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WARING'S PROBLEM FOR LINEAR POLYNOMIALS AND LAURENT POLYNOMIALS

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ABSTRACT. Waring's problem is about representing any function in a class of functions as a sum of kth powers of nonconstant functions in the same class. We allow complex coefficients in these kind of problems. Consider $\sum_{i=1}^{p_1} f_i(z)^k = z$ and $\sum_{i=1}^{p_2} f_i(z)^k = 1$. Suppose that $k \geq 2$. Let p_1 and p_2 be the smallest numbers of functions that give the above identities. W.K. Hayman obtained lower bounds of p_1 and p_2 for polynomials, entire functions, rational functions and meromorphic functions. First, we consider Waring's problem for linear polynomials and get $p_1 = k$ and $p_2 = k+1$. Then, we study Waring's problem for Laurent polynomials and obtain lower bounds of p_1 and p_2.

1. Introduction.

1.1. Waring's problem. Waring's problem deals with representing any function in a class of functions as a sum of kth powers of nonconstant functions in the same class. We allow complex coefficients in these problems.

Let k and n be natural numbers. Consider the equation of the form

$$\sum_{i=1}^{n} f_i(z)^k = Q(z), \tag{1.1.1}$$

where $f_1, f_2, \ldots, f_n$ and Q are nonconstant polynomials with complex coefficients. Suppose that

$$f_1(z)^k + f_2(z)^k + \cdots + f_n(z)^k = z. \tag{1.1.2}$$

Then we obtain

$$f_1(Q(z))^k + f_2(Q(z))^k + \cdots + f_n(Q(z))^k = Q(z) \tag{1.1.3}$$

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by the substitution of $Q(z)$ for z. Hence any nonconstant polynomial $Q(z)$ can be represented by the sum of n kth powers of nonconstant polynomials. Therefore studying the form (1.1.2) is important.

By applying finite differences, Newman and Slater [**11**] showed that the identity function z can be always represented as a sum of k kth powers of nonconstant polynomials. The following theorem about the $(k-1)$th differences of x^k can be found in the book by Hardy and Wright [**6**].

Theorem 1.1.1. *We have*

$$\sum_{r=0}^{k-1}(-1)^{k-1-r}\binom{k-1}{r}(x+r)^k = k!\,x + d, \tag{1.1.4}$$

where d is an integer independent of x. In fact $d = (1/2)(k-1)(k!)$.

By substituting $(z-d)/k!$ for x in the equation (1.1.4), we can represent z as a sum of k kth powers of nonconstant linear polynomials. Newman and Slater credit this finite difference argument to S. Hurwitz who has conjectured that the number k of kth powers of nonconstant polynomials is the minimum needed for the representation of the identity function z. Also, Heilbronn has conjectured that k is minimal even if entire functions are allowed [**8**].

We can consider other classes for Waring's problem. We denote the sets of polynomials, entire functions, rational functions, and meromorphic functions by P, E, R and M respectively as in [**9**]. By a meromorphic function we mean a meromorphic function in the whole complex plane. The finite difference argument implies that Waring's problems for P, E, R and M are solvable.

Theorem 1.1.2 [**9**]. *Suppose that $k \geq 2$ and that $n \geq 2$. Let f_1, $f_2, \ldots, f_n$ be nonconstant functions in class C satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = z, \tag{1.1.5}$$

where C is one of the classes P, E, R and M. Suppose that $p_C(k)$ is the smallest number n satisfying (1.1.5). *Then*

$$p_P(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \quad k \geq 3, \tag{1.1.6}$$

$$p_E(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \quad k \geq 2, \tag{1.1.7}$$

$$p_R(k) > \sqrt{k+1}, \quad k \geq 2, \tag{1.1.8}$$

$$p_M(k) \geq \sqrt{k+1}, \quad k \geq 2. \tag{1.1.9}$$

The inequality (1.1.6) was found first by Newman and Slater [**11**], and (1.1.8) was found first by Green [**4**]. Hayman used Cartan's theorem [**1**] to prove his theorem [**9**].

1.2. Fermat type of Waring's problem. We consider the case that $Q(z)$ in the equation (1.1.1) is a nonzero constant. Without loss of generality, $Q \equiv 1$. Then

$$f_1(z)^k + f_2(z)^k + \cdots + f_n(z)^k = 1. \tag{1.2.1}$$

Equations of the form (1.2.1) are called Fermat type equations.

The finite difference argument implies that any nonzero constant can be represented by a sum of $(k+1)$ kth powers of nonconstant polynomials [**11**].

Theorem 1.2.1 [**9**]. *Suppose that $k \geq 2$ and that $n \geq 2$. Let f_1, $f_2, \ldots, f_n$ be nonconstant functions in class C satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = 1, \tag{1.2.2}$$

where C is one of the classes P, E, R and M. Suppose that $P_C(k)$ is *the smallest number* n *satisfying* (1.2.2). *Then*

$$P_P(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \tag{1.2.3}$$

$$P_E(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \tag{1.2.4}$$

$$P_R(k) > \sqrt{k+1}, \tag{1.2.5}$$

$$P_M(k) \geq \sqrt{k+1}. \tag{1.2.6}$$

Newman and Slater noted that, together with the construction of Molluzzo, the inequality (1.2.3) gives the correct order of magnitude for $P_P(k)$ [**11**].

Theorem 1.2.2 [**10**]. *We have*

$$P_P(k) \leq \left[(4k+1)^{1/2}\right].$$

Suppose that $n \geq 2$ and that $k \geq 2$. For given n and a class C of functions, consider the following question: For which integers k, do there exist nonconstant functions $f_1, f_2, \dots, f_n$ in the class C satisfying (1.2.1)? We quote what Gundersen summarized in his survey paper [**5**].

Consider the equation

$$f_1{}^k + f_2{}^k + f_3{}^k = 1. \tag{1.2.7}$$

The following theorem is a collection of results of Toda [**12**], Fujimoto [**3**], Green [**4**], Newman and Slater [**11**] and Hayman [**9**].

Theorem 1.2.3. *For the four classes P, E, R, M, we have the following results for equation* (1.2.7):

(a) *If $k \geq 9$, then there do not exist three nonconstant meromorphic functions f_1, f_2, f_3 that satisfy* (1.2.7).

(b) *If $k \geq 8$, then there do not exist three nonconstant rational functions f_1, f_2, f_3 that satisfy* (1.2.7).

(c) *If* $k \geq 7$, *then there do not exist three nonconstant entire functions* f_1, f_2, f_3 *that satisfy* (1.2.7).

(d) *If* $k \geq 6$, *then there do not exist three nonconstant polynomials* f_1, f_2, f_3 *that satisfy* (1.2.7).

If the number n in the equation (1.2.1) increases, then the number of open questions increases [**5**].

2. Waring's problem for linear polynomials.

2.1. The representation of a function by linear polynomials.

Theorem 2.1.1 [**11**]. *Suppose that* ${f_1}^k + {f_2}^k + \cdots + {f_n}^k = Q$ *where* Q *is a polynomial, not identically zero, and* $f_1, f_2, \dots, f_n$ *are nonconstant polynomials. Then we have*

$$\deg Q \geq D(k - n^2 + n) + \frac{n(n-1)}{2}$$

where D *is the largest of the degrees of the* f_i*'s. Further if all the* f_i*'s are linear we have*

$$\deg Q \geq k - \frac{n(n-1)}{2}. \tag{2.1.1}$$

If $Q(z) = z$ and $D = 1$, then, from the inequalities (2.1.1) and (1.1.6), we get

$$n \geq \frac{1 + \sqrt{8k-7}}{2} > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \qquad k \geq 3. \tag{2.1.2}$$

For example, if $k = 5$ then we have $n \geq 4$. We improve (2.1.2) as follows.

Theorem 2.1.2. *Suppose that* $k \geq 2$ *and that* $n \geq 2$. *Let* f_1, $f_2 \dots, f_n$ *be nonconstant linear polynomials satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = z. \tag{2.1.3}$$

Suppose that p is the smallest number n satisfying (2.1.3). Then $p = k$.

Proof. The finite difference argument implies that the identity function z can be always represented by a sum of k kth powers of nonconstant linear polynomials. Thus, $p \le k$. If $k = 2$ and $p = 1$, then we get ${f_1}^2 = z$, which is impossible. Hence $p \ge 2$ and so $p = 2$. Consider the case $k \ge 3$. We suppose that $p < k$ and will obtain a contradiction. Write $f_i(z) = a_i + b_i z$ for all i. According to the minimality of p, all the ${f_i}^k$ are linearly independent. Thus we can have $a_i = 0$ for at most one i. Then $f_i(z)^k = (a_i + b_i z)^k = {a_i}^k\,(1 + (b_i/a_i)z)^k$ if $a_i \ne 0$. Suppose that ${a_i}^k = \alpha_i$ and that $b_i/a_i = \beta_i$ for each i.

Suppose that $a_p = 0$ and $a_i \ne 0$ for $1 \le i \le p-1$. Then

$$\begin{aligned}
\sum_{i=1}^{p} f_i(z)^k &= \sum_{i=1}^{p} (a_i + b_i z)^k \\
&= {b_p}^k z^k + \sum_{i=1}^{p-1} \alpha_i (1 + \beta_i z)^k \\
&= {b_p}^k z^k + \sum_{i=1}^{p-1} \alpha_i \left(\sum_{j=0}^{k} \binom{k}{j} {\beta_i}^j z^j \right) \\
&= {b_p}^k z^k + \sum_{j=0}^{k} \binom{k}{j} z^j \left(\sum_{i=1}^{p-1} \alpha_i {\beta_i}^j \right).
\end{aligned}$$

Since the righthand side is equal to z, we get, in particular, the system of equations

$$\sum_{i=1}^{p-1} {\beta_i}^j \alpha_i = 0 \quad \text{for} \quad 2 \le j \le k-1. \tag{2.1.4}$$

Since $p < k$, we use $p-1$ equations. Now consider α_i for $1 \le i \le p-1$ as unknowns. Then the coefficients form a square matrix M_1 whose determinant is given by

$$|M_1| = \begin{vmatrix} {\beta_1}^2 & {\beta_2}^2 & \cdots & {\beta_{p-1}}^2 \\ {\beta_1}^3 & {\beta_2}^3 & \cdots & {\beta_{p-1}}^3 \\ \vdots & \vdots & \ddots & \vdots \\ {\beta_1}^p & {\beta_2}^p & \cdots & {\beta_{p-1}}^p \end{vmatrix}$$
$$= {\beta_1}^2{\beta_2}^2\cdots{\beta_{p-1}}^2 \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_{p-1} \\ {\beta_1}^2 & {\beta_2}^2 & \cdots & {\beta_{p-1}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\beta_1}^{p-2} & {\beta_2}^{p-2} & \cdots & {\beta_{p-1}}^{p-2} \end{vmatrix}.$$

Since the last determinant for M_1 is the van der Monde determinant [**2**], we get

$$|M_1| = {\beta_1}^2{\beta_2}^2\cdots{\beta_{p-1}}^2 \prod_{i<j}(\beta_j - \beta_i).$$

Since all the ${f_i}^k$ are linearly independent, we have $\beta_i \neq \beta_j$ for $i \neq j$ and we get $|M_1| \neq 0$. Hence the system (2.1.4) of homogeneous linear equations has only the trivial solution and so $\alpha_i = 0$ for all i with $1 \leq i \leq p-1$. This is a contradiction.

Suppose that $a_i \neq 0$ for all i. Then

$$\sum_{i=1}^{p} f_i(z)^k = \sum_{j=0}^{k} \binom{k}{j} z^j \left(\sum_{i=1}^{p} \alpha_i {\beta_i}^j\right).$$

Since the righthand side is equal to z, we get

$$\sum_{i=1}^{p} {\beta_i}^j \alpha_i = 0 \quad \text{for} \quad 2 \leq j \leq k. \tag{2.1.5}$$

By using p equations, we have a coefficient matrix M_2 whose determinant is given by

$$|M_2| = \begin{vmatrix} {\beta_1}^2 & {\beta_2}^2 & \cdots & {\beta_p}^2 \\ {\beta_1}^3 & {\beta_2}^3 & \cdots & {\beta_p}^3 \\ \vdots & \vdots & \ddots & \vdots \\ {\beta_1}^{p+1} & {\beta_2}^{p+1} & \cdots & {\beta_p}^{p+1} \end{vmatrix}.$$

Since $|M_2| \neq 0$, the system (2.1.5) has only the trivial solution and so $\alpha_i = 0$ for all i. This is a contradiction. $\square$

2.2. The representation of a constant by linear polynomials. If $Q(z) = 1$ and $D = 1$ in Theorem 2.1.1, then from the inequalities (2.1.1) and (1.2.3) we get

$$n \geq \frac{1+\sqrt{8k+1}}{2} > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \quad k \geq 2. \tag{2.2.1}$$

For example, if $k = 3$, then we have $n \geq 3$. We improve (2.2.1) as follows.

Theorem 2.2.1. *Suppose that $k \geq 2$ and that $n \geq 2$. Let $f_1, f_2, \ldots, f_n$ be nonconstant linear polynomials satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = 1. \tag{2.2.2}$$

Suppose that p is the smallest number n satisfying (2.2.2). *Then $p = k+1$.*

Example. We define $\omega = e^{2\pi i/(k+1)}$. Then

$$\sum_{\nu=1}^{k+1} \left(\frac{1+\omega^\nu z}{(k+1)^{1/k}} \right)^k = 1.$$

Thus $p \leq k+1$.

The proof of Theorem 2.2.1 is similar to that of Theorem 2.1.2, and we omit it.

3. Waring's problem for Laurent polynomials.

3.1. Definitions and Cartan's theory.

Definition 3.1.1. A Laurent polynomial is a function of the form

$$f(z) = \sum_{n=s}^{t} a_n\, z^n, \tag{3.1.1}$$

where s and t are integers with $s \leq t$ and $a_n \in \mathbf{C}$ with $a_s \neq 0 \neq a_t$.

Hence a Laurent polynomial need not be a polynomial, in spite of the name. There is some interest in the representation. For example, if

$${f_1}^k + {f_2}^k + \cdots + {f_n}^k = 1 \tag{3.1.2}$$

holds for nonconstant Laurent polynomials $f_1,\ f_2, \dots, f_n$, we can replace z by e^z to get (3.1.2) in terms of transcendental entire functions.

We use notations from the Nevanlinna theory that can be found in Hayman's book [**7**]. We define $\log^+ x = \max\{0, \log x\}$ for $x \geq 0$. We write $n(t, f)$ for the number of poles of $f(z)$ in $|z| \leq t$ counting multiplicities, and

$$\begin{aligned}
m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta,\\
N(r, f) &= \int_0^{r} \frac{n(t, f) - n(0, f)}{t}\, dt \,+\, n(0, f)\, \log r,\\
T(r, f) &= m(r, f) + N(r, f).
\end{aligned}$$

Theorem 3.1.2 [**1**]. *Suppose that $p \geq 2$. Let $F_1,\ F_2, \dots, F_p$ be p linearly independent entire functions. Suppose that there is no point z such that $F_i(z) = 0$ for all i, where $1 \leq i \leq p$. Set*

$$F_{p+1} = \sum_{i=1}^{p} F_i(z), \tag{3.1.3}$$

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} \sup_{1\le i\le p} \log|F_i(re^{i\theta})|\,d\theta - \sup_{1\le i\le p} \log|F_i(0)|. \tag{3.1.4}$$

Let $n_i(r)$ *be the number of zeros of* $F_i(z)$ *in* $|z| \le r$*, where a zero of order* d *is counted exactly* $\min\{d, p-1\}$ *times. For* $1 \le i \le p+1$*, set*

$$N_i(r) = \int_0^r \frac{n_i(t) - n_i(0)}{t}\,dt + n_i(0)\log r. \tag{3.1.5}$$

Then

$$T(r) \le \sum_{i=1}^{p+1} N_i(r) + S(r) \tag{3.1.6}$$

holds with

$$S(r) = \frac{1}{2\pi}\int_0^{2\pi} \max\log|\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}(re^{i\theta})|\,d\theta + O(1), \tag{3.1.7}$$

where the maximum is taken for all choices of $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ *of distinct numbers* $\alpha_1, \alpha_2, \dots, \alpha_p$ *from* $\{1, 2, \dots, p+1\}$*, and where the Wronskian*

$$W(F_{\alpha_1}, F_{\alpha_2}, \dots, F_{\alpha_p}) = F_{\alpha_1}F_{\alpha_2}\cdots F_{\alpha_p}\,\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}$$

with

$$\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (F'_{\alpha_1})/(F_{\alpha_1}) & (F'_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F'_{\alpha_p})/(F_{\alpha_p}) \\ (F''_{\alpha_1})/(F_{\alpha_1}) & (F''_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F''_{\alpha_p})/(F_{\alpha_p}) \\ \vdots & \vdots & \ddots & \vdots \\ (F^{(p-1)}_{\alpha_1})/(F_{\alpha_1}) & (F^{(p-1)}_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F^{(p-1)}_{\alpha_p})/(F_{\alpha_p}) \end{vmatrix}. \tag{3.1.8}$$

3.2. The representation of a function by Laurent polynomials.

Theorem 3.2.1. *Suppose that $k \geq 3$ and that $n \geq 2$. Let $f_1, f_2, \dots, f_n$ be nonconstant Laurent polynomials satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = z. \tag{3.2.1}$$

Suppose that at least one of the f_i is not a polynomial and that p is the smallest number n satisfying (3.2.1)*. Then $p^2 - p > k$ for $k \geq 3$, i.e., $p > 1/2 + \sqrt{k + 1/4}$ for $k \geq 3$.*

Proof. We proceed with the proof similarly as in the proof of Theorem 1.1.2. Let p be the smallest number n satisfying (3.2.1). Consider

$$f_1{}^k + f_2{}^k + \cdots + f_p{}^k = z. \tag{3.2.2}$$

For each i with $1 \leq i \leq p$, we can rewrite

$$f_i(z) = \sum_{n=s_i}^{t_i} a_n\, z^n = g_i(z)\, z^{s_i},$$

where $g_i(z)$ is a polynomial. We can suppose that there exists an i such that $s_i \leq -1$. Otherwise, all $f_i(z)$ are polynomials. Let $m = \min_{1 \leq i \leq p} s_i$. Then $m \leq -1$. From (3.2.2), we get

$$h_1{}^k + h_2{}^k + \cdots + h_p{}^k = z^{1-mk}, \tag{3.2.3}$$

where $h_i(z) = f_i(z)\, z^{-m}$. Note that each $h_i(z)$ is a polynomial. We write $h_i{}^k = F_i$, where $1 \leq i \leq p$, and $F_{p+1} = z^{1-mk}$. Then (3.2.3) becomes

$$F_1 + F_2 + \cdots + F_p = F_{p+1}. \tag{3.2.4}$$

The functions $F_1, F_2, \dots, F_p$ have no common zero. Otherwise, there exists $z_0 \in \mathbf{C}$ such that $0 = F_1(z_0) = F_2(z_0) = \cdots = F_p(z_0)$. If $z_0 \neq 0$, then (3.2.4) does not hold. This is a contradiction. Assume that $z_0 = 0$. There exists i such that $s_i = m$. Now, by choosing i so that $s_i = m$, we get

$$\begin{aligned} h_i(z) = f_i(z)z^{-s_i} &= \left(a_{t_i}z^{t_i} + a_{t_i-1}z^{t_i-1} + \cdots + a_{s_i}z^{s_i}\right)\, z^{-s_i} \\ &= a_{t_i}z^{t_i-s_i} + a_{t_i-1}z^{t_i-s_i-1} + \cdots + a_{s_i}. \end{aligned}$$

Since $a_{s_i} \neq 0$, we have $F_i(z_0) = h_i(z_0)^k = h_i(0)^k = {a_{s_i}}^k \neq 0$, which is a contradiction.

The functions $F_1, F_2, \ldots, F_p$ form a linearly independent set. Otherwise, one of the functions is a linear combination of the others. Then z can be represented as a sum of $p-1$ kth powers. This contradicts the minimality of p.

There exists i with $t_i \geq 1$. Otherwise, we get $t_i \leq 0$ for all i. Then we obtain ${f_1}^k + {f_2}^k + \cdots + {f_p}^k = O(1)$ as $z \to \infty$. But the righthand side is z, which tends to ∞ by (3.2.2). This is a contradiction.

We write $\alpha = \deg h_1 = \max_{1 \leq i \leq p} \deg h_i$ without loss of generality. There exists h_j with $j \neq 1$ such that $\deg h_j = \alpha$. Otherwise, $\deg h_1 > \max_{1<j\leq p} \deg h_j$. We write $h_1(z) = a_\alpha z^\alpha + \cdots$. Then $h_1(z)^k = {a_\alpha}^k\, z^{\alpha k} + \cdots$. Then ${a_\alpha}^k\, z^{\alpha k}$ cannot be canceled by ${h_2}^k + {h_3}^k + \cdots + {h_p}^k$ and z^{1-mk} since $1 - mk \neq \alpha k$.

If $F_i(z_0) = 0$ with the order of zero d, then we count this zero $\min\{d, p-1\}$ times instead of the order of z_0 for $n_i(r)$. Each zero of $F_i(z) = (f_i(z)\, z^{-m})^k$ has order k at least. Thus d is a multiple of k.

If $k \leq p-1$, then we do not need to check this case because z always can be represented by k polynomials.

Suppose that $k > p-1$. Since $p-1 < k \leq d$, we have $\min\{d, p-1\} = p-1$ and the point z_0 is counted $p-1$ times for $N_i(r)$. Then we get

$$\begin{aligned} \frac{N_i(r)}{p-1} &= \frac{1}{p-1}\left(\int_0^r \frac{n_i(t) - n_i(0)}{t}\, dt + n_i(0)\, \log r\right) \\ &\leq \frac{1}{k}\left(\int_0^r \frac{n\left(t, (1/F_i)\right) - n\left(0, (1/F_i)\right)}{t}\, dt + n\left(0, \frac{1}{F_i}\right)\, \log r\right) \\ &= \frac{1}{k}\, N\left(r, \frac{1}{F_i}\right) \end{aligned}$$

since z_0 is counted once for $(N_i(r))/(p-1)$ and d/k times for $(1/k)N\,(r,(1/F_i))$. So we have

$$N_i(r) \leq \frac{p-1}{k}\, N\left(r, \frac{1}{F_i}\right).$$

By Jensen's formula, we have

$$N\left(r,\frac{1}{F_i}\right) = T\left(r,\frac{1}{F_i}\right) - m\left(r,\frac{1}{F_i}\right) = T(r,F_i) + O(1) - m\left(r,\frac{1}{F_i}\right).$$

Since F_i is entire, we get

$$T(r,F_i) = N(r,F_i) + m(r,F_i) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |F_i(re^{i\theta})|\, d\theta.$$

Hence

$$\begin{aligned} N\left(r,\frac{1}{F_i}\right) &= \frac{1}{2\pi}\int_0^{2\pi} \left(\log^+ |F_i(re^{i\theta})| - \log^+ \frac{1}{|F_i(re^{i\theta})|}\right) d\theta + O(1) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log |F_i(re^{i\theta})|\, d\theta + O(1) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \sup_{1\leq i\leq p} \log |F_i(re^{i\theta})|\, d\theta - \sup_{1\leq i\leq p} \log |F_i(0)| + O(1) \\ &= T(r) + O(1). \end{aligned}$$

Therefore, for each i with $1 \leq i \leq p$, we have

$$N_i(r) \leq \frac{p-1}{k}\, T(r) + O(1). \tag{3.2.5}$$

Consider

$$N_{p+1}(r) = \int_0^r \frac{n_{p+1}(t) - n_{p+1}(0)}{t}\, dt + n_{p+1}(0)\, \log r.$$

Since $F_{p+1}(z) = z^{1-mk}$ and $1-mk \geq 1+k > p$, we get $n_{p+1}(0) = p-1$ and

$$N_{p+1}(r) = (p-1)\log r. \tag{3.2.6}$$

Now, recall

$$S(r) = \frac{1}{2\pi}\int_0^{2\pi} \max \log |\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}(re^{i\theta})|\,d\theta + O(1),$$

where the maximum is taken for all choices of $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ of distinct numbers $\alpha_1, \alpha_2, \dots, \alpha_p$ from $\{1, 2, \dots, p+1\}$. Consider

$$\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (F'_{\alpha_1})/(F_{\alpha_1}) & (F'_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F'_{\alpha_p})/(F_{\alpha_p}) \\ (F''_{\alpha_1})/(F_{\alpha_1}) & (F''_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F''_{\alpha_p})/(F_{\alpha_p}) \\ \vdots & \vdots & \ddots & \vdots \\ (F^{(p-1)}_{\alpha_1})/(F_{\alpha_1}) & (F^{(p-1)}_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F^{(p-1)}_{\alpha_p})/(F_{\alpha_p}) \end{vmatrix}.$$

First, consider the case $\{\alpha_1, \alpha_2, \dots, \alpha_p\} = \{1, 2, \dots, p\}$. We write $d_i = \deg F_i$, where $1 \le i \le p$. Then we can rewrite $F_i = A_i \prod_{j=1}^{d_i} (z - z_j)$, where A_i is a constant. Then $F'_i/F_i = \sum_{j=1}^{d_i} (z - z_j)^{-1} = O(z^{-1})$. For $1 \le l \le p-1$,

$$\frac{F_i^{(l)}}{F_i} = \frac{F_i^{(l)}}{F_i^{(l-1)}}\,\frac{F_i^{(l-1)}}{F_i^{(l-2)}}\cdots\frac{F'_i}{F_i} = O(z^{-l}).$$

The function $\Delta_{(\alpha_1,\alpha_2,\cdots,\alpha_p)}$ is a sum of products of the form $\pm F_i^{(j)}/F_i$, where $1 \le i \le p$ and $1 \le j \le p-1$. Therefore we get

$$\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}(z) = \sum O(z^{-1})O(z^{-2})\cdots O(z^{-(p-1)}) = O(z^{-(p(p-1)/2)}).$$

Now consider the other cases. Choose p different numbers $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ from $\{1, 2, \dots, p+1\}$ such that $\alpha_p = p+1$. Then, since $F_{p+1} = z^{1-mk}$, we get

$$\begin{aligned}\Delta_{(\alpha_1,\alpha_2,\dots,\alpha_p)}(z) &= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ (F'_{\alpha_1})/(F_{\alpha_1}) & (F'_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F'_{\alpha_{p-1}})/(F_{\alpha_{p-1}}) & O(z^{-1}) \\ (F''_{\alpha_1})/(F_{\alpha_1}) & (F''_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F''_{\alpha_{p-1}})/(F_{\alpha_{p-1}}) & O(z^{-2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (F^{(p-1)}_{\alpha_1})/(F_{\alpha_1}) & (F^{(p-1)}_{\alpha_2})/(F_{\alpha_2}) & \cdots & (F^{(p-1)}_{\alpha_{p-1}})/(F_{\alpha_{p-1}}) & O(z^{-(p-1)}) \end{vmatrix} \\ &= \sum O(z^{-1})O(z^{-2})\cdots O(z^{-(p-1)}) = O\left(z^{-(p(p-1)/2)}\right).\end{aligned}$$

Thus, considering all choices of $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, we obtain

$$\max \left|\Delta_{(\alpha_1,\alpha_2,\cdots,\alpha_p)}\right| = O\left(z^{-p(p-1)/2}\right). \tag{3.2.7}$$

The equation (3.1.7) together with (3.2.7) gives

$$S(r) \leq \frac{-p(p-1)}{2} \log r + O(1). \tag{3.2.8}$$

From Cartan's theorem, (3.2.5), (3.2.6), and (3.2.8), we get

$$T(r) \leq \frac{p(p-1)}{k} T(r) + (p-1)\log r - \frac{p(p-1)}{2} \log r + O(1). \tag{3.2.9}$$

Thus

$$\left(1 - \frac{p(p-1)}{k}\right) T(r) \leq (p-1)\left(1 - \frac{p}{2}\right) \log r + O(1). \tag{3.2.10}$$

We define $d_j = \deg h_j$, where $1 \leq j \leq p$. Since there exists j with $t_j \geq 1$ and $m \leq -1$, we have $\alpha = \max_{1\leq j\leq p} \deg h_j = \max_{1\leq j\leq p} (t_j - m) \geq 2$. Then we have

$$|h_j(re^{i\theta})| \geq c_j r^{\deg h_j} = c_j r^{d_j}$$

as $r \to \infty$, where c_j is a positive constant. Hence

$$\log|F_j| = k\log|h_j| \geq k\log c_j r^{d_j} \geq k\, d_j \log r + O(1)$$

as $r \to \infty$. Thus

$$\begin{aligned}\sup_{1\leq j\leq p} \log|F_j| &= \sup_{1\leq j\leq p} k\log|h_j| \geq k\left(\max_{1\leq j\leq p} d_j\right)\log r + O(1)\\ &= k\,\alpha\, \log r + O(1)\end{aligned}$$

as $r \to \infty$. Hence we get

$$T(r) \geq k\,\alpha\, \log r + O(1) \tag{3.2.11}$$

as $r \to \infty$. Now, the inequality (3.2.10) together with (3.2.11) gives

$$(k - p(p-1))\,\alpha\, \log r \leq (p-1)\left(1 - \frac{p}{2}\right) \log r + O(1)$$

as $r \to \infty$. If $p = 2$, we obtain $k \leq p(p-1) = 2$. If $p > 2$, the righthand side is negative for large r, so that $k < p(p-1)$. This completes the proof of Theorem 3.2.1. $\square$

Since the class L of Laurent polynomials is not closed under composition, we cannot deduce the representability of all functions in L from that of z. On the other hand if $W_C(k)$ denotes the minimum number n, such that any element of the class C can be represented as the sum of n kth powers of functions in C, then

$$W_L(k) \leq W_P(k) \leq k.$$

In fact if $f(z) = P(z)/z^{nk}$ is a Laurent polynomial, where $P(z)$ is an ordinary polynomial, then we have

$$P(z) = \sum_{\nu=1}^{p} (P_\nu(z))^k$$

where $p = W_P(k)$ and the P_ν are polynomials, Then

$$f(z) = \sum_{\nu=1}^{p} \left(\frac{P_\nu(z)}{z^n}\right)^k$$

is the required representation of $f(z)$ in terms of Laurent polynomials. Thus $W_L(k) \leq p = W_P(k)$.

3.3. The representation of a constant by Laurent polynomials.

Theorem 3.3.1. *Suppose that $k \geq 3$ and that $n \geq 2$. Let $f_1, f_2, \dots, f_n$ be nonconstant Laurent polynomials satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = 1. \tag{3.3.1}$$

Suppose that at least one of the f_i is not a polynomial and that p is the smallest number n satisfying (3.3.1). *Then $p^2 - p > k$ for $k \geq 3$, i.e., $p > 1/2 + \sqrt{k + 1/4}$ for $k \geq 3$.*

The proof of Theorem 3.3.1 is similar to that of Theorem 3.2.1, so we omit it.

3.4. The proof of Hayman's theorem. We conclude this paper by discussing the proof of Hayman's theorem [**9**, p. 2], which is Theorem 1.1.2 in this paper, for polynomials. Suppose that $k \geq 2$ and that $n \geq 2$. Let $f_1, f_2, \ldots, f_n$ be nonconstant polynomials satisfying

$$\sum_{i=1}^{n} f_i(z)^k = z.$$

Suppose that p is the smallest number n satisfying the above equation. Since $F_1 = {f_1}^k,\ F_2 = {f_2}^k,\ \ldots,\ F_p = {f_p}^k$ and $F_{p+1} = z$, we get

$$N_i(r) \leq \frac{p-1}{k}\, T(r) + O(1) \quad \text{for} \quad 1 \leq i \leq p, \tag{3.4.1}$$

and

$$N_{p+1}(r) = \log r. \tag{3.4.2}$$

Hayman considered only

$$\Delta(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (F_1'/F_1) & (F_2'/F_2) & \cdots & (F_p'/F_p) \\ (F_1''/F_1) & (F_2''/F_2) & \cdots & (F_p''/F_p) \\ \vdots & \vdots & \ddots & \vdots \\ (F_1^{(p-1)}/F_1) & (F_2^{(p-1)}/F_2) & \cdots & (F_p^{(p-1)}/F_p) \end{vmatrix}$$

and set

$$S(r) = \frac{1}{2\pi}\int_0^{2\pi} \log\, |\Delta(re^{i\theta})|\, d\theta + O(1). \tag{3.4.3}$$

Then

$$\Delta(z) = O\left(z^{-(p(p-1))/2}\right).$$

Now, we estimate $\Delta(z)$ more precisely. We define $d_i = \deg F_i$, where $1 \leq i \leq p$, and $F_i = a_{d_i} z^{d_i} + a_{d_i - 1} z^{d_i - 1} + \cdots$. Then $F_i' = a_{d_i} d_i z^{d_i - 1} + a_{d_i - 1}(d_i - 1) z^{d_i - 2} + \cdots$. Hence,

$$\frac{F_i'}{F_i} = \frac{a_{d_i} d_i z^{d_i - 1} + a_{d_i - 1}(d_i - 1) z^{d_i - 2} + \cdots}{a_{d_i} z^{d_i} + a_{d_i - 1} z^{d_i - 1} + \cdots} = \frac{d_i}{z} + O(z^{-2}).$$

Similarly,

$$\begin{aligned}\frac{F_i''}{F_i} &= \frac{a_{d_i} d_i(d_i-1)z^{d_i-2} + a_{d_i-1}(d_i-1)(d_i-2)z^{d_i-3}+\cdots}{a_{d_i}z^{d_i}+a_{d_i-1}z^{d_i-1}+\cdots}\\ &= \frac{d_i(d_i-1)}{z^2}+O(z^{-3}).\end{aligned}$$

Similarly, we can write

$$\frac{F_i^{(l)}}{F_i} = \frac{d_i(d_i-1)\cdots(d_i-(l-1))}{z^l}+O\left(\frac{1}{z^{l+1}}\right),$$

where $1 \le l \le p-1$. Consider the determinant δ where each (i,j)-entry is the leading term of the (i,j)-entry in Δ. Then

$$\delta(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ d_1/z & d_2/z & \cdots & d_p/z \\ (d_1(d_1-1))/z^2 & (d_2(d_2-1))/z^2 & \cdots & (d_p(d_p-1))/z^2 \\ \vdots & \vdots & \ddots & \vdots \\ (d_1(d_1-1)\cdots(d_1-p+2))/z^{p-1} & (d_2(d_2-1)\cdots(d_2-p+2))/z^{p-1} & \cdots & (d_p(d_p-1)\cdots(d_p-p+2))/z^{p-1} \end{vmatrix}.$$

By applying elementary row operations to δ we get $\delta = \left(z^{-(p(p-1))/2}\right)\delta'$, where

$$\delta' = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_p \\ {d_1}^2 & {d_2}^2 & \cdots & {d_p}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {d_1}^{p-1} & {d_2}^{p-1} & \cdots & {d_p}^{p-1} \end{vmatrix}.$$

Since the last determinant δ' is the van der Monde determinant [**2**], we get

$$\delta' = \prod_{i<j}(d_j-d_i).$$

Since there exist d_i and d_j such that $d_i = d_j = \max_{1\leq n\leq p} d_n$, where $i \neq j$, we obtain $\delta' = 0$ and we get

$$\Delta(z) = O\left(z^{(-p(p-1)/2)-1}\right). \tag{3.4.4}$$

The equation (3.4.3) together with (3.4.4) gives

$$S(r) \leq \left(\frac{-p(p-1)}{2} - 1\right)\log r + O(1). \tag{3.4.5}$$

If it were sufficient to consider only this Δ as Hayman [**9**, p. 4] claimed, it would follow from Cartan's theorem as stated by Hayman [**9**, p. 3], (3.4.1), (3.4.2), and (3.4.5) that

$$T(r) \leq \frac{p(p-1)}{k}\, T(r) + \log r + \left(\frac{-p(p-1)}{2} - 1\right)\log r + O(1). \tag{3.4.6}$$

Thus

$$\left(1 - \frac{p(p-1)}{k}\right) T(r) \leq -\frac{p(p-1)}{2}\log r + O(1). \tag{3.4.7}$$

For the case $k = 2$ and $p = 2$, we know that $(z + (1/4))^2 - (z - (1/4))^2 = z$. Let $F_1(z) = (z + (1/4))^2$, $F_2(z) = -(z - (1/4))^2$, and $F_3(z) = z$. Then we obtain

$$\begin{aligned}\Delta(z) &= \begin{vmatrix} 1 & 1 \\ F_1'/F_1 & F_2'/F_2 \end{vmatrix} = \frac{2}{(z-(1/4))} - \frac{2}{(z+(1/4))} \\ &= \frac{1}{(z-(1/4))\,(z+(1/4))} = O\left(\frac{1}{z^2}\right)\end{aligned}$$

as $z \to \infty$. Thus, by the inequality (3.4.7), we get

$$0 \leq -\log r + O(1)$$

as $r \to \infty$. This is a contradiction. Thus we need to consider the maximum of $|\Delta(z)|$ for all choices of $\{\alpha_1, \alpha_2\}$ from $\{1, 2, 3\}$ to use Cartan's theorem. Since

$$\begin{aligned}\Delta_{(1,3)}(z) &= \begin{vmatrix} 1 & 1 \\ F_1'/F_1 & F_3'/F_3 \end{vmatrix} = -\frac{1}{(z+(1/4))} + \frac{1/4}{z\,(z+(1/4))} \\ &= O\left(\frac{1}{z}\right) + O\left(\frac{1}{z^2}\right) = O\left(\frac{1}{z}\right)\end{aligned}$$

as $z \to \infty$ and

$$\Delta_{(2,3)}(z) = \begin{vmatrix} 1 & 1 \\ F_2'/F_2 & F_3'/F_3 \end{vmatrix} = -\frac{1}{(z-(1/4))} + \frac{-1/4}{z\,(z-(1/4))}$$
$$= O\left(\frac{1}{z}\right) + O\left(\frac{1}{z^2}\right) = O\left(\frac{1}{z}\right)$$

as $z \to \infty$, the maximum of $|\Delta(z)|$ for all choices of $\{\alpha_1, \alpha_2\}$ from $\{1,2,3\}$ is $O(1/z)$ and not $O(z^{-2})$.

The above consideration shows the sharpness of Cartan's theorem: it is really necessary to define w in the proof of Cartan's theorem [**1**, p. 13] and $S(r)$ in (3.1.7) as the maximum involving several determinants, not only using one determinant. Hayman [**9**, p. 4] incorrectly quoted Cartan's theorem in terms of one determinant only. Luckily, he estimated this one determinant in a cruder fashion than he could have, and his cruder upper bound is, in fact, an upper bound for the required maximum. Therefore the final conclusions of Hayman remain valid.

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