines

$\underline{h}^{\max}$ in only a few special cases. The paper [10] also raises the question of whether $\underline{h}^{\min}$ always exists; i.e., whether there exists a Z' such that h_{2Z} is at least as big in every degree as $h_{2Z'}$ for every Z with $h_Z = h_{Z'}$. This question remains open.

Below, we will consider all possible Hilbert functions h_Z, where $Z = p_1 + \cdots + p_r$ for points p_i on a conic, and for each possible Hilbert function h, we will determine which configuration types Z have $h_Z = h$. We will also determine h_{2Z} for each configuration type Z. One could, as shown in Example 3.3, also determine the graded Betti numbers. Thus, for points on a conic, this completes the program begun in [10] of determining the Hilbert functions (and graded Betti numbers) of double point schemes whose support has given Hilbert function. Moreover, by an inspection of the results below, one sees that among the configuration types of reduced schemes $Z = p_1 + \cdots + p_r$ for points on a conic with a given Hilbert function there is always one type which specializes to each of the others, and there is always one which is a specialization of each of the others. It follows by semincontinuity that $\underline{h}^{\max}$ and $\underline{h}^{\min}$ exist, as do the analogous functions *for any* m; i.e., not only for the schemes $2Z$ but for the schemes mZ for every $m \geq 1$. Not only do these minimal and maximal Hilbert functions exist, but because there are only finitely many configuration types for points on a conic they can be found explicitly for any specific m: in particular, pick any subscheme $Z = p_1 + \cdots + p_r$ of distinct points p_i such that $h_Z(2) < 6$. Since $h_Z(2) < 6$, for any $Z' = p'_1 + \cdots + p'_r$ such that $h_{Z'} = h_Z$, the points p'_i must lie on a conic. Now, for any given m, compute $h_{mZ'}$ (as demonstrated in Example 3.3) for each configuration type and choose those Z' for which $h_{2Z'}$ is maximal, or minimal, as desired, from among all Z' for which $h_{Z'} = h_Z$. In addition to finding the maximal and minimal Hilbert functions, this method also determines which configuration types give rise to them.

Now we determine the configuration types corresponding to each Hilbert function h for reduced 0-dimensional subschemes with $h(2) < 6$ (i.e., $h = h_Z$ for subschemes $Z = p_1 + \cdots + p_r$ for distinct points p_i on a conic), following which we give the Hilbert function h_{2Z} for each type Z. We do not give detailed proofs; instead we note that in case the points p_i are smooth points on a reducible conic, the Hilbert functions of Z and of $2Z$ can be easily written down using the results of [9] for Z or using [10] (for $\mathrm{char}(k) = 0$) or [6] (for $\mathrm{char}(k)$ arbitrary) for either Z or $2Z$. With somewhat more effort, the case of points on an irreducible conic and the case of points on a reducible conic when one of the points p_i is the singular point of the conic can be analyzed using the method of Example 3.3.

It is convenient to specify a Hilbert function h using its first difference, which we will denote Δh. Using this notation, the possible Hilbert functions for $Z = p_1 + \cdots + p_r$ for distinct points p_i on a conic are precisely those of the form

$$\Delta h = 1\ \underbrace{2\ \cdots\ 2}_{i \text{ times}}\ \underbrace{1\ \cdots\ 1}_{j \text{ times}}\ 0\ \cdots,$$

where $i \geq 0$ and $j \geq 0$. It turns out that every Hilbert function h of r points on a conic of $\mathbb{P}^2$ corresponds to either one, two, three or four configuration types of r points. We now consider each of the possibilities.

We begin with the case that $i = 0$ and $j \geq 0$:

$$\Delta h = 1\ \underbrace{1\ \cdots\ 1}_{j}\ 0\ \cdots.$$

Here $h(1) < 3$, so in this case the r points are collinear (with $r = j + 1$), hence there is only one configuration type, it has $\mathrm{neg}(X) = \emptyset$ for $r < 3$ and $\mathrm{neg}(X) = \{L - E_1 - \cdots - E_r\}$ for $r \geq 3$, and $\underline{h}^{\min} = \underline{h}^{\max}$.

For the remaining cases we have $i > 0$, so $h(1) = 3$ but $h(2) < 6$, hence the r points p_i lie on a conic, but not on a line. We begin with $i = 1$ and $j > 0$, so $r = j + 3$:

$$\Delta h = 1\ 2\ \underbrace{1\ \cdots\ 1}_{j}\ 0\ \cdots\ .$$

We obtain this Hilbert function either for $r = 4$ general points (in which case $\mathrm{neg}(X) = \emptyset$), or in case we have $r-1$ points on one line and one point off that line (in which case, up to equivalence, $\mathrm{neg}(X) = \{L - E_1 - \cdots - E_{r-1}\}$). Thus, when $r = 4$ this Hilbert function arises from two configuration types, and for $r > 4$ it arises from exactly one configuration type. Thus except when $r = 4$, we have $\underline{h}^{\min} = \underline{h}^{\max}$, and, when $r = 4$, $\underline{h}^{\max}$ occurs when the four points are general while $\underline{h}^{\min}$ occurs when exactly three of the four are collinear.

Next consider $r = 2i + j + 1$, $i \geq 2$, $j \geq 2$.

$$\Delta h = 1\ \underbrace{2\ \cdots\ 2}_{i}\ \underbrace{1\ \cdots\ 1}_{j}\ 0\ \cdots\ .$$

This can occur for points only on a reducible conic; in this case, recalling the formula $a + b = r + \epsilon$, we have $b = i + j + 1$ and $a = i + \epsilon$. If $i = 2$ and $\epsilon = 0$, then $\mathrm{neg}(X) = \{L - E_{i_1} - \cdots - E_{i_b}\}$. Otherwise, we have $\mathrm{neg}(X) = \{L - E_{i_1} - \cdots - E_{i_b}, L - E_{i_{b+1}} - \cdots - E_{i_{b+a}}\}$, with the indices being distinct if $\epsilon = 0$ but with $i_1 = i_{b+1}$ if $\epsilon = 1$. Thus this Hilbert function arises from two configuration types for each given i and j, and $\underline{h}^{\min}$ occurs when $\epsilon = 1$ and $\underline{h}^{\max}$ when $\epsilon = 0$.

Now consider $i > 0$ and $j = 0$, so $r = 2i + 1$:

$$\Delta h = 1\ \underbrace{2\ \cdots\ 2}_{i}\ 0\ \cdots\ .$$

This case comes either from the configuration type with $r \geq 3$ points (with r odd) on an irreducible conic (in which case $\mathrm{neg}(X) = \emptyset$ if $i = 1$ or 2, and $\mathrm{neg}(X) = \{2L - E_1 - \cdots - E_r\}$ if $i > 2$), or from the configuration types with $r = a + b - \epsilon$ points on a pair of lines, for which $b = i + 1$, $a = i > 1$ and $\epsilon = 0$ (in which case, up to equivalence, $\mathrm{neg}(X) = \{L - E_1 - \cdots - E_3\}$ if $i = 2$ and $\mathrm{neg}(X) = \{L - E_1 - \cdots - E_b, L - E_{b+1} - \cdots - E_r\}$ if $i > 2$) or for which $b = a = i + 1 > 2$ and $\epsilon = 1$ (in which case $\mathrm{neg}(X) = \{L - E_1 - \cdots - E_b, L - E_1 - E_{b+1} - \cdots - E_r\}$). Thus this Hilbert function arises from a single configuration type if $i = 1$ (in which case $\underline{h}^{\min} = \underline{h}^{\max}$), and from three configuration types for each $i > 1$ (in which case $\underline{h}^{\min}$ occurs when the conic is reducible with $b = a$, and $\underline{h}^{\max}$ occurs when the conic is irreducible).

Finally, consider $i > 1$ and $j = 1$ (note that $i = j = 1$ is subsumed by the previous case), so $r = 2i + 2$:

$$\Delta h = 1\ \underbrace{2\ \cdots\ 2}_{i}\ 1\ 0\ \cdots\ .$$

This case comes from either the configuration type with $r \geq 6$ points (with r even) on an irreducible conic (in which case $\mathrm{neg}(X) = \{2L - E_1 - \cdots - E_r\}$), or from the configuration types with $r = a + b - \epsilon$ points on a pair of lines for which either: $b = i + 2$, $a = i$ and $\epsilon = 0$ (in which case $\operatorname{neg}(X) = \{L - E_{i_1} - \cdots - E_{i_b}\}$ if $i = 2$ and $\operatorname{neg}(X) = \{L-E_{i_1}-\cdots-E_{i_b}, L-E_{i_{b+1}}-\cdots-E_{i_{b+a}}\}$ if $i > 2$); $b = a = i+1$ and $\epsilon = 0$ (in which case $\operatorname{neg}(X) = \{L-E_{i_1}-\cdots-E_{i_b}, L-E_{i_{b+1}}-\cdots-E_{i_{b+a}}\}$); or $b = i+2$, $a = i+1$ and $\epsilon = 1$ (in which case $\operatorname{neg}(X) = \{L-E_1-E_{i_1}-\cdots-E_{i_b}, L-E_1-E_{i_{b+1}}-\cdots-E_{i_{b+a}}\}$). Thus this Hilbert function arises from four configuration types for each i, and $\underline{h}^{\min}$ occurs when the conic is irreducible while $\underline{h}^{\max}$ occurs when the conic is reducible with $\epsilon = 1$.

We now write down the Hilbert functions of double point schemes for each configuration type. In the case of r points on a line, the corresponding double point scheme $2Z$ has Hilbert function

$$\Delta h_{2Z} = 1\ \underbrace{2\ \cdots\ 2}_{r}\ \underbrace{1\ 1\ \cdots\ 1}_{r-1}\ 0\ \cdots .$$

The possible Hilbert functions for $2Z$ for $Z = p_1 + \cdots + p_r$ contained in a conic but not on a line are all of the form

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i}\ \underbrace{3\ \cdots\ 3}_{j}\ \underbrace{2\ \cdots\ 2}_{k}\ \underbrace{1\ \cdots\ 1}_{l}\ 0\ \cdots ,$$

for various nonnegative integers i, j, k and l.

First suppose the conic is irreducible. Since $r = 2$ is the case of points on a line, we may assume that $r \geq 3$. If $r = 3$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ 3\ 0\ \cdots .$$

If $r = 4$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ 4\ 2\ 0\ \cdots .$$

If $r \geq 5$ is odd, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=\frac{r-1}{2}}\ \underbrace{2\ \cdots\ 2}_{k=\frac{r-5}{2}}\ 1\ 0\ \cdots .$$

If $r \geq 6$ is even, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=\frac{r}{2}-1}\ 3\ \underbrace{2\ \cdots\ 2}_{k=\frac{r-6}{2}}\ 1\ 0\ \cdots .$$

Now assume the points lie on a reducible conic, consisting of two distinct lines, and that no line nor any irreducible conic contains the points. Thus $4 \leq r = a + b - \epsilon$, and we may assume that $1 \leq a \leq b$.

If $2a < b$ and $\epsilon = 0$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a}\ \underbrace{3\ \cdots\ 3}_{j=a-1}\ \underbrace{2\ \cdots\ 2}_{k=b-2a-1}\ \underbrace{1\ \cdots\ 1}_{l=b-1}\ 0\ \cdots .$$

If $a < b < 2a$ and $\epsilon = 0$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a}\ \underbrace{3\ \cdots\ 3}_{j=b-a-1}\ \underbrace{2\ \cdots\ 2}_{k=2a-b-1}\ \underbrace{1\ \cdots\ 1}_{l=2b-2a-1}\ 0\ \cdots .$$

If $b = 2a$ and $\epsilon = 0$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a}\ \underbrace{3\ \cdots\ 3}_{j=a-2}\ 2\ \underbrace{1\ \cdots\ 1}_{l=2(a-1)}\ 0\ \cdots.$$

The only remaining case with $\epsilon = 0$ is $b = a \geq 3$, in which case

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a-1}\ 3\ \underbrace{2\ \cdots\ 2}_{k=a-3}\ 1\ 0\ \cdots.$$

If $\epsilon = 1$, we may as well assume that $3 \leq a \leq b$, since $a = 2$ and $\epsilon = 1$ is the same as $a = 1$ and $\epsilon = 0$ (for appropriate b's). If $a = b$ and $\epsilon = 1$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a-2}\ 3\ \underbrace{2\ \cdots\ 2}_{k=a-2}\ 0\ \cdots.$$

If $a < b \leq 2a - 1$ and $\epsilon = 1$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a-1}\ \underbrace{3\ \cdots\ 3}_{j=b-a-1}\ \underbrace{2\ \cdots\ 2}_{k=2a-b-1}\ \underbrace{1\ \cdots\ 1}_{l=2(b-a)}\ 0\ \cdots.$$

If $2a - 1 \leq b$ and $\epsilon = 1$, then

$$\Delta h_{2Z} = 1\ 2\ 3\ \underbrace{4\ \cdots\ 4}_{i=a-1}\ \underbrace{3\ \cdots\ 3}_{j=a-2}\ \underbrace{2\ \cdots\ 2}_{k=b-2a+1}\ \underbrace{1\ \cdots\ 1}_{l=b-1}\ 0\ \cdots.$$

4. The case of $r \leq 8$ distinct points of $\mathbb{P}^2$

We now consider the case of $r \leq 8$ distinct points of $\mathbb{P}^2$. As in the case of points on a conic, we define some finite families of special divisors on the surface X obtained by blowing up the points $p_1, \ldots, p_r$, where $r \leq 8$. As in earlier sections we will let $E_i \in \mathrm{Cl}(X)$ be the divisors corresponding to the blown up points and L the divisor class which is the preimage of a general line in $\mathbb{P}^2$. The families will be denoted:

- $\mathcal{B}_r = \{E_1, \ldots, E_r\}$;
- $\mathcal{L}_r = \{L - E_{i_1} - \cdots - E_{i_j} \mid 2 \leq j \leq r\}$;
- $\mathcal{Q}_r = \{2L - E_{i_1} - \cdots - E_{i_j} \mid 5 \leq j \leq r\}$;
- $\mathcal{C}_r = \{3L - 2E_{i_1} - E_{i_2} - \cdots - E_{i_j} \mid 7 \leq j \leq 8, j \leq r\}$; and
- $\mathcal{M}_8 = \{4L - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - E_{i_4} - \cdots - E_{i_8}, 5L - 2E_{i_1} - \cdots - 2E_{i_6} - E_{i_7} - E_{i_8}, 6L - 3E_{i_1} - 2E_{i_2} - \cdots - 2E_{i_8}\}$.

Let $\mathcal{N}_r = \mathcal{B}_r \cup \mathcal{L}_r \cup \mathcal{Q}_r \cup \mathcal{C}_r \cup \mathcal{M}_8$. Notice that the elements of $\mathcal{B}_r$ and $\mathcal{M}_8$ all have self-intersection -1, hence are never elements of $\mathrm{neg}(X)$. They will be elements of $\mathrm{Neg}(X)$ when they are classes of irreducible curves and as such are involved in determining Hilbert functions via the role they play in determining the fixed and free parts of linear systems $|F(Z, t)|$. (See Example 3.3 for an explicit demonstration of how $\mathrm{Neg}(X)$ is involved in computing Hilbert functions.)

Proposition 4.1

Let X *be obtained by blowing up* $2 \le r \le 8$ *distinct points of* $\mathbb{P}^2$.

(a) *Then* $\mathrm{NEF}(X) \subset \mathrm{EFF}(X)$*;*

(b) *if* $r = 8$*, then every element of* $\mathrm{EFF}(X)$ *is a nonnegative rational linear combination of elements of* $\mathrm{Neg}(X)$*;*

(c) *if* $r < 8$*, then every element of* $\mathrm{EFF}(X)$ *is a nonnegative integer linear combination of elements of* $\mathrm{Neg}(X)$*;*

(d) $\mathrm{Neg}(X) \subset \mathcal{N}_r$*;*

(e) $\mathrm{Neg}(X) = \mathrm{neg}(X) \cup \{C \in \mathcal{N}_r \mid C^2 = -1, C \cdot D \ge 0$ *for all* $D \in \mathrm{neg}(X)\}$*;*

(f) *for every nef divisor* F*,* $|F|$ *is fixed component free; and*

(g) *the divisor* A_r *is ample.*

Proof. Part (a) is [18, Theorem 8]. Part (f) follows from [19]. For (d), apply adjunction: if C is the class of a prime divisor with $C^2 < 0$, then $C^2 = -C \cdot K_X + 2g - 2$, where g is the arithmetic genus of C. Since $g \ge 0$, we see that either $-C \cdot K_X < 0$ (and hence C is a fixed component of $|-K_X|$), or $-C \cdot K_X \ge 0$ and hence $g = 0$ (so C is smooth and g is actually the genus of C) and either $-C \cdot K_X = -1 = C^2$ or $-C \cdot K_X = 0$ and $C^2 = -2$. But it is easy to check that $K_X^\perp$ is negative definite so it is not hard to list every class satisfying $-C \cdot K_X = 0$, $C^2 = -2$ and $L \cdot C \ge 0$. If one does so, one gets the subset of $\mathcal{N}_r$ of elements of self-intersection -2; see [22, Chapter IV] for details. Similarly, the set of all solutions to $C^2 = -1 = -C \cdot K_X$ consists of the elements in $\mathcal{N}_r$ of self intersection -1; again, we refer the reader to [22, Chapter IV]. If $-C \cdot K_X < 0$, then $-K_X - C$ is effective so $C \cdot L \le -K_X \cdot L = 3$. It is easy to check that the only such curves $C = dL - m_1E_1 - \cdots - m_rE_r$ with $d \le 3$ which are effective, reduced and irreducible, come from lines through 4 or more points (i.e., up to indexation, $C = L - E_1 - \cdots - E_j$, $j \ge 4$), or conics through 7 or 8 points (i.e., $C = 2L - E_1 - \cdots - E_j$, $7 \le j \le 8$). There can be no cubic, since a reduced irreducible cubic has at most one singular point, so would be either $-K_X$ itself or $3L - 2E_1 - E_2 - \cdots - E_8$, neither of which meets $-K_X$ negatively. Thus if $-C \cdot K_X < 0$, then C must be among the elements of $\mathcal{N}_r$ with self-intersection less than -2. This proves (d). The proof of (e) is essentially the same as that of Proposition 3.1(c). Parts (b) and (c) are well known; see [12] for a detailed proof in case $r = 6$. The proofs for other values of r are similar. Finally, the proof of (g) is essentially the same as the proof of Proposition 3.1(e). □

With Proposition 4.1, the procedure for computing $h^0(X, F)$ for any class F on a blow-up X of $\mathbb{P}^2$ at $r \le 8$ points is identical to the procedure if the points lie on a conic. Given $\mathrm{neg}(X)$, find $\mathrm{Neg}(X)$. Let $H = F$. Loop through the elements C of $\mathrm{Neg}(X)$. Whenever $F \cdot C < 0$, replace H by $H - C$. Eventually either $H \cdot A_r < 0$, in which case $h^0(X, F) = h^0(X, H) = 0$, or H is nef and $F = H + N$, where N is the sum of the classes subtracted off, and we have $h^0(X, F) = h^0(X, H) = (H^2 - K_X \cdot H)/2 + 1$ by Theorem 2.3. In case $r \le 6$, we can also compute the graded Betti numbers, as discussed above, since when H is nef, μ_H has maximal rank by Theorem 2.3.

It follows, in the same way as for points on a conic, that the negative curve type of sets of $r \le 8$ points is the same as the configuration type.

5. Configuration types for $r = 7$ or 8 points of $\mathbb{P}^2$

Given r distinct points of $\mathbb{P}^2$, if either $r \leq 8$ or the points lie on a conic, then in previous sections we saw how to compute the Hilbert function, and in some cases the graded Betti numbers, of any fat point subscheme with support at the r points, if we know the configuration type of the points (or, equivalently, if we know $\mathrm{neg}(X)$, where X is the surface obtained by blowing up the points). The types were easy to enumerate in the case of points on a conic. We now consider the case of $r \leq 8$ points. (If $r > 8$, then typically $\mathrm{Neg}(X)$ is infinite, and $\mathrm{neg}(X)$ can also be infinite. In fact, for each $r \geq 9$, there are infinitely many configuration types of r points; see Example 5.9. Moreover, for $r > 9$, $\mathrm{Neg}(X)$ is known only in special cases, such as, for example, when X is obtained by blowing up points on a conic.) Since any $r \leq 5$ points lie on a conic, and since the case $r = 6$ is done in detail in [12], we will focus on the cases $7 \leq r \leq 8$.

We begin by formalizing the notion of negative curve type: by Proposition 4.1, the elements of $\mathrm{neg}(X)$ are all in $\mathcal{N}_r$, and moreover if C and D are distinct elements of $\mathrm{neg}(X)$, then, being prime divisors, $C \cdot D \geq 0$. Thus we define a subset S of $\mathcal{N}_r$ to be *pairwise nonnegative* if whenever $C \neq D$ for elements C and D of S, then $C \cdot D \geq 0$.

DEFINITION 5.1 A *formal configuration type* for $r = 7$ or 8 points in $\mathbb{P}^2$ is a pairwise nonnegative subset S of $\mathcal{N}_r$. We say two types S_1 and S_2 are equivalent if by permuting $E_1, \ldots, E_r$ we can transform S_1 to S_2. If $S = \mathrm{neg}(X)$ for some surface X, we say S is *representable*; i.e., there is a bijective correspondence between configuration types and representable formal configuration types.

The formal configuration types for $r \leq 5$ points are a special case of the types for points on a conic, which are easy to enumerate and was done in Section 3. In this section we will explicitly list the formal configuration types S for sets of 6, 7 and 8 points, determine which are representable, and for each such formal configuration type, represented say by points $p_1, \ldots, p_r$, we will answer some of the questions raised in [10] by applying our procedure to compute Hilbert functions (and, for $r \leq 6$, the graded Betti numbers too), for subschemes $Z = p_1 + \cdots + p_r$ and $2Z$. Also, not only do we find all the Hilbert functions that arise, we also determine exactly what arrangements of the points are needed to give each Hilbert function. We note that the same procedure can be used to compute the Hilbert function for $m_1p_1 + \cdots + m_rp_r$ for any m_i, and what arrangements of the points give that Hilbert function, as long as $r \leq 8$.

Remark 5.2 Formal configuration types generalize the notion [7] of combinatorial geometries of rank up to 3. A combinatorial geometry is a formal specification of the linear dependencies on a set of "points." Formally, a combinatorial geometry on r "points" of rank at most 3 (hence formally the points are in the plane) can be defined to be a matrix with r columns. Each row contains only zeroes and ones, the sum of the entries in each row is at least 3, and the dot product of two different rows is never bigger than 1. The columns represent points, and each row represents a line. A 1 in row i and column j means that line i goes through point j; a 0 means that it doesn't. We allow a matrix with no rows, which just means the given combinatorial geometry consists of r points, no three of which are collinear. However, writing down a matrix with the required properties doesn't guarantee that any point set with the specified dependencies, and no other dependencies, actually exists. Determining whether there exists an actual set of points exhibiting a given combinatorial geometry is a separate question. When there does exist one over a given field k, one says that the combinatorial geometry is *representable* over k. A list of all combinatorial geometries on $r \leq 8$ points is given in [2], without dealing with the problem of representability.

Note that if one takes a combinatorial geometry of rank at most 3 on $r \leq 8$ points, each row of its matrix gives an element of $\mathcal{N}_r$ of self intersection < -1, as follows. Let m_{ij} be the entry in column j of row i. Then $C_i = L - m_{i1}E_1 - \cdots - m_{ir}E_r \in \mathcal{N}_r$, and the set S consisting of all of the C_i is a formal configuration type. Conversely, if S is a formal configuration type such that every $C \in S$ has $C \cdot L = 1$, then we can reverse the process and obtain a combinatorial geometry from S. However, we are also interested in cases where the points lie on conics and cubics, so $C \cdot L$ can, for us, be 2 or 3. (We could also allow infinitely near points, in which case we could have $C \cdot L = 0$.)

Since $S \subset \mathcal{N}_r$ and $\mathcal{N}_r$ is a finite set, it is clear that with enough patience one can write down every formal configuration type. The list given in [2] gives us all formal configuration types which do not involve curves of degree bigger than 1. To these we add, in all possible ways, classes $C \in \mathcal{N}_r$ with $C \cdot L = 2$ or 3. Each time we find a class that can be added to a previous formal configuration type without violating pairwise nonnegativity, we obtain another type. We must then check to see if it is equivalent to one which has already occurred. Eventually we obtain a complete list. We now give these lists.

Table 1 gives the list of the eleven formal configuration types (up to equivalence) for $r = 6$ points. It is not hard to see that each type is in fact representable. In the table, the 6 points $p_1, \ldots, p_6$ are denoted alphabetically by the lower case letters "a" through "f". Whenever a type S contains a class $L - E_{i_1} - \cdots - E_{i_j}$, the letters corresponding to the points $p_{i_1}, \ldots, p_{i_j}$ are listed in the table, preceded by a 1 to indicate that the coefficient of L is 1. Intuitively, configuration type 7, for example, is a specification that the points p_1, p_2, p_3 and p_4 are to be collinear, and so are the points p_1, p_5 and p_6. For type 11, no three of the points are to be collinear, but there is to be an irreducible conic containing the points $p_1, \ldots, p_6$. Thus the type 11 lists "2: abcdef", and S in this case consists of $2L - E_1 - \cdots - E_6$. Our notation mimics that of [2]. (Table 1 also gives the types for $r < 6$, using the following convention. The types for $r = 5$ correspond exactly to the types listed in Table 1 which do not involve the letter "f"; the types for $r = 4$ are those which do not involve the letters "e" or "f", etc.)

Table 2 gives both h_Z and the graded Betti numbers for $Z = m(p_1 + \cdots + p_r)$ for each configuration type with $r \leq 6$ for $m = 1$ and $m = 2$. The configuration types are listed using the same numbers as in the previous table. The table gives the Hilbert function h_Z by listing $h_Z(t)$ for every t from 0 up to the degree t for which $h_Z(t) = \deg(Z)$. The graded Betti numbers are specified using the same notation explained in Example 3.3.

Table 3 lists the configuration types (up to equivalence) for $r = 7$ points. There are 29 types, and each is representable over some field, although as we shall see in some cases representability depends on the characteristic of k.

Table 4 gives the Hilbert functions for $Z = p_1 + \cdots + p_7$ and $2Z$, for each type.

The configuration types are listed by the same item number used in the table above. Five different Hilbert functions occur for 7 distinct points of multiplicity 1. For three of these five Hilbert functions h_Z, only one Hilbert function is possible for h_{2Z}. For one of these five Hilbert functions, three different Hilbert functions occur for double points, and for the other, two different Hilbert functions occur for double points. All together, there are thus eight different Hilbert functions which occur for 7 distinct points in the plane of multiplicity 2. For each Hilbert function h of 7 simple points, we see from the table that there is both a maximum and a minimum Hilbert function among all Hilbert functions of double points whose support has the given Hilbert function h. The table groups together Hilbert functions of double point schemes whose support schemes have the same Hilbert function.

Tables 5 and 6 list the 146 formal configuration types (up to equivalence) for $r = 8$ points. Note that in every case of a configuration type involving a cubic, the cubic has a double point which is always assumed to be at the last point; i.e., the notation 3: abcdefgh denotes a cubic through all 8 points, always with a double point at h.

Table 7 gives h_Z and h_{2Z} for each reduced scheme Z corresponding to a formal configuration type of $r = 8$ points. Six different Hilbert functions occur for 8 distinct points of multiplicity 1. Again we see that there is both a maximum and a minimum Hilbert function among the Hilbert functions of double point schemes whose support scheme's Hilbert function is one of these six. For four of the six Hilbert functions h_Z, only one Hilbert function is possible for h_{2Z}. For both of the other two of these six Hilbert functions, three different Hilbert functions occur for double points. All together, there are thus ten different Hilbert functions which occur for 8 distinct points in the plane of multiplicity 2. (Note that types 30, 45 and 96 are not representable over any field k. Our procedure can still be run, however, so we show the result our procedure gives in these cases too. This is what would happen, if these types were representable.)

EXAMPLE 5.3 We give an example to demonstrate how we generated the lists of configuration types. Consider the case of 8 points. From the list of the 69 simple eight point matroids of rank at most three given in [2], we can immediately write down all formal configuration types contained in $\mathcal{L}_8$. This gives 69 configuration types.

For each of these 69 we then check to see which classes corresponding to either conics through 6 or more points or cubics through all 8 points and singular at one are compatible with the classes in the formal configuration type of the given matroid.

For example, take configuration 10. Configuration 10 is contained in $\mathcal{L}_8$ and thus comes directly from one of the 69 matroids. In terms of the basis $L, E_1, \cdots, E_8$, the classes in configuration type 10 are L_i for $i = 1, \ldots, 4$ given as follows:

$$\begin{array}{llrrrrrrrrr} L_1: & = & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ L_2: & = & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ L_3: & = & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ L_4: & = & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \end{array} .$$

After permuting the coefficients of the E_i, there are 28 classes in the orbit of $2L - E_1 - \cdots - E_6$, eight of the form $2L - E_1 - \cdots - E_7$, one of the form $2L - E_1 - \cdots - E_8$, and eight of the form $3L - 2E_1 - E_2 - \cdots - E_8$. Of these 45 classes, the only ones which meet all four classes of configuration type 10 nonnegatively are Q_1, Q_2 and C given as follows:

$$\begin{array}{lcrrrrrrrrrr} Q_1: & = & 2 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 \\ Q_2: & = & 2 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ C\ : & = & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{array}.$$

Since $Q_1 \cdot Q_2 \geq 0$ but $Q_i \cdot C < 0$ for $i = 1, 2$, the formal configuration types one gets from configuration 10 are: $\{L_1, L_2, L_3, L_4\}$ (i.e., type 10 itself), $\{L_1, L_2, L_3, L_4, Q_1\}$ (this has type 77), $\{L_1, L_2, L_3, L_4, Q_2\}$ (this also has type 77), $\{L_1, L_2, L_3, L_4, Q_1, Q_2\}$ (this has type 89) and $\{L_1, L_2, L_3, L_4, C\}$ (this has type 40). Of course, none of these are isomorphic to any obtained starting from a matroid not isomorphic to the one giving configuration type 10, but in fact $\{L_1, L_2, L_3, L_4, Q_1\}$ is isomorphic to $\{L_1, L_2, L_3, L_4, Q_2\}$ (since permuting the basis $L, E_1, \ldots, E_8$ by transposing indices 2 and 4 and at the same time transposing indices 3 and 5 permutes the E_i in such a way as to convert $\{L_1, L_2, L_3, L_4, Q_1\}$ into $\{L_1, L_2, L_3, L_4, Q_2\}$).

We now consider the problem of representability. Note that every 7 point formal configuration type is also, clearly, an 8 point formal configuration type. For example, type 24 for 7 points is (after a permutation of the points; just swap d and g) type 30 for 8 points. This type is also a combinatorial geometry: it is the well known Fano plane; that is, the projective plane over the integers modulo 2. In particular, regarded as a 7 point formal configuration type, it is representable in characteristic 2. Regarded as an 8 point type, however, it is never representable, since by the proof of Theorem 5.7 no matter where one picks the eighth point to be, it is either on one of the lines through pairs of the first seven points or there is a singular cubic through the first seven points with its singularity at the eighth point.

Theorem 5.4

Let k be an algebraically closed field. Type 23 for $r = 7$ points is representable if and only if $\mathrm{char}(k) \neq 2$*. Type 24 for $r = 7$ points is representable if and only if* $\mathrm{char}(k) = 2$*. The remaining types for $r = 7$ are representable over every algebraically closed field.*

Some lemmas will be helpful for the proof. Let S be a formal configuration type for r points. If $C = dL - m_1E_1 - \cdots - m_rE_r \in S$ and $i \leq r$, let $C_i = dL - m_1E_1 - \cdots - m_iE_i$ (i.e., C_i is obtained from C by truncation). Let $S(i)$ be the set of all classes C_i such that $C \in S$ and $C_i^2 \leq -2$.

Lemma 5.5

Let k be an algebraically closed field, and let S be a formal configuration type for $r = 7$ points. If $S(i)$ is k-representable for some $i < r$, and if the number $\#(S(i+1) - S(i))$ of elements in $S(i+1)$ but not in $S(i)$ is at most 1, then $S(i+1)$ is k-representable. In particular, if $\#(S(i+1) - S(i)) \leq 1$ for each $i < r$, then S is representable over k.

Proof. Every element of $S(i)$ is the truncation of an element of $\mathcal{N}_{i+1}$. Thus $S(1)$ is always empty, because there are no elements of $\mathcal{N}_2$ of self intersection < -1. Thus $S(1)$ is a k-representable 1 point formal configuration type. Suppose $S(i)$ is a k-representable i point formal configuration type. Thus there are points $p_1, \ldots, p_i$ in the projective plane $\mathbb{P}^2$ over k such that for the surface X_i obtained by blowing them up we have $S(i) = \mathrm{neg}(X)$.

Now assume $S(i+1) = S(i)$. Suppose for some point p_{i+1}, the surface X_{i+1} obtained by blowing up $p_1, \ldots, p_{i+1}$ has a prime divisor $C = dL - m_1E_1 - \cdots - m_{i+1}E_{i+1} \in \mathrm{neg}(X_{i+1})$ not in $S(i+1)$. Then $C \in \mathcal{N}_{i+1}$ with $C^2 < -1$. But the coefficients m_j are always 0 or 1 (since $i + 1 \leq r = 7$). Since $C \notin S(i+1) = S(i)$, we see $m_{i+1} = 1$. Thus $C' = dL - m_1E_1 - \cdots - m_iE_i$ is the class of a prime divisor on X_i with $(C')^2 \leq -1$. Thus $C' \in \mathrm{Neg}(X_i)$. Since $\mathrm{Neg}(X_i)$ is finite, the union of all such C' is a proper closed subset of X_i. If we pick p_{i+1} to avoid this closed subset, then $\mathrm{neg}(X_{i+1}) = S(i+1)$, so $S(i+1)$ is representable over k.

Finally, assume $\#(S(i+1)-S(i)) = 1$. Let $D = dL-m_1E_1-\cdots-m_{i+1}E_{i+1}$ be the class in $S(i+1)$ which is not in $S(i)$. Thus $m_{i+1} = 1$ and $D' = dL-m_1E_1-\cdots-m_iE_i$ is in $\mathcal{N}_i$. If $(D')^2 < -1$, then (keeping in mind how $S(i)$ is constructed) $D' \in S(i)$, and hence D' is the class of a prime divisor on X_i. If $(D')^2 = -1$, then since $D' \cdot C \geq 0$ for all $C \in S(i)$, by Proposition 4.1(e) D' again is the class of a prime divisor on X_i. Thus we must choose p_{i+1} to be a point on D', which, as before, is not on any C' in $\mathrm{Neg}(X_i)$ other than D'. These C' comprise a proper closed subset meeting D' in a finite subset, so choosing p_{i+1} to be any point of D' not in this finite subset results in $\mathrm{neg}(X_{i+1}) = S(i+1)$, showing that $S(i+1)$ is k-representable. □

Lemma 5.6

Let k be an algebraically closed field, and let S be a formal configuration type for $r \leq 8$ points. Assume that $S \subset K^\perp$, where $K = -3L+E_1+\cdots+E_r$ (i.e., $D \cdot K = 0$ for every element $D \in S$). Let T be the torsion subgroup of $K^\perp/\langle S\rangle$, where $\langle S\rangle$ is the subgroup of $K^\perp$ generated by S. If for some reduced irreducible plane cubic $C \subset \mathbb{P}^2(k)$ either $K^\perp/\langle S\rangle = T$ and T is isomorphic to a subgroup of $\mathrm{Pic}^0(C)$ or there are infinitely many positive integers l such that $T \times \mathbf{Z}/l\mathbf{Z}$ is isomorphic to a subgroup of $\mathrm{Pic}^0(C)$, then S is representable over k. In particular, if the number of elements $\#(T)$ of T is square-free, then S is representable over k.

Proof. Let G be the free Abelian group generated by $L, E_1, \ldots, E_r$. We can regard T as a subgroup of $G/\langle S\rangle$. If there are infinitely many l such that $T\times\mathbf{Z}/l\mathbf{Z}$ is isomorphic to a subgroup of $\mathrm{Pic}^0(C)$, then since $\mathcal{N}_r$ is finite, we can pick such an l that also has the property that there is a surjective homomorphism $\phi : K^\perp \to T \times \mathbf{Z}/l\mathbf{Z}$ factoring through the canonical quotient $K^\perp \to K^\perp/\langle S\rangle$ such that the only elements of $\mathcal{N}_r$ in $\ker(\phi)$ are the elements of $\mathcal{N}_r \cap \langle S\rangle$. Identifying $T \times \mathbf{Z}/l\mathbf{Z}$ with a subgroup of $\mathrm{Pic}^0(C)$, we may regard ϕ as giving a homomorphism to $\mathrm{Pic}^0(C)$. If $K^\perp/\langle S\rangle = T$, we proceed as before using $l = 1$. Since $K^\perp$ and E_1 generate G freely, we can extend ϕ to a homomorphism $\Phi : G \to \mathrm{Pic}(C)$ by mapping E_1 to an arbitrary smooth point p_1 of C, and taking $\Phi|_{K^\perp} = \phi$. The images $\Phi(E_i)$ now give points p_i of C since $E_i - E_1 \in K_X^\perp$, so $\Phi(E_i) = \Phi(E_1+(E_i-E_1)) = p_1+\phi(E_i-E_1)$ is linearly equivalent to a unique point of C, which we define to be p_i. Blowing up the points p_i gives a morphism $X \to \mathbb{P}^2$ of surfaces such that X has no prime divisor D of self-intersection less than -2. Indeed, suppose there were such a D. Note that $(-K_X)^2 = 9 - r > 0$ and that $-K_X$ is the class of the proper transform of C. Let us denote this proper transform by C'. Since $-K_X$ is the class of C', a reduced irreducible curve of positive self-intersection, we see that $-K_X$ is nef, but by adjunction $D^2 < -2$ implies $-K_X \cdot D < 0$, so D cannot be effective.

We want to show that $S = \text{neg}(X)$. To do this, we will use two facts about S. The first is that if $v \in \langle S \rangle$ has $v^2 = -2$ and $v \cdot L \geq 0$, then v is a nonnegative integer linear combination of elements of S. First we justify this fact. Since $r \leq 8$, $K_X^\perp$ is negative definite and even (i.e., $0 \neq v \in K_X^\perp$ implies v^2 is negative and even). Suppose $v \in \langle S \rangle$ has $v^2 = -2$. Write $v = \sum_i P_i - \sum_j N_j$ for elements P_i and N_j of S, where $P_i \neq N_j$ for all i and j. If $\sum_i P_i = 0$ or $\sum_j N_j = 0$, then of course either v or $-v$ is a nonnegative integer linear combination of elements of S. But $S \subseteq \mathcal{N}_r \cap K_X^\perp$, and every element of $\mathcal{N}_r \cap K_X^\perp$ meets L positively, so $-v$ cannot be a nonnegative linear combination of elements of S. I.e., v is a nonnegative integer linear combination of elements of S, as claimed. To finish, we have one case remaining to consider: suppose $\sum_i P_i \neq 0$ and $\sum_j N_j \neq 0$. Then, since S is contained in $K_X^\perp$ and is pairwise nonnegative, we have $-2 = v^2 = (\sum_i P_i)^2 + (\sum_j N_j)^2 - 2(\sum_i P_i)(\sum_j N_j) \leq -2 - 2 - 0$, which is impossible.

The second fact about S is that elements of S are effective. Recall that $-K_X$ is the class of C', which above we noted is nef of positive self-intersection. So if $D \in S$, then $(D - C') \cdot C' < 0$, hence $h^0(X, D - C') = 0$. Moreover, $h^2(X, D - C') = h^0(X, -D) = 0$ since $D \cdot L > 0$, and now $h^1(X, D - C') = 0$ follows by Riemann-Roch. It follows that the restriction morphism $\mathcal{O}_X(D) \to \mathcal{O}_{C'}(D)$ is surjective on global sections, but $h^0(C', \mathcal{O}_{C'}(D)) = 1$ since $D \in S \subset \ker(f)$; i.e., $h^0(X, D) = 1$ so D is effective.

We now show that $\text{neg}(X) \subseteq S$. If D is a prime divisor with $D^2 = -2$, then $-K_X \cdot D = 0$ and $D \in \ker(\phi)$, so by construction $D \in \langle S \rangle$. As shown above, we thus have $D = \sum_i P_i$ for elements P_i of S. Since each P_i is effective and D is prime of negative self-intersection, D must be one of the P_i, hence $D \in S$ so $\text{neg}(X) \subseteq S$.

Finally, we show that $S \subseteq \text{neg}(X)$. Say $D \in S$, so D is effective. Since D has negative self-intersection, D must meet an element of $\text{neg}(X) \subseteq S$ negatively. Since S is pairwise nonnegative, this element can only be D itself, so $D \in \text{neg}(X)$, hence $S \subseteq \text{neg}(X)$. Thus $S = \text{neg}(X)$ so S is representable.

If $\#(T)$ is square-free, then for any square-free l relatively prime to $\#(T)$, the group $T \times \mathbf{Z}/l\mathbf{Z}$ is cyclic and its order is square-free. Since any smooth non-supersingular plane cubic C has cyclic torsion subgroups of all square-free orders, $T \times \mathbf{Z}/l\mathbf{Z}$ is isomorphic to a subgroup of $\text{Pic}^0(C)$. □

We can now prove Theorem 5.4:

Proof. The proof is either by Lemma 5.6 or by reduction to previous cases using Lemma 5.5, so we begin by verifying that every formal configuration type for $r \leq 6$ is representable. This is obvious for $r = 1$, and now Lemma 5.5 applies up through $r = 5$. All formal types for $r = 6$ also follow from $r = 5$ by Lemma 5.5 except for type 10, since $\#(S(6) - S(5)) = 2$ for type 10 if we take the points in the order given in our list of types for $r = 6$. In this case $\#(T) = 2$, however, so representability follows by Lemma 5.6.

Now consider $r = 7$. Checking the table of types for $r = 7$ we see, taking the points in the order given in the table, that Lemma 5.5 applies for all the types except 21, 23, 24, 28 and 29. For types 21, 28 and 29, T has order 0, so these types are representable by Lemma 5.6. For types 23 and 24, note that up to the general linear group, we may choose the points $p_1, \dots, p_7$ to be, respectively, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$. If we blow these points up to get X, and if S has type 23, then clearly $S \subset \mathrm{neg}(X)$. Moreover, the only element of $\mathcal{N}_7$ which meets every element of S nonnegatively is $L - E_4 - E_5 - E_6$ (i.e., the proper transform of the line through the points $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$); adding this class to S gives type 24. Thus type 23 is representable if and only if these three points are not collinear, and type 24 is representable if and only if they are collinear. But they are collinear if and only if k has characteristic 2. □

For types with $r = 8$ points we have:

Theorem 5.7

Let k be an algebraically closed field. Consider formal configuration types for $r = 8$ points. Types 23, 31, 44, 90, 112, 128, 131 are representable if and only $\mathrm{char}(k) \neq 2$. *Types 46 and 130 are representable if and only if* $\mathrm{char}(k) = 2$. *Types 30, 45 and 96 are never representable. The rest are always representable.*

We will need a version of Lemma 5.5 for $r = 8$.

Lemma 5.8

Let k be an algebraically closed field, and let S be a formal configuration type for $r = 8$ points. If $S(7)$ is k-representable, $\#(S - S(7)) \leq 1$, and if for each class $D \in \mathcal{N}_8$ with $D \cdot L = 3$ and $D^2 < -1$ there is a $C \in S$ such that $C \neq D$ but $C \cdot D < 0$, then S is k-representable.

Proof. The proof is the same as for Lemma 5.5. That proof assumes that $D \in \mathcal{N}_r$ with $D^2 < -1$ implies that $D \cdot E_i$ is always either 0 or 1. For $r = 8$, this can fail since, for example, $3L - 2E_1 - E_2 - \cdots - E_8 \in \mathcal{N}_r$. Thus it is possible a priori that such a class is in $\mathrm{neg}(X)$ for the surface obtained in the proof of Lemma 5.5. The hypothesis that for each such D there is a $C \in S$ such that $C \cdot D < 0$ guarantees that this does not happen. □

We now prove Theorem 5.7:

Proof. For the first 32 types, we have $S \subset K^\perp$. If S is representable, then $S = \mathrm{neg}(X)$ for some X, and for this X the class $-K_X$ is nef. The torsion subgroups T, when nonzero, are as follows: $\mathbf{Z}/2\mathbf{Z}$ for types 13, 16, 19, 24, 25, and 29, $(\mathbf{Z}/2\mathbf{Z})^2$ for types 23 and 31, $(\mathbf{Z}/2\mathbf{Z})^3$ for type 30, $\mathbf{Z}/3\mathbf{Z}$ for types 27 and 28, and $(\mathbf{Z}/3\mathbf{Z})^2$ for type 32. Thus, by Lemma 5.6, possibly except for types 23, 31, 30 and 32, they are always representable.

For type 32 it turns out that $K^\perp/\langle S \rangle = T$. This T embeds in $\mathrm{Pic}^0(C)$ for any smooth cubic C if $\mathrm{char}(k) \neq 3$ and for any cuspidal cubic if $\mathrm{char}(k) = 3$. Thus type 32 is always representable, by Lemma 5.6.

By Lemma 5.6, types 23 and 31 are representable if $\mathrm{char}(k) \neq 2$. If either of these types S were representable in characteristic 2, then $S(7)$ would be representable, but $S(7)$ in either case is the $r = 7$ point type numbered 23 in our table, which is not representable in characteristic 2.

For type 30, S is never representable. Suppose it is representable. Then $-K_X$ is nef, hence $|-K_X|$ is a pencil with no fixed components, by Proposition 4.1. Thus $|-K_X|$ contains an integral divisor; i.e., X is obtained by blowing up smooth points on a reduced irreducible cubic C, where the proper transform of C is the integral element of $|-K_X|$. But the points $p_i \in C$ blown up to give X are distinct, hence the images $E_1 - E_i$ for $i \geq 1$ under $\Phi : \mathrm{Pic}(X) \to \mathrm{Pic}(C)$ are distinct. The images of $E_1 - E_i$ for $i \leq 8$ factor through T. Since the points p_i are distinct, the image of T in $\mathrm{Pic}^0(C)$ has order at least 7, whereas $\#(T) = 8$, so in fact Φ gives an injection of T into $\mathrm{Pic}^0(C)$. But the only cubic curve whose Picard group contains $(\mathbf{Z}/2\mathbf{Z})^3$ is the cuspidal cubic, in characteristic 2. Thus $\Phi(K^\perp)$ has pure 2-torsion. We can write $D = 3L - E_1 - \cdots - E_7 - 2E_8$ in terms of $K^\perp$ and the elements of S as given in the table of 8 point types, i.e., $D \in \langle S\rangle + 2K^\perp$, hence $\Phi(D) = 0$. Thus D is effective. Since $D \notin \langle S\rangle$, this means that $\mathrm{neg}(X) \neq S$.

Types 33 through 50 are exactly those for which S contains a cubic; in particular, $C' = 3L - E_1 - \cdots - E_7 - 2E_8 \in S$. If C' and D are in $\mathrm{neg}(X)$ and $C' \neq D$, then $D \cdot E_8 = 0$, since a check of all elements $D \in \mathcal{N}_8$ shows that if $D \cdot E_8 > 0$, then $D \cdot C' < 0$. Thus for any type S containing C' we have $S(7) = S - \{C'\}$, and S is representable if and only if the surface X which represents S comes from blowing up points $p_1, \ldots, p_8$, where $p_1, \ldots, p_7$ are smooth points on a reduced irreducible singular cubic C with p_8 being the singular point of C. Thus, if $S(7)$ is not representable, neither is S. And if $S(7)$ is representable by smooth points on a reduced irreducible singular cubic C, then so is S.

Let T be the torsion group for $S(7)$. For types S numbered 33 through 50 it turns out that T is 0 except as follows: it is $\mathbf{Z}/2\mathbf{Z}$ for types 41, 44; $(\mathbf{Z}/2\mathbf{Z})^2$ for type 45; and $(\mathbf{Z}/2\mathbf{Z})^3$ for type 46. When T is zero, $S(7)$ is (by applying Lemma 5.6) representable by blowing up smooth points of a cubic with a node, since in that case $\mathrm{Pic}^0(C)$ is the multiplicative group k^*.

As for type 46, by taking 7 smooth points on a cuspidal cubic in characteristic 2 we can, by Lemma 5.6, represent $S(7)$, hence as observed above, S is representable in characteristic 2 for type 46. Since $S(7)$ is not representable in characteristic not 2, neither is S.

Types 41 and 44 are representable by Lemma 5.6 if the characteristic is not 2. Type 41 is representable also in characteristic 2, since taking C to be a cuspidal cubic, the homomorphism $K^\perp \to \mathrm{Pic}^0(C)$ factors through $K^\perp/(\langle S(7)\rangle + 2K^\perp) = (\mathbf{Z}/2\mathbf{Z})^4$, but by explicitly checking the map, no element of $\mathcal{N}_7$ is in the kernel except those of $S(7)$. On the other hand, type 44 is not representable in characteristic 2, since if it were we would need to blow up 7 smooth points on a cuspidal cubic C, but then $L_{123}, L_{145}, L_{257} \in S(7)$ and L_{347} and L_{356} would be in the kernel of $K^\perp \to \mathrm{Pic}^0(C)$ even though neither is in $S(7)$; e.g., $L_{347} \equiv L_{123} + L_{145} + L_{257} \pmod 2$, where $L_{i_1i_2i_3} = L - E_{i_1} - E_{i_2} - E_{i_3}$.

Type 45 is not representable in characteristic 2, since $S(7)$ is not. It is not representable in any other characteristic either. If it were, we would need to find smooth points on a singular cubic C such that for the induced homomorphism $\mathrm{Pic}(X_7) \to \mathrm{Pic}(C)$, where X_7 is the blow-up of the first 7 points and p_8 is the singular point of C, the only (-2)-classes in the kernel are the elements of $S(7)$. But $K_{X_7}^\perp/\langle S(7)\rangle$ is $(\mathbf{Z}/2\mathbf{Z})^2 \oplus \mathbf{Z}$. Since the characteristic is not 2 but C is singular, the torsion of the image of this quotient in $\mathrm{Pic}(C)$ is cyclic. Thus some element of the torsion subgroup of $K_{X_7}^{\perp}/\langle S(7)\rangle$ must map to 0. But there are three such elements, and if x is any one of them, an explicit check shows that $\langle S(7), x\rangle$ contains (-2)-classes not in $S(7)$, hence X_7 would have $\mathrm{neg}(X_7) \neq S(7)$.

For types 51 through 96, $S \subset K^{\perp}$, so we can apply Lemma 5.6. The torsion subgroup T of $K^{\perp}/\langle S\rangle$ is zero except as follows: $\mathbf{Z}/2\mathbf{Z}$ for types 58, 60, 78, 82, 86, 89, 94; $(\mathbf{Z}/2\mathbf{Z})^2$ for types 90, 95; $(\mathbf{Z}/2\mathbf{Z})^3$ for type 96; and $\mathbf{Z}/3\mathbf{Z}$ for type 62. Thus all are always representable except possibly types 90, 95 and 96. Types 90 and 95 are representable (taking points on a smooth non-supersingular cubic, by Lemma 5.6) except possibly in characteristic 2. Type 90 is not representable in characteristic 2. We cannot take smooth points on a cuspidal cubic C, since then the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(C)$ contains $\langle S\rangle + 2K^{\perp}$, which by explicit check contains (-2)-classes other than those in S. If S is representable by choosing smooth points on either a nodal cubic or a smooth cubic, then the 2-torsion is at most $\mathbf{Z}/2\mathbf{Z}$, so the map $(\mathbf{Z}/2\mathbf{Z})^2 = T \to \mathrm{Pic}^0(C)$ induced by $\mathrm{Pic}(X) \to \mathrm{Pic}(C)$ must kill some of the 2-torsion of T. But by brute force check, every 2-torsion element of T is the image of a (-2)-class, hence killing any 2-torsion makes $\mathrm{neg}(X) \neq S$. Type 95 is representable in characteristic 2. We cannot take smooth points on a cuspidal cubic C, since then the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(C)$ contains $\langle S\rangle + 2K^{\perp}$, which by explicit check contains (-2)-classes other than those in S, but there is an element x of $K^{\perp}$ such that x maps to a 2-torsion element of $K^{\perp}/\langle S\rangle$ but such that the only (-2)-classes in $\langle S, x\rangle$ are those in S. Thus $K^{\perp}/\langle S, x\rangle$ has torsion group T' which embeds in $\mathrm{Pic}(C)$ for either a nodal or smooth and non-supersingular C in characteristic 2. Thus, as in Lemma 5.6, S is representable in characteristic 2.

Type 96 is never representable. If the characteristic is not 2, we must kill some of the 2-torsion, but by the same method as in the case of type 90, doing so introduces extra (-2)-classes. Thus the only hope for representability is in characteristic 2, with points on a cuspidal cubic. But $\langle S\rangle + 2K^{\perp}$ by explicit check contains (-2)-classes other than those in S, so even this does not work.

We need to treat types 111, 112, 119, 121, 126, 128, 129, 130, 131 specially. For type 111, a brute force check shows that there is no (-2)-class which meets the elements of S nonnegatively. Thus if we can choose points such that $S \subset \mathrm{neg}(X)$, then $S = \mathrm{neg}(X)$. But in this case it is easy to check that by choosing our points from among the intersections of three general lines and a general conic we do indeed get $S \subset \mathrm{neg}(X)$. For type 112, perform a quadratic transformation centered at the points c, d, and g. This transforms type 112 into type 128. Thus the one is representable if and only if the other is. (Alternatively, we can regard types 112 and 128 as giving the same surface X, but with respect to different morphisms $X \to \mathbb{P}^2$.) Since for type 128, $S(7)$ is not representable in characteristic 2, neither is type 128 (nor 112). But by Lemma 5.8, type 128 (and hence 112) is representable if the characteristic is not 2. Type 119 can be handled in the same way that 111 was. Type 121 can be handled by applying a quadratic transformation centered at the points a, b and c. This transforms 121 into type 144, which can be handled by Lemma 5.8. Type 126 can be handled as was 111, choosing the 8 points from among the 10 intersections of five general lines (thus it is easy to ensure that $S \subset \mathrm{neg}(X)$, but a brute force check shows that there is no (-2)-class which meets the elements of S nonnegatively.) For type 129, since all of the elements of S come from lines through either the point a or the point b, and since e is on the line through a and d, as long as the points are distinct, e cannot be on any of the lines but the line through a and d. Likewise, h can only be on the line containing c, d and g and the line containing b and e. Thus if we choose the points so that e is general but so that $S(7)$ is representable (which we can, since $S(7)$ is the 7 point type numbered 22), then $S \subset \text{neg}(X)$. The only possible additional element of $\text{neg}(X)$ besides S allowed by pairwise nonnegativity would come from a line through c, e and f. But since e is general, e cannot be on the line through c and f. Thus S is representable. For type 130, $S(7)$ is representable if and only if the characteristic is 2. Thus 130 is not representable if the characteristic is not 2, and by Lemma 5.8, it is representable if the characteristic is 2. For type 131, we may choose coordinates such that a is the point $(0, 0, 1)$, c is $(0, 1, 0)$, g is $(1, 0, 0)$, e is $(1, 1, 1)$, hence b is $(0, 1, 1)$, and f is $(1, 0, 1)$. Now d is forced to be the point $(1, 1, -2)$. Since in characteristic 2 this is a point on the line through c and g, but S does not allow the points c, d and g to be collinear, we S is not representable in characteristic 2. But if the characteristic is not 2, then we can check explicitly that $S \subset \text{neg}(X)$. No other additional element of $\text{neg}(X)$ besides S is allowed by pairwise nonnegativity, so $S = \text{neg}(X)$ here.

The remaining types are handled by Lemma 5.8 in a way similar to what was done applying Lemma 5.5 in the proof of Theorem 5.4. For example, for type 97, the type S' given by the points a, b, c, d, e, g, h is the 7 point type 29. It is representable, and $\#(S - S') = 0$, so type 97 is representable, by Lemma 5.8. □

Example 5.9 Here we show briefly that there are infinitely many configuration types among sets of r points, for each $r \geq 9$. In fact, it is clear by the definition of configuration type that if there are infinitely many types for $r = 9$, then there are infinitely many for all $r > 9$. So pick points $p_1, \dots, p_9$ on a smooth cubic curve C'. Let $Z = p_1 + \cdots + p_9$, and let X be the surface obtained by blowing up the points, and let C be the proper transform of C' on X; note that the class of C is $-K_X$. Using the group law on the cubic, it is not hard to see that among all choices of the points p_i there arise infinitely many different positive integers m_Z such that m_Z is the least positive integer for which the restriction of $m_Z K_X$ to C is trivial (as a line bundle). By results of [19], it follows that $h_{m_Z Z}(3i)$ is $\binom{3i+2}{2} - 1$ for $i < m_Z$, while $h_{m_Z Z}(3m_Z) = \binom{3i+2}{2} - 2$. I.e., there are infinitely many configuration types of 9 points.

Remark 5.10 It is possible for Z and Z' to have different configuration types but nonetheless for mZ and mZ' to have the same Hilbert functions for all m. In this situation it is convenient to say that Z and Z' have the same *uniform* configuration type. For example, the 6 point types 8 and 11 have the same uniform type, since they are both a complete intersection of a conic and a cubic. We also note that in order for two nonequivalent types to have the same uniform type, it need not be true that they be complete intersections. In particular, adding a general seventh point to the 6 point types 8 and 11 gives the 7 point types 11 and 26, neither of which is a complete intersection but for each $m \geq 1$ the Hilbert function of mZ is the same whether Z has type 11 or 26. Type 26 consists of 6 points on an irreducible conic with a general seventh point. Type 11 consists of two sets of three collinear points with a seventh general point. For type 11 it turns out that whenever one of the lines through 3 collinear points is a fixed component for forms of degree t vanishing on mZ, the other line through 3 collinear points is, by symmetry, also a fixed component. Thus the two lines are always taken together, so things for mZ when Z has type 11 behave the same as when Z has type 26, where the two lines are replaced by an irreducible conic. Finally, it is interesting to mention that one can find reduced finite subschemes Z and Z' of the plane such that $2Z$ and $2Z'$ have the same Hilbert function, but where the Hilbert functions of Z and Z' are different; see [10, Example 7.1].

Remark 5.11 Because there are only finitely many configuration types of r points for each $r \leq 8$, it follows that there is a number N_r such that if one knows the Hilbert function of mZ (i.e., of $(I(Z))^{(m)}$) for each $m \leq N_r$ for some reduced scheme Z consisting of r points in the plane, then one can deduce the uniform configuration type of Z and hence the Hilbert function of mZ for all $m > 0$. For example, by examining the Hilbert function of $p_1 + p_2 + p_3$ one can tell if the points are collinear or not, and hence $N_3 = 1$. By checking Hilbert functions for each type, and by straightforward arguments to show which different types have the same uniform type, we determined that $N_r = 1$ for $r \leq 3$, $N_4 = 2$, $N_r = 3$ for $r = 5, 6$ and $N_7 = 7$. We have not bothered to determine exactly which 8 point types have the same uniform types, and so we do not know for sure what the value of N_8 is, but it is not less than 16, and we suspect that it is exactly 16. As an interesting sidelight, it turns out in fact for each $r \leq 7$ that the Hilbert function of N_rZ alone already determines the Hilbert function of mZ for all m, since for $r \leq 7$, the Hilbert function of N_rZ distinguishes the uniform type. Thus if I is the ideal of a reduced set of $r \leq 7$ points, then the Hilbert function of $I^{(N_r)}$ determines the Hilbert functions of $I^{(m)}$ for all $m > 0$. (For the case of $r = 8$, the least N for which the Hilbert function of $I^{(N)}$ could by itself determine the Hilbert functions of $I^{(m)}$ for all $m > 0$ is $N = 22$, and we suspect that $N = 22$ in fact works.)

6. The Tables

Table 1. Configuration types for $r = 6$ points

*N*o.	*Type*	*N*o.	*Type*	*N*o.	*Type*
1.	$\emptyset$	5.	1: abc, ade	9.	1: abc, ade, bdf
2.	1: abc	6.	1: abcdef	10.	1: abc, ade, bdf, cef
3.	1: abcd	7.	1: abcd, aef	11.	2: abcdef
4.	1: abcde	8.	1: abc, def		

TABLE 2. Hilbert functions by configuration type for $r \leq 6$ points

r	m	*Type(s)*	h_Z	F_0	F_1
1	1	1	1	1^2	2^1
1	2	1	1	2^3	3^2
2	1	1	1, 2	$1^1, 2^1$	3^1
2	2	1	1, 3, 5, 6	$2^1, 3^1, 4^1$	$4^1, 5^1$
3	1	1	1, 3	2^3	3^2
3	2	1	1, 3, 6, 9	$3^1, 4^3$	5^3
3	1	2	1, 2, 3	$1^1, 3^1$	4^1
3	2	2	1, 3, 5, 7, 8, 9	$2^1, 4^1, 6^1$	$5^1, 7^1$
4	1	1	1, 3, 4	2^2	4^1
4	2	1	1, 3, 6, 10, 12	4^3	6^2
4	1	2	1, 3, 4	$2^2, 3^1$	$3^1, 4^1$
4	2	2	1, 3, 6, 10, 11, 12	$4^4, 6^1$	$5^3, 7^1$
4	1	3	1, 2, 3, 4	$1^1, 4^1$	1^5
4	2	3	1, 3, 5, 7, 9, 10, 11, 12	$2^1, 5^1, 8^1$	$6^1, 9^1$
5	1	1, 2	1, 3, 5	$2^1, 3^2$	4^2
5	2	1	1, 3, 6, 10, 14, 15	$4^1, 5^3$	$6^2, 7^1$
5	2	2	1, 3, 6, 10, 14, 15	$4^1, 5^3, 6^1$	$6^3, 7^1$
5	1	3	1, 3, 4, 5	$2^2, 4^1$	$3^1, 5^1$
5	2	3	1, 3, 6, 10, 12, 13, 14, 15	$4^3, 5^1, 8^1$	$5^2, 6^1, 9^1$
5	1	4	1, 3, 4, 5	$1^1, 5^1$	6^1
5	2	4	1, 3, 5, 7, 9, 11, 12, 13, 14, 15	$2^1, 6^1, 10^1$	$7^1, 11^1$
5	1	5	1, 3, 5	$2^1, 3^2$	4^2
5	2	5	1, 3, 6, 10, 13, 15	$4^2, 6^2$	$6^1, 7^2$
6	1	1, 2, 5, 9, 10	1, 3, 6	3^4	4^3
6	2	1, 2	1, 3, 6, 10, 15, 18	$5^3, 6^1$	7^3
6	2	5	1, 3, 6, 10, 15, 18	$5^3, 6^2$	$6^1, 7^3$
6	2	9	1, 3, 6, 10, 15, 18	$5^3, 6^3$	$6^2, 7^3$
6	2	10	1, 3, 6, 10, 14, 18	$4^1, 6^4$	7^4
6	1	3, 7	1, 3, 5, 6	$2^1, 3^1, 4^1$	$4^1, 5^1$
6	1	8, 11	1, 3, 5, 6	$2^1, 3^1$	5^1
6	2	3	1, 3, 6, 10, 14, 16, 17, 18	$4^1, 5^2, 8^1$	$6^1, 7^1, 9^1$
6	2	7	1, 3, 6, 10, 14, 16, 17, 18	$4^1, 5^2, 6^1, 8^1$	$6^2, 7^1, 9^1$
6	2	8, 11	1, 3, 6, 10, 14, 17, 18	$4^1, 5^1, 6^1$	$7^1, 8^1$
6	1	4	1, 3, 4, 5, 6	$2^2, 5^1$	$3^1, 6^1$
6	2	4	1, 3, 6, 10, 12, 14, 15, 16, 17, 18	$4^3, 6^1, 10^1$	$5^2, 7^1, 11^1$
6	1	6	1, 2, 3, 4, 5, 6	$1^1, 6^1$	7^1
6	2	6	1, 3, 5, 7, 9, 11, 13, 14, 15, 16, 17, 18	$2^1, 7^1, 12^1$	$8^1, 13^1$

TABLE 3. Seven point configuration types

N°.	*Type*	*N°.*	Type
1	empty	16	1: abc, ade, cef
2	1: abcdefg	17	1: abcg, ade, bdf, cef
3	1: abcdef	18	1: abc, ade, bdf, ceg
4	1: abcde	19	1: abc, ade, cef, afg
5	1: abcd	20	1: abc, adf, cef, bde
6	1: abc	21	1: abc, def, adg, beg, cfg
7	1: abcde, afg	22	1: abc, adf, cef, bde, aeg
8	1: abcd, efg	23	1: abc, adf, cef, bde, aeg, cdg
9	1: abcd, defg	24	1: abc, adf, cef, bde, aeg, cdg, bfg
10	1: abcd, def	25	2: abcdefg
11	1: abc, def	26	2: abcdef
12	1: abc, ade	27	1: abg; 2: abcdef
13	1: abcd, def, ceg	28	1: abg, cdg; 2: abcdef
14	1: abc, def, adg	29	1: abg, cdg, efg; 2: abcdef
15	1: abc, ade, afg		

TABLE 4. Hilbert functions by configuration type for $r = 7$ points

r	m	*Type(s)*	h_Z
7	1	1, 10, 29	1, 3, 5, 7
7	2	1, 10	1, 3, 6, 10, 14, 18, 20, 21
7	2	29	1, 3, 6, 10, 14, 17, 19, 21
7	1	2, 3, 4, 5, 11, 12, 13, ..., 27, 28	1, 3, 6, 7
7	2	2, 3, 4, 5, 15, 16, 17, 18	1, 3, 6, 10, 15, 20, 21
7	2	11, 12, 13, 14	1, 3, 6, 10, 15, 19, 20, 21
7	2	19, 20, 21, 22, 23, 24, 25, 26, 27, 28	1, 3, 6, 10, 15, 21
7	1	6	1, 2, 3, 4, 5, 6, 7
7	2	6	1, 3, 5, 7, 9, 11, 13, 15, 16, 17, 18, 19, 20, 21
7	1	7	1, 3, 4, 5, 6, 7
7	2	7	1, 3, 6, 10, 12, 14, 16, 17, 18, 19, 20, 21
7	1	8, 9	1, 3, 5, 6, 7
7	2	8, 9	1, 3, 6, 10, 14, 17, 18, 19, 20, 21

TABLE 5. Eight point configuration types (part 1)

N°.	*Type*	*N°.*	*Type*
1	empty	51	2: abcdef
2	1: abc	52	2: abcdef, abcdgh
3	1: abc, def	53	2: abcdef, abcdgh, abefgh
4	1: abc, ade	54	2: abcdef, abcdgh, abefgh, cdefgh
5	1: abc, ade, afg	55	1: abc, ade, fgh, 2: bcdegh
6	1: abc, ade, bdf	56	1: abc, ade, bdf, cgh, 2: abefgh
7	1: abc, ade, bfg	57	1: abc, ade, afg, bdf, 2: cdefgh
8	1: abc, ade, fgh	58	1: abc, ade, afg, bdf, beg, 2: cdefgh
9	1: abc, ade, bdf, cgh	59	1: abc, ade, afg, bdf, beh, 2: cdefgh
10	1: abc, ade, bdf, ceg	60	1: abc, ade, afg, bdh, ceh, 2: bcdefg
11	1: abc, ade, bdf, cef	61	1: abc, ade, afg, bdh, cfh, 2: bcdefg
12	1: abc, ade, bfg, dfh	62	1: abc, ade, afg, bdh, cfh, egh, 2: bcdefg
13	1: abc, ade, afg, bdf	63	1: abc, 2: cdefgh
14	1: abc, ade, afg, bdh	64	1: abc, ade, 2: bcdegh
15	1: abc, ade, afg, bdf, ceg	65	1: abc, ade, afg, 2: bcdefg
16	1: abc, ade, afg, bdf, beg	66	1: abc, ade, bdf, 2: cdefgh
17	1: abc, ade, afg, bdf, ceh	67	1: abc, ade, bfg, 2: cdefgh
18	1: abc, ade, afg, bdf, beh	68	1: abc, ade, afg, bdh, 2: cdefgh
19	1: abc, ade, afg, bdh, ceh	69	1: abc, 2: bcdefg
20	1: abc, ade, afg, bdh, cfh	70	1: abc, ade, 2: cdefgh
21	1: abc, ade, bdf, cgh, efg	71	1: abc, ade, afg, 2: cdefgh
22	1: abc, ade, afg, bdf, ceg, beh	72	1: abc, ade, bdf, 2: bcdegh
23	1: abc, ade, afg, bdf, beg, cdg	73	1: abc, ade, bfg, 2: bcdegh
24	1: abc, ade, afg, bdf, beg, dgh	74	1: abc, ade, afg, bdh, 2: bcdefg
25	1: abc, ade, afg, bdf, beg, cdh	75	1: abc, ade, 2: acefgh
26	1: abc, ade, afg, bdf, ceh, bgh	76	1: abc, def, 2: bcefgh
27	1: abc, ade, afg, bdh, cfh, egh	77	1: abc, ade, bdf, ceg, 2: acdfgh
28	1: abc, ade, afg, bdf, ceg, beh, cfh	78	1: abc, ade, bdf, cef, 2: bcdegh
29	1: abc, ade, afg, bdf, ceg, beh, cdh	79	1: abc, ade, bfg, dfh, 2: bcdegh
30	1: abc, ade, afg, bdf, beg, cdg, cef	80	1: abc, 2: cdefgh, abefgh
31	1: abc, ade, afg, bdf, beg, cdg, ceh	81	1: abc, ade, 2: bcdegh, acefgh
32	1: abc, ade, afg, bdf, ceg, beh, cfh, dgh	82	1: abc, ade, bdf, 2: bcdegh, acdfgh
33	3: abcdefgh	83	1: abc, 2: acdefg, abefgh
34	1: abc, 3: abcdefgh	84	1: abc, ade, 2: bdefgh, acefgh
35	1: abc, def, 3: abcdefgh	85	1: abc, ade, bdf, 2: cdefgh, abefgh
36	1: abc, ade, 3: abcdefgh	86	1: abc, ade, 2: acefgh, abdfgh
37	1: abc, ade, afg, 3: abcdefgh	87	1: abc, def, 2: bcefgh, acdfgh
38	1: abc, ade, bdf, 3: abcdefgh	88	1: abc, ade, bfg, 2: bcdegh, acefgh
39	1: abc, ade, bfg, 3: abcdefgh	89	1: abc, ade, bdf, ceg, 2: acdfgh, abefgh
40	1: abc, ade, bdf, ceg, 3: abcdefgh	90	1: abc, ade, bdf, cef, 2: bcdegh, acdfgh
41	1: abc, ade, bdf, cef, 3: abcdefgh	91	1: abc, ade, bfg, dfh, 2: bcdegh, acefgh
42	1: abc, ade, afg, bdf, 3: abcdefgh	92	1: abc, 2: bcdfgh, acdefg, abefgh
43	1: abc, ade, afg, bdf, ceg, 3: abcdefgh	93	1: abc, def, 2: bcefgh, acdfgh, abdegh
44	1: abc, ade, afg, bdf, beg, 3: abcdefgh	94	1: abc, ade, 2: bcdegh, acefgh, abdfgh
45	1: abc, ade, afg, bdf, beg, cdg, 3: abcdefgh	95	1: abc, ade, bdf, 2: bcdegh, acdfgh, abefgh
46	1: abc, ade, afg, bdf, beg, cdg, cef, 3: abcdefgh	96	1: abc, ade, bdf, cef, 2: bcdegh, acdfgh, abefgh
47	2: abcdef, 3: abcdefgh	97	1: abc, ade, afgh, 2: bcdegh
48	1: abc, 2: bcdefg, 3: abcdefgh	98	1: abc, defg
49	1: abc, ade, 2: bcdefg, 3: abcdefgh	99	1: abc, ade, afgh
50	1: abc, ade, afg, 2: bcdefg, 3: abcdefgh	100	1: abc, ade, bdf, afgh

TABLE 6. Eight point configuration types (part 2)

N^o.	*Type*	*N^o.*	*Type*
101	1: abc, ade, afg, bdf, cegh	124	1: abc, ade, bdf, ceg, afgh
102	1: abc, adeh, 2: bcdefg	125	1: abc, ade, bdf, cef, afgh
103	1: abc, ade, bdf, afgh, 2: bcdegh	126	1: abc, ade, bfg, dfh, cegh
104	1: abc, adef	127	1: abc, ade, afg, bdh, cefh
105	1: abc, ade, bdfg	128	1: abc, ade, afg, bdf, beg, cdgh
106	1: abc, ade, bdf, cegh	129	1: abc, ade, afg, bdf, beh, cdgh
107	1: abc, ade, afg, bdf, begh	130	1: abc, ade, afg, bdf, beg, cdg, cefh
108	1: abc, ade, bdfg, 2: acefgh	131	1: abc, ade, afg, bdf, beg, dgh, cefh
109	1: abc, ade, bfgh	132	1: abcde
110	1: abc, ade, bdf, cefg	133	1: abcde, afgh
111	1: abc, def, adgh, 2: bcefgh	134	1: abcde, fgh
112	1: abc, ade, bdf, cef, afgh, 2: bcdegh	135	1: abcde, afg
113	1: abcd	136	1: abcde, afg, bfh
114	1: abc, ade, afg, bdfh	137	1: abcde, afg, bfh, cgh
115	1: abcd, efgh	138	1: abcdef
116	1: abcd, aefg	139	1: abcdef, agh
117	1: abc, adef, bdgh	140	1: abcdefg
118	1: abc, ade, bdfg, cefh	141	1: abcdefgh
119	1: abc, ade, afg, bdfh, cegh	142	2: abcdefg
120	1: abgh, 2: abcdef	143	1: abh, 2: abcdefg
121	1: efgh, 2: abcdef, abcdgh	144	1: abh, cdh, 2: abcdefg
122	1: abc, def, adgh	145	1: abh, cdh, efh, 2: abcdefg
123	1: abc, ade, bfg, cdfh	146	2: abcdefgh

TABLE 7. Hilbert functions by configuration type for $r = 8$ points

r	m	*Type(s)*	h_Z
8	1	1, ..., 114, 116, ..., 131, 142, ..., 145	1, 3, 6, 8
8	2	1, ..., 96	1, 3, 6, 10, 15, 21, 24
8	2	97, ..., 110, 112, 113, 114, 120, 122, ..., 125, 127, ..., 131, 142, ..., 145	1, 3, 6, 10, 15, 21, 23, 24
8	2	111, 121, 126	1, 3, 6, 10, 15, 20, 23, 24, 24
8	1	115, 133, 134, 146	1, 3, 5, 7, 8
8	2	115, 146	1, 3, 6, 10, 14, 18, 21, 23, 24
8	2	133	1, 3, 6, 10, 14, 18, 20, 22, 23, 24
8	2	134	1, 3, 6, 10, 14, 18, 21, 22, 23, 24
8	1	132, 135, 136, 137	1, 3, 6, 7, 8
8	2	132, 135, 136, 137	1, 3, 6, 10, 15, 20, 21, 22, 23, 24
8	1	138, 139	1, 3, 5, 6, 7, 8
8	2	138, 139	1, 3, 6, 10, 14, 17, 19, 20, 21, 22, 23, 24
8	1	140	1, 3, 4, 5, 6, 7, 8
8	2	140	1, 3, 6, 10, 12, 14, 16, 18, 19, 20, 21, 22, 23, 24
8	1	141	1, 2, 3, 4, 5, 6, 7, 8
8	2	141	1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 20, 21, 22, 23, 24

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