ON GENERIC IDENTIFIABILITY OF SYMMETRIC TENSORS OF SUBGENERIC RANK

LUCA CHIANTINI, GIORGIO OTTAVIANI, AND NICK VANNIEUWENHOVEN

ABSTRACT. We prove that the general symmetric tensor in $S^d\mathbb{C}^{n+1}$ of rank r is identifiable, provided that r is smaller than the generic rank. That is, its Waring decomposition as a sum of r powers of linear forms is unique. Only three exceptional cases arise, all of which were known in the literature. Our original contribution regards the case of cubics ($d = 3$), while for $d \geq 4$ we rely on known results on weak defectivity by Ballico, Ciliberto, Chiantini, and Mella.

1. INTRODUCTION

We denote by $S^d\mathbb{C}^{n+1}$ the space of symmetric tensors on $\mathbb{C}^{n+1}$; such tensors can be identified with homogeneous polynomials of degree d in $n+1$ variables, which are also referred to as *forms*. In this symmetric setting, the most natural tensor rank decomposition is the classical Waring decomposition, which expresses a symmetric tensor as a sum of powers of linear forms. Precisely, every form $f \in S^d\mathbb{C}^{n+1}$ has a minimal expression

$$f = \sum_{i=1}^{r} l_i^d, \tag{1}$$

where $l_i \in \mathbb{C}^{n+1}$ are linear forms [21]; the minimal number of summands r is called the symmetric rank of f, since in the correspondence between forms and symmetric tensors, powers of linear forms correspond to tensors of rank 1. A natural question concerns the number of summands required for representing a general form in $S^d\mathbb{C}^{n+1}$. This problem is elementary for $d = 2$, which corresponds to the case of symmetric matrices. For $d \geq 3$, the question was answered by Alexander and Hirschowitz in [2]. Letting

$$r_{d,n} = \frac{\binom{n+d}{d}}{n+1}, \tag{2}$$

they proved that the general $f \in S^d\mathbb{C}^{n+1}$ with $d \geq 3$ has rank $\lceil r_{d,n} \rceil$, which is called the *generic rank*, unless the space $S^d\mathbb{C}^{n+1}$ is one of the so-called defective cases $S^4\mathbb{C}^{n+1}$ for $n = 2, 3, 4$ and $S^3\mathbb{C}^5$, where the generic rank is $\lceil r_{d,n} \rceil + 1$. When the rank of a Waring decomposition is strictly smaller than $r_{d,n}$, we say that this decomposition is of *subgeneric rank*. It could be worthy noticing that, in our notation, being of *subgeneric rank* is not always equivalent to being of *rank smaller than*

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the one of a general tensor, because in the defective cases above a general tensor has rank strictly bigger than $r_{d,n}$.

The Alexander–Hirschowitz theorem implies that the generic tensor of subgeneric rank admits only a finite number of alternative Waring decompositions [24]. In this paper, we shall be concerned with proving that the generic tensor of subgeneric rank admits precisely one Waring decomposition, modulo permutations of the summands and scaling by d-roots of unity. More precisely, the main result of this paper is the following theorem.

Theorem 1.1. *Let $d \geq 3$. The general tensor in $S^d\mathbb{C}^{n+1}$ of subgeneric rank $r < r_{d,n}$ with $r_{d,n}$ as in (2) has a* unique *Waring decomposition, i.e., it is* identifiable, *unless it is one of the following cases:*

1. *$d = 6$, $n = 2$, and $r = 9$;*
2. *$d = 4$, $n = 3$, and $r = 8$;*
3. *$d = 3$, $n = 5$, and $r = 9$.*

In all of these exceptional cases, there are exactly two Waring decompositions.

The three exceptional cases were already known in the literature. The first two cases are classical; see Remark 4.4 in [26] and Remark 6.5 in [13]. The third case was recently found by Ranestad and Voisin; see the proof of Lemma 4.3 in [30]. A uniform treatment of these cases is presented in Proposition 2.1. Our original contribution establishes that there are no more exceptions to identifiability for cubics.

The proof of Theorem 1.1 is based on the study of the geometric concepts of *weak defectivity*, developed in [8], and *tangentially weak defectivity*, developed in [6]. Indeed, from this point of view, the theorem can be reformulated in the following way, which is a result of independent interest.

Theorem 1.2. *Let $d \geq 3$, $r_{d,n}$ as in (2), and $r < r_{d,n}$. Then, the common singular locus of the space of hypersurfaces of degree d in $\mathbb{P}^n$ that are singular at r general points, consists of exactly these r points, except in the following cases:*

1. *$d = 6$, $n = 2$, and $r = 9$. The unique sextic plane curve singular at 9 general points is a double cubic, so that its singular locus is an elliptic cubic curve;*
2. *$d = 4$, $n = 3$, and $r = 8$. The net of quartic surfaces singular at 8 points consists of reducible quadrics, so that the common singular locus is the base locus of the pencil of quadrics through 8 general points, which is an elliptic normal curve of degree 4;*
3. *$d = 3$, $n = 5$, and $r = 9$. The common singular locus of the pencil of cubic 4-folds singular at 9 general points is the unique elliptic normal curve of degree 6 through these 9 points.*

Furthermore, the above exceptional cases are the only instances where there exists a unique elliptic normal curve of degree $n + 1$ in $\mathbb{P}^n$ through r general points.

In this formulation, the theorem was already partially proved: the case $n \leq 2$ was proved by Chiantini and Ciliberto [9]; for $d \geq 4$ it was proved by Ballico [4, Theorem 1.1]; and for $d = 3$ with $r < r_{d,n} - \frac{n+2}{3} + 1$ it was proved by Mella [26, Theorem 4.1]. Consequently, the original contribution of this paper concerns the case of cubics, i.e., $d = 3$, which we solve completely in the subgeneric case. This answers the question posed in Remark 4.4 in [26].

We notice that Ballico [4] proved an even stronger result for $d \geq 4$. Namely, he showed that a general hypersurface of degree $d \geq 4$ in $\mathbb{P}^n$ that is singular in r general points, is singular only at these r points (except for the exceptional cases (1) and (2) of Theorem 1.2). This is equivalent to showing that the Veronese variety $v_d(\mathbb{P}^n)$ is not r-weakly defective, while our result only says that it is not r-*tangentially* weakly defective. We wonder whether the above list also gives the classification of all r-weakly defective Veronese varieties $v_d(\mathbb{P}^n)$, even for $d = 3$.

Symmetric tensors of general rank are not expected to admit only a finite number of Waring decompositions, because the expected dimension $\lceil r_{d,n} \rceil (n+1)$ of the $\lceil r_{d,n} \rceil$-secant variety of the Veronese variety $v_d(\mathbb{P}^n)$ may exceed the dimension $\binom{n+d}{d}$ of the embedding space $S^d\mathbb{C}^{n+1}$. Therefore, at least a curve's worth of alternative Waring decompositions of a general symmetric tensor is anticipated in these cases. However, if $r_{d,n} = \lceil r_{d,n} \rceil$ is integer, then a general symmetric tensor is still expected to admit only a finite number of Waring decompositions. The approach pursued in this paper, i.e., proving not tangential weak defectivity, cannot handle tensors of the generic rank. Other approaches need to be considered in this setting. In fact, Mella [27] formulated a conjecture about the cases where the expression in (1) is still expected to be unique even for general symmetric tensors. In [20], further evidence for this conjecture was given; in addition, the analogous problem for nonsymmetric tensors was also considered.

Even though the general symmetric tensor is not of subgeneric rank, the setting considered in this paper is nevertheless important in applications where one is mostly interested in the identifiability of symmetric tensors of subgeneric rank. For instance, Anandkumar, Ge, Hsu, Kakade, and Telgarsky [3] consider statistical parameter inference algorithms based on decomposing symmetric tensors for a wide class of latent variable models. The identifiability of the Waring decomposition then ensures that the recovered parameters, which correspond with the individual symmetric rank-1 terms in Waring's decomposition, are unique, and, thus, admit an interpretation in the application domain. The rank of the Waring decomposition, in these applications, is invariably much smaller than the generic rank. As general sources on tensor decomposition, we refer to [13, 16, 21, 24, 32].

In analogy to Theorem 1.1, we mention that the results in [6, 11, 12] give broad evidence to the analogous problem in the setting of nonsymmetric tensors, i.e., that a general nonsymmetric tensor of subgeneric rank admits a unique tensor rank decomposition, unless it is one of the exceptional cases that have already been proved in [1, 5, 6, 10, 11].

The content of the paper is the following. In section 2, we present a uniform treatment of the exceptional cases appearing in Theorems 1.1 and 1.2. Remark 2.6 also discusses our initial motivation for studying the topic of this paper. Section 3 contains the proof of the main theorem. Thereafter, the connection between weakly defective varieties and the dual varieties to secant varieties, including a description of the dual varieties of all weakly defective examples appearing in Theorem 1.2, is explored in section 4. In particular, Theorem 4.2 contains the description of cubic hypersurfaces in $\mathbb{P}^5$ that can be written as the determinant of a 3×3 matrix with linear entries. In section 5, we give an explicit criterion allowing to check if a given Waring decomposition is unique. This algorithm is an extension to the symmetric case of the one provided in [12] for general tensors.

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2. The exceptional cases

The following classical result shows that the values $n = 2, 3, 5$, which appear in Theorems 1.1 and 1.2, have a special role for elliptic normal curves.

Proposition 2.1 (Coble [15]). *Assume that there are only finitely many elliptic normal curves passing through k general points in $\mathbb{P}^n$. Then, $n = 2$, 3, or 5 and, correspondingly, $k = \frac{(n+1)^2}{n-1}$. In these three cases, there is a unique elliptic normal curve in $\mathbb{P}^n$ passing through $\frac{(n+1)^2}{n-1}$ general points.*

Proof. Elliptic normal curves of degree $(n+1)$ in $\mathbb{P}^n$ depend on $(n+1)^2$ parameters, which is the dimension of the space of sections of the normal bundle. The passage of the curve through a point in $\mathbb{P}^n$ imposes $n-1$ conditions, which is the codimension of the curve. Therefore, we may expect finitely many elliptic normal curves through k *general* points in $\mathbb{P}^n$ only if $k(n-1) = (n+1)^2$. This implies that $(n-1)$ divides $(n+1)^2 = (n-1)(n+3) + 4$, hence $(n-1)$ divides 4, which gives the values $n = 2, 3, 5$. Moreover, $k = (n+1)^2/(n-1)$.

In case $n = 2$ and $k = 9$, the elliptic curve is a plane cubic, and it is unique.

In case $n = 3$ and $k = 8$, an elliptic normal curve is a complete intersection of two quadrics. Thus, if $\langle Q_1, Q_2 \rangle$ is the pencil of quadrics through 8 general points $p_1, \ldots, p_8$, then $C = Q_1 \cap Q_2$ is the unique elliptic normal curve through the p_i's.

In case $n = 5$ and $k = 9$, the existence and the uniqueness of the curve was found by Coble [15, Theorem 19] by applying a Gale transform—see [18] for a nice review—and reducing to the case $n = 2$ and $k = 9$; a modern treatment was given by Dolgachev [17, Theorem 5.2]. □

Next, we analyze the case of cubic hypersurfaces in $\mathbb{P}^5$ that are singular at 9 general points.

Proposition 2.2 (Veneroni [34, Section 1], Coble [15, p. 16], Room [31, Sections 9–22], Fisher [19, Lemma 2.9]). *We have the following two results.*

- *(i) The 2-minors of a 3×3 matrix with linear entries on $\mathbb{P}^5$ define a (sextic) elliptic normal curve in $\mathbb{P}^5$.*
- *(ii) If $\mathcal{C}$ is a (sextic) elliptic normal curve in $\mathbb{P}^5$, then the variety of secant lines $\sigma_2(\mathcal{C})$ is a complete intersection of two cubic hypersurfaces on $\mathbb{P}^5$, each one being the determinant of a 3×3 matrix with linear entries on $\mathbb{P}^5$.*

Proof. Part (i) is well known: The curve is obtained by cutting the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^9$, i.e., the variety of 3×3 matrices of rank 1, with a linear space $\mathbb{P}^5$. Claim (ii) follows by [19, Lemma 2.9]. □

Theorem 2.3. *Let $p_1, \ldots, p_9$ be general points in $\mathbb{P}^5$. Let $\mathcal{C}$ be the elliptic normal sextic curve through these points. A cubic that is singular at $p_1, \ldots, p_9$ contains $\sigma_2(\mathcal{C})$ and is singular on $\mathcal{C}$.*

Proof. By Proposition 2.2, in the pencil of cubics containing $\sigma_2(\mathcal{C})$, the general element is singular along $\mathcal{C}$. This pencil fills the space of cubics that are singular at $p_1, \dots, p_9$, which is 2-dimensional by the Alexander–Hirschowitz theorem [2]. $\square$

These observations lead to a different proof of the third exceptional case in Theorem 1.1.

Proposition 2.4 (Ranestad–Voisin [30])**.** *The general tensor in* $S^3\mathbb{C}^6$ *of rank* 9 *has exactly two Waring decompositions as sum of* 9 *powers of linear forms.*

Proof. In the language of [9], we have to prove that the secant order of $\sigma_9(v_3(\mathbb{P}^5))$ is 2. By [9, Theorem 2.4], this is equal to the secant order of the 9-contact locus $\mathcal{C}$, which corresponds to the third Veronese embedding of an elliptic normal sextic curve in $\mathbb{P}^5$, by Theorem 2.3. Thus, $\mathcal{C}$ is an elliptic curve of degree 18 in $\mathbb{P}^{17}$, whose secant order is 2 by [9, Proposition 5.2]. $\square$

Corollary 2.5. *Let* $n = 2$, 3*, or* 5*. The unique elliptic normal curve that is mentioned in Proposition 2.1, which passes through* $\frac{(n+1)^2}{n-1}$ *general points in* $\mathbb{P}^n$*, can be constructed as the singular locus of a general hypersurface of degree* $\frac{2(n+1)}{n-1}$ *that is singular in the* $\frac{(n+1)^2}{n-1}$ *points.*

Proof. The cases $n = 2, 3$ have already been considered in the proof of Proposition 2.1. The case $n = 5$ follows from Theorem 2.3. $\square$

Remark 2.6. *It was only at the completion of this manuscript that we became aware of Ranestad and Voisin's proof of the third exceptional case that appears in Theorems 1.1 and 1.2. Our initial motivation for studying this problem arose because the third case was unexpectedly—in our minds—suggested by a computational analysis performed by the third author, who ran the algorithm that we present in section 5, for hypersurfaces of degree* d *in* $\mathbb{P}^n$ *singular at the maximal number of random points, i.e.,* $r = r_{d,n} - 1$ *with* $r_{d,n}$ *as in (2), for all reasonably small values of* d, n*. It took a while to realize what happened, because this third case was missing in [25, Theorem 6.1.2]. Actually, Theorem 6.1.2 of [25] only intended to collect previous results by Ballico [4], Ciliberto and Chiantini [8], and Mella [26,27], which are individually correct. The second author takes the responsibility to have first overlooked the assumption* $d \geq 4$ *in summarizing and reporting the results of [4,26]. For* $d = 3$*, Theorem 1.1 was known with the additional assumption* $r < r_{3,n} - \frac{n+2}{3} + 1$*; see [26, Theorem 4.1]. From the theoretical proof that we present in section 3, we can conclude that the third case was the last exception. Therefore, Theorem 6.1.2 in [25] remains true if the third case* $(k, d, n) = (9, 5, 3)$ *is added to the list of exceptions. Exactly the same remark applies to the formulation of Theorem 2.3 in [28] and Theorem 12.3.4.3 in [24]. We informed the coauthors of [25,28] of the problem, and they accepted the above conclusion.*

3. Cubics singular at the maximum number of points.

We turn our attention to the proof of Theorem 1.1 in the case of cubics, i.e., $d = 3$. Given n, we define

$$k_n = \left\lceil \frac{\binom{n+3}{3}}{n+1} \right\rceil = \left\lceil \frac{(n+3)(n+2)}{6} \right\rceil;$$

it is the generic rank for cubic polynomials for $n \neq 4$. In other words, a cubic polynomial on $\mathbb{P}^n$ singular at k_n general points vanishes identically for $n \neq 4$ [2]. Some elementary algebra shows that $k_n = \frac{(n+3)(n+2)}{6}$ if $n \not\equiv 2 \mod 3$, while $k_n = \frac{(n+3)(n+2)}{6} + \frac{2}{3} = \frac{(n+4)(n+1)}{6} + 1$ if $n \equiv 2 \mod 3$.

For the sake of future reference, let us state explicitly the following consequence of the Alexander–Hirschowitz theorem [2].

Theorem 3.1 (Alexander–Hirschowitz [2])**.** *The space of cubic hypersurfaces on $\mathbb{P}^n$ that are singular at $k_n - 1$ general points has dimension*

(i) $n+1$ *if* $n \not\equiv 2 \mod 3$, *or*
(ii) $\frac{n+1}{3}$ *if* $n \equiv 2 \mod 3$.

In addition, the space of cubic hypersurfaces on $\mathbb{P}^n$, $n \neq 4$, that are singular at k_n general points is empty.

To complete the proof of the main theorems, it remains to show the following result, which refines Theorem 3.1.

Theorem 3.2. *The space of cubic hypersurfaces in $\mathbb{P}^n$ that are singular at $k_n - 1$ general points has dimension*

(i) $n+1$ *if* $n \not\equiv 2 \mod 3$, *or*
(ii) $\frac{n+1}{3}$ *if* $n \equiv 2 \mod 3$,

and, in addition, its common singular locus consists only of these $k_n - 1$ points, provided that $n \neq 5$.

In order to prove Theorem 3.2, we may assume $n \geq 6$, since the cases with $n \leq 4$ (as well as the case of cubics in $\mathbb{P}^5$ singular at 8 points) can be checked separately using the approach described in section 5. The proof of Theorem 3.2 will follow the following steps. In section 3.1, we will prove case (i) by induction on subspaces of codimension 3, following an approach that is mainly inspired by [7, Section 5], where an alternative proof of Theorem 3.1 was given. To prove case (ii), the aforementioned technique needs a modification. We will construct an inductive proof on subspaces of codimension 3 and 4; in the inductive step, we will rely, additionally, on the argument of case (i). This strategy will be presented in section 3.2.

In the rest of this section, if S is a set of simple points in $\mathbb{P}^n$ and $P \subset \mathbb{P}^n$ is a linear subspace, we denote with $I_{S,P}(d)$ the space of degree d polynomials in P vanishing at all of the points in S. Moreover, if $\mathbf{X}$ is a a set of double (singular) points, we denote by $I_{\mathbf{X} \cup S,P}(d)$ the space of degree d polynomials in P vanishing on all of the points in $S \cup \mathbf{X}$ and whose derivatives vanish on all of the points in $\mathbf{X}$. The notation $L = (x_i \dots x_{i+d})$ denotes the subspace of codimension $d+1$ whose ideal is $\langle x_i, x_{i+1}, \dots, x_{i+d} \rangle$.

3.1. **Proof of Theorem 3.2 (i) by induction on codimension** 3**.** We start by proving three auxiliary results.

Proposition 3.3. *Let $n \geq 6$, and let $L, M, N \subset \mathbb{P}^n$ be general subspaces of codimension 3. Let l_i, respectively m_i, with $i = 1, 2, 3$ be three general points on L, respectively M. Let n_i with $i = 1, 2$ be two general points on N. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M \cup N$ and that are singular at the eight points $\mathbf{X} = \{l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2\}$ has dimension 3. Furthermore, the common singular locus is contained in $L \cup M \cup N$.*

Proof. The base cases $n = 6$, 7 and 8 can be proved with the Macaulay2 script `generic-identifiability.m2` that is provided as an ancillary file to the arXiv version of this paper. Using this software, we may compute the following dimensions:

$$\dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^6}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^6}(3) = 3, \text{ and}$$
$$\dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^7}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^7}(3) = 3,$$

so that the claim on the codimension follows. The code also proves the statement about the singular locus.

For $n \geq 9$, the statement follows by induction on n. Indeed, we may choose coordinates such that $L = (x_0 \dots x_2)$, $M = (x_3 \dots x_5)$, $N = (x_6 \dots x_8)$. In this setting it is clear that there are no quadrics that contain $L \cup M \cup N$, and moreover every cubic containing $L \cup M \cup N$ is a cone with vertex in $L \cap M \cap N$. Thus, for a general hyperplane $H \subset \mathbb{P}^n$, the Castelnuovo sequence (see [7, Equation (1)]) induces an inclusion

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3).$$

Hence, if we specialize the eight points to the hyperplane H, we get an inclusion

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M\cup N)\cap H,H}(3).$$

Then, our statement follows by induction. The singular locus is a cone with vertex $L \cap M \cap N$ over the singular locus of the base case $n = 8$. □

Remark 3.4. *Following the output of the software for the case $n = 8$, we can guess the common singular locus of the cubic hypersurfaces in $I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3)$, for $n \geq 8$. It turns out that in some examples—but we believe in general—it is given by the union of the three linear subspaces $L \cap M$, $L \cap N$, $M \cap N$ and by 8 linear subspaces of codimension 7, each containing one of the 8 points, and three of them contained in L, three of them contained in M, and two of them contained in N.*

Proposition 3.5. *Let $n \geq 5$, and let $L, M \subset \mathbb{P}^n$ be subspaces of codimension three. Let l_i, respectively m_i, with $i = 1, \dots, n-2$ be general points on L, respectively M. Let $p_1, p_2 \in \mathbb{P}^n$ be general points. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ containing $L \cup M$ and singular along the set of $2n-2$ points $\mathbf{X} = \{l_1, l_2, \dots, l_{n-2}, m_1, m_2, \dots, m_{n-2}, p_1, p_2\}$ has dimension $n+1$. Its common singular locus contains the linear space $L \cap M$ and is 0-dimensional at the points p_1 and p_2.*

Proof. The base cases $n = 5$, 6, and 7 can be proved with the Macaulay2 script `generic-identifiability.m2`. Running the software, we find the following dimensions

$$\dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^5}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^5}(3) = 6,$$
$$\dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^6}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^6}(3) = 7, \text{ and}$$
$$\dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^7}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^7}(3) = 8.$$

These values indeed correspond to the claimed dimensions.

For $n \geq 8$, the statement follows by induction from $n-3$ to n. Indeed, given a third general subspace N of codimension 3, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3),$$

where the dimensions of the three spaces in the sequence are respectively 27, $9(n-1)$, and $9(n-4)$. Let us specialize $n-5$ of the points $l_i \in L$ to $L \cap N$, $n-5$ of the points $m_i \in M$ to $M \cap N$, and the two points p_1, p_2 to N. Then, we obtain a sequence

$$0 \longrightarrow I_{\mathbf{X} \cup L \cup M \cup N, \mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X} \cup L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X} \cup L \cup M) \cap N, N}(3),$$

where the trace $(\mathbf{X} \cup L \cup M) \cap N$ satisfies the assumptions on $N = \mathbb{P}^{n-3}$, so that we can apply the induction. Notice that the residual (left) space satisfies the hypotheses of Proposition 3.3 and has dimension 3. Since the common singular locus of the cubics containing $L \cup M$ and singular at $\mathbf{X}$ must be contained in the common singular locus of the leftmost 3-dimensional space, it follows by Proposition 3.3 that its components through p_1 and p_2 must be contained in N. After the degeneration, the space of cubics $I_{\mathbf{X} \cup L \cup M, \mathbb{P}^n}(3)$ still has dimension at most $3 + (n-2) = n+1$ by induction. Hence, by semicontinuity it follows that its dimension is indeed equal to $n+1$. The common singular locus cannot be positive dimensional at points p_1 and p_2, because otherwise it should be of positive dimension in the trace (right space), where by induction we know that it is 0-dimensional. □

Proposition 3.6. *Let $n \geq 6$, and let $L \subset \mathbb{P}^n$ be a subspace of codimension 3. If $n \not\equiv 2 \mod 3$, then the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain L and that are singular at $\frac{n(n-1)}{6}$ general points $l_i \in L$ and at n general points $p_i \in \mathbb{P}^n$ has dimension $n+1$. Moreover, its common singular locus is 0-dimensional at the n points p_i.*

Proof. The statement can be checked for $n = 6, 7$ with the Macaulay2 script `generic-identifiability.m2`.

Let $n \geq 9$ and $n \not\equiv 2 \mod 3$. Consider the sequence

$$0 \longrightarrow I_{L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{L, \mathbb{P}^n}(3) \longrightarrow I_{L \cap M, M}(3),$$

where M is a general subspace of codimension 3. Denoting by $\mathbf{X}$ the union of the double points supported at the points l_i and p_i, we get

$$0 \longrightarrow I_{\mathbf{X} \cup L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X} \cup L, \mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X} \cup L) \cap M, M}(3).$$

We specialize $\frac{(n-3)(n-4)}{6}$ of the points l_i to $L \cap M$ and $n-2$ of the points p_i to M. We can assume that at least one point p_i that is a contained in a positive dimensional component of the singular locus is not specialized. Thus, we left $n-2$ general points on L and 2 general points in $\mathbb{P}^n$. Let us note that we cannot apply induction from $n-3$ to n to determine the dimension of the right space—contrary to the strategy that was employed in the proof of the foregoing propositions in this section—because then we would have to specialize $n-3$ of the points p_i to M, hereby losing control over the singular locus. Instead, we note that we can immediately use Proposition 5.4 of [7] (in $\mathbb{P}^{n-3}$) on the trace (right space); it turns out to be empty. On the residual (left space), Proposition 3.5 can be invoked, proving that it has dimension $n+1$. Now, if the singular locus would have a positive dimensional component, then, since the dimension of the space of cubics is constant along the specialization (it equals $n+1$), we would get a deformation of the singular locus, which should be of positive dimension at every point. This, however, contradicts Proposition 3.5, hereby concluding the proof. □

We are now ready to prove the first part of Theorem 3.2.

Proof of Theorem 3.2, case (i). We fix a linear subspace $L \subset \mathbb{P}^n$ of codimension 3 and consider the exact sequence

$$0 \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow S_{\mathbb{P}^n}(3) \longrightarrow S_L(3),$$

where $S_{\mathbb{P}^n}(3)$ is the space of cubic polynomials on $\mathbb{P}^n$ and the quotient space $S_L(3)$ is isomorphic to the space of cubic polynomials on L. Then, we specialize to L as many points as possible in such a way that the trace with respect to L imposes independent conditions on the cubics of L. To be precise, we have $k_n - 1 = \frac{(n+3)(n+2)}{6} - 1$ double points and we specialize $k_{n-3} = \frac{n(n-1)}{6}$ of them to L, leaving n points outside. Then, the result follows from Theorem 5.1 of [7] on the trace (right space), which turns out to be empty, and by Proposition 3.6 on the residual (left space), which has dimension $n+1$. If the contact locus has positive dimension, then, since the dimension of the space of cubics is constant and equal to $(n+1)$ in the degeneration, we would get a deformation of the singular locus with a positive dimension at every point, contradicting Proposition 3.6 and concluding the proof. □

3.2. **Proof of Theorem 3.2 (ii) by induction on codimension** 3 **and** 4. For proving the second case in Theorem 3.2, we need to introduce several other auxiliary results on configurations that involve subspaces of codimension three and four. These configurations are covered in Propositions 3.7 through 3.12.

3.2.1. *Codimension* 4*,* 3*,* 3*.*

Proposition 3.7. *Let $n \geq 6$, and let $L, M, N \subset \mathbb{P}^n$ be general subspaces of codimension* 4*,* 3*, and* 3*, respectively. Let l_1, l_2, l_3 be general points on L. Let m_i, respectively, n_i with $i = 1, \ldots, 4$ be four general points on M, respectively N. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M \cup N$ and are singular at the* 11 *points $\mathbf{X} = \{l_1, l_2, l_3, m_1, \ldots, m_4, n_1, \ldots, n_4\}$ is empty.*

Proof. The proof is similar to the proof of Proposition 3.3. The Macaulay2 code proves the base cases $n = 6$, 7, 8 and 9. For $n \geq 9$, we may choose coordinates such that $L = (x_0 \ldots x_3)$, $M = (x_4 \ldots x_6)$, and $N = (x_7 \ldots x_9)$. Then, the statement follows by induction on n. Indeed, as in the proof of Proposition 3.3, the space $I_{L\cup M\cup N,\mathbb{P}^n}(2)$ is empty, thus for a general hyperplane $H \subset \mathbb{P}^n$, the Castelnuovo sequence induces an embedding

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3),$$

and, moreover, every cubic in the left space is a cone with vertex at $L \cap M \cap N$. Hence, by specializing the 11 points to the hyperplane H, we get:

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M\cup N)\cap H,H}(3).$$

Then, the statement follows by induction. □

Proposition 3.8. *Let $n \geq 7$, let $n \equiv 1 \mod 3$, and let $L, M \subset \mathbb{P}^n$ be subspaces of codimension* 4 *and* 3*, respectively. Let l_i with $i = 1, \ldots, n-3$ be general points on L. Let m_i with $i = 1, \ldots, \frac{4n-10}{3}$ be general points on M. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M$ and are singular at all the l_i's and m_i's and at four general points $p_1, p_2, p_3, p_4 \in \mathbb{P}^n$, is empty.*

Proof. The Macaulay2 script proves the base case $n = 7$.

For $n = 3k + 1$ with $k \geq 3$, the statement follows by induction from $n - 3$ to n. Indeed, given a third general subspace N of codimension 3, we get the exact sequence

$$0 \longrightarrow I_{L \cup M \cup N, \mathbb{P}^n}(3) \longrightarrow I_{L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{(L \cup M) \cap N, N}(3),$$

where the dimensions of the three spaces in the sequence are respectively 36, $12n-18$ and $12n - 54$. Let $\mathbf{X}$ denote the union of the double points supported at the p_i's, l_i's and m_i's. Assume that we specialize $n - 6$ of the points $l_i \in L$ to $L \cap N$, $\frac{4n-22}{3}$ of the points $m_i \in M$ to $M \cap N$, and the four points $p_1, \ldots, p_4$ to N. Then, we obtain a sequence

$$0 \longrightarrow I_{\mathbf{X} \cup L \cup M \cup N, \mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X} \cup L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X} \cup L \cup M) \cap N, N}(3),$$

where the trace $(\mathbf{X} \cup L \cup M) \cap N$ satisfies the assumptions on $N = \mathbb{P}^{n-3}$, so that we can apply induction. Then, we may conclude, as the residual (left space) satisfies the hypotheses of Proposition 3.7, and, consequently, it is empty. □

Proposition 3.9. *Let $n \geq 7$, $n \equiv 1 \mod 3$, and $L \subset \mathbb{P}^n$ be a subspace of codimension four. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that are singular at $k_{n-4} = \frac{(n-1)(n-2)}{6}$ general points l_i on L (and, thus, contain L, by Theorem 3.1) and at $\frac{4n+2}{3}$ general points $p_i \in \mathbb{P}^n$ is empty.*

Proof. The Macaulay2 script `generic-identifiability.m2` proves the case $n = 7$.

For $n = 3k + 1$ with $k \geq 3$, the statement follows by the sequence

$$0 \longrightarrow I_{L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{L, \mathbb{P}^n}(3) \longrightarrow I_{L \cap M, M}(3),$$

where M is a general subspace of codimension 3. If we denote by $\mathbf{X}$ the union of the double points supported at the points l_i and p_i, then we get the sequence

$$0 \longrightarrow I_{\mathbf{X} \cup L \cup M, \mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X} \cup L, \mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X} \cup L) \cap M, M}(3).$$

Then, we specialize k_{n-7} of the points l_i to $L \cap M$ and $\frac{4n-10}{3}$ of the points p_i to M. The trace (right space) contains exactly k_{n-3} double points and turns out to be empty by induction. Thus, there remain $n - 3$ general points on L and 4 general points on $\mathbb{P}^n$; we can then use Proposition 3.8 on the residual (left space) to conclude. □

3.2.2. *Codimension 4, 4, 3.*

Proposition 3.10. *Let $n \geq 8$, and let $L, M, N \subset \mathbb{P}^n$ be general subspaces of codimension respectively 4, 4, and 3. Let l_i, respectively m_i, with $i = 1, \ldots, 4$ be general points on L, respectively M. Finally, let n_i with $i = 1, \ldots, 5$ be general points on N. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M \cup N$ and that are singular at the 13 points $\mathbf{X} = \{l_1, \ldots, l_4, m_1, \ldots, m_4, n_1, \ldots, n_5\}$ has dimension 1. In other words, there is a unique cubic hypersurface W through $L \cup M \cup N$ and singular at $\mathbf{X}$. Furthermore, the singular locus of W is contained in $L \cup M \cup N$.*

Proof. The proof is similar to the proof of Proposition 3.3. The Macaulay2 code `generic-identifiability.m2` proves the base cases $n = 8$, 9, and 10.

For $n \geq 11$, we may choose coordinates such that $L = (x_0 \ldots x_3)$, $M = (x_4 \ldots x_7)$, and $N = (x_8 \ldots x_{10})$; then, the statement follows by induction on n. Indeed, as in the proofs of Proposition 3.3 and Proposition 3.7, we let $H \subset \mathbb{P}^n$ be a general hyperplane, so that the Castelnuovo sequence induces the inclusion

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3),$$

because the space $I_{L\cup M\cup N,\mathbb{P}^n}(2)$ is empty. Hence, by specializing the 13 points on the hyperplane H, we get an exact sequence:

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M\cup N)\cap H,H}(3).$$

Now the statement follows by induction. □

Proposition 3.11. *Let $n \geq 8$, $n \equiv 2 \mod 3$, and $L, M \subset \mathbb{P}^n$ be subspaces of codimension 4. Let l_i and m_i, where $i = 1, \ldots, \frac{4n-14}{3}$, be general points on L and M, respectively. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M$ and are singular at the $\frac{8n-28}{3}$ points l_i, m_i, $i = 1, \ldots, \frac{4n-14}{3}$, and at an additional set of five general points $p_i \in \mathbb{P}^n$, $i = 1, \ldots, 5$, has dimension $\frac{n+1}{3}$. Furthermore, its common singular locus, which contains the linear space $L \cap M$, is 0-dimensional at each of the points $p_1, \ldots, p_5$.*

Proof. The case $n = 8$ is handled in the `generic-identifiability.m2` Macaulay2 script.

For $n = 3k + 2$ with $k \geq 3$, the statement follows by induction on k. Given a third general subspace N of codimension 3, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3),$$

where the dimensions of the three spaces in the sequence are respectively 48, $16(n-2)$ and $16(n-5)$.

Let $\mathbf{X}$ denote the union of the double points supported at $p_1, \ldots, p_5$, l_i and m_i with $i = 1, \ldots, \frac{4n-14}{3}$. Then, we specialize $\frac{4n-26}{3}$ of the points $l_i \in L$ to $L \cap N$, $\frac{4n-26}{3}$ of the points $m_i \in M$ to $M \cap N$, and the points $p_1, \ldots, p_5$ to N. We thus obtain a sequence

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M)\cap N,N}(3),$$

where the trace $(\mathbf{X} \cup L \cup M) \cap N$ satisfies the assumptions on $N = \mathbb{P}^{n-3}$, so that we can apply induction. Then, the residual (left space) satisfies the hypotheses of Proposition 3.10 and has dimension one. Moreover, the common singular locus has to be contained in the common singular locus of the left 1-dimensional space. After the degeneration, the space $I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3)$ still has dimension less than or equal to $1 + \frac{n-2}{3} = \frac{n+1}{3}$, by induction, and, therefore, its dimension equals $\frac{n+1}{3}$, by semicontinuity. The common singular locus cannot be positive dimensional at the points $p_1, \ldots, p_5$, because otherwise it should be positive dimensional in the trace (right space), while we know that it is 0-dimensional there by induction. □

Proposition 3.12. *Let $n \geq 8$, $n \equiv 2 \mod 3$, and $L \subset \mathbb{P}^n$ be a subspace of codimension 4. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain L and are singular at $k_{n-4} = \frac{(n-1)(n-2)}{6}$ general points $l_i \in L$ and at $\frac{4n+1}{3}$ general points $p_i \in \mathbb{P}^n$ has dimension $\frac{n+1}{3}$. Furthermore, its singular locus is of dimension 0 at all of the points p_i.*

Proof. The statement follows by the sequence

$$0 \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow I_{L\cap M,M}(3),$$

where M is a general subspace of codimension 4. Denoting by $\mathbf{X}$ the union of the double points supported at the points l_i's and p_i's, we get

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L)\cap M,M}(3).$$

Suppose that the singular locus would be of positive dimension at one of the p_i's, say, at $q = p_j$. Then, we specialize k_{n-8} of the points l_i to $L \cap M$ and $\frac{4n-14}{3}$ of the points $\{p_i\}_{i\neq j}$ to M. The trace (right space) contains exactly k_{n-4} double points and is empty because of Proposition 3.9. There remain $\frac{4n-14}{3}$ general points on L and 5 general points—one of which is q—on $\mathbb{P}^n$. We can use Proposition 3.11 on the residual (left space), which has dimension $\frac{n+1}{3}$. By assumption, the singular locus has positive dimension at q; however, the dimension of the space of cubics is constant and equal to $\frac{n+1}{3}$ through the degeneration, so that we have a deformation of the singular locus, which is of positive dimension at every point. In particular, the singular locus will be of positive dimension at all the points that were not specialized to M, hereby contradicting Proposition 3.11. We conclude that our initial assumption must have been false, so that no general points p_j can exist where the singular locus is of positive dimension. □

Proof of Theorem 3.2, part (ii). We fix a codimension four linear subspace $L \subset \mathbb{P}^n$ and we use the exact sequence

$$0 \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow S_{\mathbb{P}^n}(3) \longrightarrow S_L(3),$$

where, as above, $S_{\mathbb{P}^n}(3)$ is the space of cubic polynomials on $\mathbb{P}^n$ and the quotient space $S_L(3)$ is isomorphic to the space of cubic polynomials on L. We specialize k_{n-4} points on L, leaving $\frac{4n+1}{3}$ points outside. Then, the result follows from Theorem 5.1 of [7] on the trace (right space), which turns out to be empty and by Proposition 3.12 on the residual (left space). If the contact locus would have a positive dimension, then, since the dimension of the space of cubics is constant and equal to $\frac{n+1}{3}$ in the degeneration, we would get a deformation of the singular locus, which should be of positive dimension at every point; however, this contradicts Proposition 3.12, hereby concluding the proof. □

4. Dual varieties to the relevant secant varieties

Denote by $\mathrm{T}_x\mathcal{X}$ the tangent space to the projective variety $\mathcal{X} \subset \mathbb{P}^n$ at the point $x \in \mathcal{X}$. Following the notation of [9] we say that $\mathcal{X}$ is *not k-weakly defective* if the general hyperplane H containing the linear span of the tangent spaces at k general points $x_1, \ldots, x_k \in \mathcal{X}$, i.e., $\langle \mathrm{T}_{x_1}\mathcal{X}, \ldots, \mathrm{T}_{x_k}\mathcal{X}\rangle \subset H$, is tangent to $\mathcal{X}$ only at finitely many points. This is equivalent with saying that the k-contact locus with respect to $x_1, \ldots, x_k$ and H is zero-dimensional.

For any projective variety $\mathcal{X}$, we will denote by $\mathcal{X}^\vee$ the dual variety to $\mathcal{X}$. Note that the dual of the secant variety $\sigma_k(v_d(\mathbb{P}^n))^\vee$ contains the points corresponding to hypersurfaces of degree d in $\mathbb{P}^n$ with k general singular points, and it has codimension $\geq k$, where k is the expected value for the codimension.

Proposition 4.1. *Let $\mathcal{X} \subset \mathbb{P}^N$ and let $\sigma_k(\mathcal{X})$ be the k-secant variety of $\mathcal{X}$. Then, the following are equivalent:*

(i) the general hyperplane H containing $\langle \mathrm{T}_{x_1}\mathcal{X}, \ldots, \mathrm{T}_{x_k}\mathcal{X}\rangle$ for general $x_1, \ldots, x_k$ is tangent to $\mathcal{X}$ only at $x_1, \ldots, x_k$, i.e., the k-contact locus with respect to $x_1, \ldots, x_k$ and H consists exactly of the points $x_1, \ldots, x_k$,

(ii) $\mathcal{X}$ is not k-weakly defective, and

(iii) $\dim [\sigma_k(\mathcal{X})]^\vee = N-k$*, that is a general hyperplane tangent to* $\sigma_k(\mathcal{X})$ *is tangent along a linear space of projective dimension* $k-1$*.*

Proof. (i) $\iff$ (ii) follows from [8, Theorem 1.4]. (ii) $\iff$ (iii) follows from Terracini's Lemma. □

It is interesting to describe the dual varieties of $\sigma_k(v_d(\mathbb{P}^n))$ in the exceptional cases of Theorems 1.1 and 1.2. They have dimension smaller than expected.

Theorem 4.2. *The following dual varieties correspond to the exceptional cases appearing in Theorems 1.1 and 1.2.*

(i) $\sigma_9(v_6(\mathbb{P}^2))^\vee$ *contains the plane sextics which are double cubics. It has codimension* 18*.*
(ii) $\sigma_8(v_4(\mathbb{P}^3))^\vee$ *contains the quartic surfaces which are reducible in a pair of quadrics. It has codimension* 16*.*
(iii) $\sigma_9(v_3(\mathbb{P}^5))^\vee$ *contains the cubic* 4*-folds which can be written as the determinant of a* 3×3 *matrix with linear entries. It has codimension* 18*.*

To compute the dimension in third case, note that the Hilbert scheme of elliptic normal sextic curves in $\mathbb{P}^5$ has dimension 36. So the cubic hypersurfaces coming from this construction have dimension 37, and $37+18=55=\binom{8}{3}-1$.

We remark that the defective Veronese varieties according to the classification of Alexander and Hirschowitz [2] (see [29] for the equations of the defective secant varieties) yield the following dual varieties

(i) $\sigma_{n(n+3)/2}(v_4(\mathbb{P}^n))^\vee$, for $n=2,3,4$, contains quartic hypersurfaces which are double quadrics. It has codimension $\binom{n+2}{3}\frac{n+7}{4}$.
(ii) $\sigma_7(v_3(\mathbb{P}^4))^\vee$ contains cubic 3-folds which can be written as the determinant of a 3×3 symmetric matrix with linear entries. It has codimension 13. Indeed, it is birational to the Hilbert scheme of quartic rational normal curves which has dimension 21.

5. Specific identifiability of symmetric tensors

While the generic symmetric tensor of subgeneric rank is expected to admit a unique Waring decomposition, *specific* tensors, whose Waring decomposition is assumed to be known, may admit multiple decompositions. We proceed by presenting an approach for certifying specific identifiability of symmetric tensors of small rank by checking not tangential weak defectivity of the r-secant variety of a Veronese variety in the given point. The strategy is an adaption of the algorithm from [12] to the setting of identifiability with respect to the Veronese variety $\mathcal{V}=v_d(\mathbb{P}^n)$. As such, the presented condition will only be a sufficient condition; that is, if the criterion does not apply, then the outcome of the test is inconclusive. On the other hand, if the criterion applies, then the given input tensor is r-identifiable and of symmetric rank r. Throughout this section, it is assumed that we are handed a Waring decomposition

$$p = p_1 + \cdots + p_r \in \sigma_r(\mathcal{V}) \subset S^d\mathbb{C}^{n+1},$$

wherein the point $p_i = \mathbf{a}_i^{\otimes d} \in \mathcal{V}$ is the degree d Veronese embedding of the vector $\mathbf{a}_i \in \mathbb{C}^{n+1}$. In other words, we know the points p_i appearing in the decomposition. The goal only consists of certifying that p is r-identifiable—the decomposition(s) are not sought. To this end, the strategy in [12] suggests a two-step procedure: Prove that p is a smooth point, and verify the Hessian criterion.

It is important to stress that we discuss the general setting of degree $d \geq 3$ Veronese embeddings. We will restrict our attention to nondefective r-secants of $\mathcal{V}$, because identifiability will not hold for general tensors on a defective r-secant variety. This is the interesting setting, because the Alexander–Hirschowitz theorem [2] stipulates that most $\sigma_r(\mathcal{V})$ are nondefective.

5.1. **The Hessian criterion.** We recall the main proposition from [12] and adapt it to the present context of symmetric tensors.

Lemma 5.1 (Sufficient condition for specific identifiability)**.** *Let $\mathcal{V} = v_d(\mathbb{P}^n)$ be a nondefective Veronese variety, and let $r \leq \lceil r_{d,n} \rceil - 1$ with $r_{d,n}$ as in (2). Assume that we are given a* nonsingular *point*

$$p = p_1 + p_2 + \cdots + p_r \in \sigma_r(\mathcal{V}).$$

If the linear span of the tangent spaces to $\mathcal{V}$ at the p_i's, i.e.,

$$\mathrm{M} = \langle \mathrm{T}_{p_1}\mathcal{V}, \ldots, \mathrm{T}_{p_r}\mathcal{V} \rangle,$$

has the expected dimension, i.e., $r(n+1)$, and if, in addition, the r-tangential contact locus

$$\mathcal{C}_r = \{p \in \mathcal{V} \mid \mathrm{T}_p\mathcal{V} \subset \mathrm{M}\} \subset \mathcal{V}$$

is zero-dimensional at every $p_1, p_2, \ldots, p_r$, then p is r-identifiable, r is its symmetric rank, and $p = \sum_{i=1}^r p_i$ is its unique decomposition.

Proof. The proof is obtained by repeating the proofs of [12, Lemma 4.3, Lemma 4.4, and Theorem 4.5], therein substituting the Segre variety with the Veronese variety $\mathcal{V}$. We present a simplification of the proof of [12, Theorem 4.5]. There is an open neighborhood of $p = p_1 + \cdots + p_r$ consisting of points for which smoothness, Terracini's lemma [33], and the absence of a contact locus will hold. The variety must thus be generally identifiable, hence the projection π onto the first factor of the usual abstract secant variety $A\sigma_r(\mathcal{V})$ is a birational morphism. After we find another decomposition $p = \sum_{i=1}^r b_i q_i$, we get that the fiber $\pi^{-1}(p)$ contains the two points $(p, (p_1, \ldots, p_r))$ and $(p, (q_1, \ldots, q_r))$. Terracini's lemma implies that the connected component of the fiber passing through $(p, (p_1, \ldots, p_r))$ cannot be positive dimensional, hence, it contains just this unique point. Since π is birational and p is a smooth point of $\sigma_r(\mathcal{V}) = \pi(A\sigma_r(\mathcal{V}))$, we have a contradiction with Zariski's Main Theorem. □

Remark 5.2. *We note a minor omission in the formulation of Theorem 4.5 in [12], where we forgot to include the condition that* M *should be of the expected dimension. It is clear from the proof of aforementioned theorem that this condition must hold, as can be understood from the invocation of [12, Lemma 4.3].*

Remark 5.3. *If one chooses r* random *points $p_i \in v_d(\mathbb{P}^n)$, then Lemma 5.1 may be invoked to prove generic r-identifiability. In this way, one can prove the cases $v_3(\mathbb{P}^2)$, $v_3(\mathbb{P}^3)$ and $v_3(\mathbb{P}^4)$, which were not covered by the proof in section 3. The case $v_3(\mathbb{P}^1)$ is trivial, because there is only one point, which is naturally identifiable.*

For practically verifying Lemma 5.1, we need a sufficiently explicit description of the r-contact locus. This is obtained as follows. Interpreting a point $p_i \in \mathcal{V}$ as a power of a linear form, say

$$p_i = (a_{0,i}x_0 + a_{1,i}x_1 + \cdots + a_{n,i}x_n)^d,$$

where $\{x_i\}_{i=0}^n$ is a basis of $\mathbb{P}^n$, it follows immediately that the tangent space is given by

$$\mathrm{T}_{p_i}\mathcal{V} = \langle x_0(a_{0,i}x_0+a_{1,i}x_1+\cdots+a_{n,i}x_n)^{d-1}, \ldots, x_n(a_{0,i}x_0+a_{1,i}x_1+\cdots+a_{n,i}x_n)^{d-1}\rangle.$$

If we choose the standard monomial basis $\{x_{i_1}x_{i_2}\cdots x_{i_d}\}_{0\le i_1\le i_2\le\cdots\le i_d\le n}$ for $v_d(\mathbb{P}^n)$, then this tangent space can be represented in a straightforward manner as a $\binom{n+d}{d}\times(n+1)$ matrix of constants, say T_i. The Cartesian equations of M may then be constructed by computing the kernel of the matrix $T = \begin{bmatrix} T_1 & T_2 & \cdots & T_r \end{bmatrix}^T$. The number of such equations should be precisely $\ell = \binom{n+d}{d} - r(n+1)$; otherwise, the first condition in Lemma 5.1 concerning the dimension of the tangent space would be violated. Let us denote the Cartesian equations as

$$q_l(\mathbf{x}) = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_d=i_{d-1}}^{n} k_{(i_1,i_2,\ldots,i_d),l}\cdot x_{i_1}x_{i_2}\cdots x_{i_d} = 0, \quad l = 1,2,\ldots,\ell, \tag{3}$$

where the vector $\mathbf{k}_l = [k_{(i_1,i_2,\ldots,i_d),l}]_{0\le i_1\le i_2\le\cdots\le i_d\le n}$ is the lth basis vector of the kernel of the matrix of constants T. For imposing that a point $\rho = (a_0x_0+\cdots+a_nx_n)^d$ is contained in M, it should obey the Cartesian equations, i.e., $q_l(a_0,a_1,\ldots,a_n) = 0$ for all $l = 1,2,\ldots,\ell$. That is, the equations (3) define the ideal-theoretic equations for $\mathcal{V}\cap\mathrm{M}$. It similarly follows that deriving the equations (3) with respect to $x_0,x_1,\ldots,x_n$ and substituting $x_0,x_1,\ldots,x_n$ for, respectively, $a_0,a_1,\ldots,a_n$ results in the ideal-theoretic equations of the intersection $\mathcal{C}_r = \mathrm{T}_p\mathcal{V}\cap\mathrm{M}$; naturally, the $a_0,a_1,\ldots,a_n$ should be treated as new variables. The number of equations thus constructed equals $\ell(n+1)$. To determine that $\mathcal{C}_r$ is zero-dimensional at each p_i, it suffices to verify that the codimension of the tangent space, i.e., the derivative of the equations of the ideal, is n at each of the p_i's. This tangent space can be represented by a matrix H of size $(n+1)\times\ell(n+1)$, which contains only constants when it is evaluated at one of the p_i's. The rank of H coincides with the dimension of the contact locus and can be computed using simple linear algebra. As was remarked in [12], H can be interpreted as a "stacked Hessian" matrix $H = \begin{bmatrix} H^1 & H^2 & \ldots & H^\ell \end{bmatrix}$, wherein H^k is the Hessian matrix of partial derivatives

$$H^k = [h^k_{i,j}]_{i,j=0}^n = \left[\frac{\partial^2}{\partial x_j\partial x_i}q_k(x_0,x_1,\ldots,x_n)\right]_{i,j=0}^n;$$

this is the reason why we call the above approach of verifying Lemma 5.1 the *Hessian criterion.*

A computer implementation of the Hessian criterion in Macaulay2 is included in the `specific-identifiability.m2` file that accompanies the arXiv version of this manuscript.

5.2. **The smoothness criterion.** The Hessian criterion in Lemma 5.1 may only be applied to smooth points of $\sigma_r(\mathcal{V})$. One approach for proving smoothness consists of verifying that the local equations of $\sigma_r(\mathcal{V})$ are of the expected degree. Such equations are known in the case when the number of terms r in the symmetric decomposition is sufficiently small. A standard nontrivial set of local equations is generated by the $(r+1)$-minors of the usual symmetric flattenings; see [24, Theorem 7.3.3.3] and [21, Theorems 4.5A and 4.10A]. For Veronese embeddings of odd degree, the Young flattenings from [25, Section 4] apply in a wider range than the standard symmetric flattenings; however, they are more involved to explain and implement. Our discussion will focus on the simple symmetric flattenings, which can still handle a respectable number of cases for Veronese embeddings of degree at least four.

The strategy that we present for proving that p corresponds to a smooth point is, essentially, based on [21, Theorems 4.10A and 4.5A] and [24, Theorem 7.3.3.3] and consists of obtaining local equations of the r-secant variety $\sigma_r(\mathcal{V})$ in p. Crucial to this approach are the *symmetric flattenings*, which, we recall, may be defined as follows. Let $p \in S^d\mathbb{C}^{n+1}$, then we can define the map

$$\begin{aligned} \phi_k^p : \qquad S^k(\mathbb{C}^{n+1})^* &\to S^{d-k}\mathbb{C}^{n+1} \\ x_{i_1}x_{i_2}\cdots x_{i_k} &\mapsto \frac{\partial^k}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}. \end{aligned}$$

We have the following.

Lemma 5.4 (Sufficient condition for smoothness)**.** *Let $\mathcal{V} = v_d(\mathbb{P}^n)$ be the Veronese variety, let $\delta = \lfloor \frac{d}{2} \rfloor$, and let $r < r_{\delta,n}$. Assume that we are given a point*

$$p = p_1 + p_2 + \cdots + p_d \in \sigma_r(\mathcal{V}).$$

Let N *be the following linear space:*

$$\mathrm{N} = \ker(\phi_\delta^p) \circ \mathrm{image}(\phi_\delta^p)^\perp \subset S^d\mathbb{C}^{n+1},$$

i.e., the symmetric product of the kernel and the complement of the image of ϕ_δ^p. If

$$\operatorname{rank} \phi_\delta^p = r, \quad \textit{and} \quad \dim \mathrm{N} = \binom{n+d}{d} - r(n+1),$$

then p is a smooth point of $\sigma_r(v_d(\mathbb{P}^n))$.

Proof. The subspace N is the normal space at p of the locus of $(r+1)$-minors of the catalecticant matrix ϕ_δ^p; see, for example, [24, Prop. 5.3.3.1]. If N has the expected dimension $\binom{n+d}{d} - r(n+1)$, then the locus of $(r+1)$-minors of the catalecticant matrix ϕ_δ^p is smooth at p and of the expected dimension $r(n+1)-1$. The r-secant variety $\sigma_r(v_d(\mathbb{P}^n))$ is contained in that locus, being of the expected dimension $r(n+1)-1$ by the Alexander-Hirschowitz theorem, so it too has to be smooth at p. □

A natural question concerning the foregoing lemma concerns the maximum value of r for which it can be applied. That is, if we pick a sufficiently general smooth point $p \in \sigma_r(\mathcal{V})$, what is the maximum value of r for which Lemma 5.4 can prove that p is, indeed, smooth? A lower bound follows immediately from the work of Iarrobino and Kanev [21, Theorem 4.10A]:

Proposition 5.5. *Let $\mathcal{V} = v_d(\mathbb{P}^n)$ be the Veronese variety, let $\delta = \lfloor \frac{d}{2} \rfloor$, and let*

$$r \le \binom{n+\delta-1}{\delta-1}.$$

Then, Lemma 5.4 can be applied to all points of an irreducible component of $\sigma_r(\mathcal{V})$ minus some Zariski-closed set.

Table 1. The maximum value r for which Lemma 5.4 applies to all points in an irreducible component of $\sigma_r(\mathcal{V})$ minus some Zariski-closed set is displayed as the middle set of columns (♣) for each degree $d = 4, 5, 6, 7, 8$ of the Veronese embedding $\mathcal{V} = v_d(\mathbb{P}^n)$. The left set of columns (♠) shows the lower bound from Proposition 5.5, for every d. The right set of columns (♢) shows the maximum value of r for which Kruskal's criterion is applicable, for every d. A $\star$ indicates that the value could not be computed within a reasonable time. Values displayed in boldface indicate the widest range for r for a particular combination of the degree d and size n.

n	d														
	4			5			6			7			8		
	♠	♣	♢	♠	♣	♢	♠	♣	♢	♠	♣	♢	♠	♣	♢
1	2	**2**	**2**	2	2	**3**	3	**3**	**3**	3	3	**4**	4	**4**	**4**
2	3	**4**	**4**	3	4	**5**	6	**6**	**6**	6	**7**	**7**	10	**10**	8
3	4	5	**6**	4	6	**8**	10	**12**	9	10	**15**	11	20	**23**	12
4	5	7	**8**	5	9	**10**	15	**21**	12	15	**27**	14	35	**47**	16
5	6	**10**	**10**	6	**14**	13	21	**33**	15	**21**	$\star$	18	56	**87**	20
6	7	**12**	**12**	7	**19**	15	28	**50**	18	**28**	$\star$	21	**84**	$\star$	24
7	8	**16**	14	8	**25**	18	36	**72**	21	**36**	$\star$	25	**120**	$\star$	28
8	9	**20**	16	9	**33**	20	**45**	$\star$	24	**45**	$\star$	28	**165**	$\star$	32
9	10	**25**	18	10	**41**	23	**55**	$\star$	27	**55**	$\star$	32	**220**	$\star$	36
10	11	**29**	20	11	$\star$	**25**	**66**	$\star$	30	**66**	$\star$	35	**286**	$\star$	40

Proof. The claim follows from [21] and the fact that the conditions on the dimension of M and the rank of ϕ_δ^p are valid on dense open sets in the Zariski topology. □

In Table 1, some values of the lower bound in Proposition 5.5 are tabulated along with a sharp maximum value of r for which the equations generated by Lemma 5.4 generate an irreducible component of $\sigma_r(\mathcal{V})$. The values of this alleged sharp upper bound were computed by taking random points on this variety and verifying Lemma 5.4; as such, they are only true with high probability. It is clear from the table that the lower bound in Proposition 5.5 is not sharp. For $d = 3$, the symmetric flattenings are only sufficient for $r = 1$ and 2. One should employ Young flattenings [25] for extending the range instead.

An implementation in Macaulay2 of the above sufficient condition for smoothness based on a symmetric flattening is included in the `specific-identifiability.m2` file that is provided with the arXiv version of this paper.

5.3. **An elementary algorithm.** For the sake of completeness, we present an algorithm that attempts to prove the identifiability of a given Waring decomposition $p = p_1 + p_2 + \cdots + p_r \in \sigma_r(\mathcal{V})$, where $\mathcal{V} = v_d(\mathbb{P}^n)$, by checking the sufficient conditions in Lemma 5.1 and Lemma 5.4. It operates as follows.

S1. Construct a matrix representation of the span $\mathrm{M} = \langle \mathrm{T}_{p_1}\mathcal{V}, \mathrm{T}_{p_2}\mathcal{V}, \ldots, \mathrm{T}_{p_r}\mathcal{V}\rangle$. If $\operatorname{rank} \mathrm{M} < r(n+1)$, then the algorithm terminates, claiming that it cannot prove the identifiability of p.

S2. Construct the symmetric flattening ϕ_δ^p for $\delta = \lfloor \frac{d}{2} \rfloor$. If $\operatorname{rank} \phi_\delta^p < r$, then the algorithm terminates, claiming that it cannot prove identifiability of p.

S3. Compute the matrix $\mathrm{N} = \ker(\phi_\delta^p) \circ \operatorname{image}(\phi_\delta^p)^\perp$. If $\operatorname{rank} \mathrm{N} > \binom{n+d}{d} - r(n+1)$, then the algorithm terminates, claiming that it cannot prove identifiability of p.

S4. Compute a basis of the kernel of M. Denote the number of equations by ℓ.

S5. For every point p_i, $i = 1, 2, \ldots, r$, perform the following:

S5a. Construct the Hessians H^k for $k = 1, 2, \ldots, \ell$ evaluated at the point p_i and stack them into the matrix H.

S5b. If $\operatorname{rank} H < n$, then the contact locus is of positive dimension at p_i. The algorithm halts, claiming that it cannot prove identifiability of p.

S6. The algorithm proclaims that the Waring decomposition $p = p_1 + \cdots + p_r$ is unique and that p is a smooth point of $\sigma_r(\mathcal{V})$.

It is instructive to investigate the largest value of r for which the above algorithm may be expected to prove identifiability of a sufficiently general point $p \in \sigma_r(\mathcal{V})$ on a generically identifiable Veronese variety $\mathcal{V}$. As the Hessian criterion applies for all tensors of subgeneric rank, it follows that the range of applicability is bounded only by the smoothness test. That is, the highlighted columns in Table 1 contain the relevant values. This range improves on the state-of-the-art criteria for identifiability of Waring decompositions. Few procedures are known for assessing that a specific point p is identifiable. To the best of our knowledge, the best bound that is applicable for Veronese embeddings of any degree $d \geq 3$ is the so-called Kruskal condition [22, 23]. Let $p_i = \mathbf{a}_i^{\otimes d} \in \mathcal{V}$ be some specific points. In the symmetric setting, Kruskal's condition states that if

$$r \leq \frac{1}{2}(dk - d + 1),$$

where k is the largest number such that every subset of $\{\mathbf{a}_i\}_i$ consisting of k vectors is linearly independent. For points in general configuration, the maximum value for k is thus $n + 1$. A comparison between the proposed criterion for specific identifiability and Kruskal's criterion is also featured in Table 1. As can be seen, the newly proposed criterion almost invariably improves on the literature.

5.4. **The algorithm at work for a specific example.** Consider the following 17-term Waring decomposition in $S^4\mathbb{C}^9$:

$$\begin{aligned} p = \sum_{i=0}^{8} x_i^4 + {} \\ & (x_0 + 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 + 7x_6 + 8x_7 + 9x_8)^4 + \\ & (9x_0 + 8x_1 + 7x_2 + 6x_3 + 5x_4 + 4x_5 + 3x_6 + 2x_7 + x_8)^4 + \\ & (x_0 + 2x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 + 3x_6 + 2x_7 + x_8)^4 + \\ & (5x_0 + 4x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 + 3x_6 + 4x_7 + 5x_8)^4 + \\ & (x_0 + 3x_1 + 3x_2 + 9x_3 + 9x_4 + 9x_5 + 3x_6 + 3x_7 + x_8)^4 + \\ & (4x_0 + 2x_1 + x_2 + 2x_3 + 4x_4 + 2x_5 + x_6 + 2x_7 + 4x_8)^4 + \\ & (x_0 + 2x_1 + 4x_2 + 8x_3 + 16x_4 + 8x_5 + 4x_6 + 2x_7 + x_8)^4 + \\ & (x_0 + x_1 + 2x_2 + 2x_3 + 3x_4 + 3x_5 + 4x_6 + 4x_7 + 5x_8)^4. \end{aligned}$$

The identifiability of this example cannot be handled with Kruskal's condition, because it only applies for Waring decompositions with up to $\frac{1}{2}(4n-3) = 16.5$ terms. The sufficient condition presented in this paper, on the other hand, is applicable up to 20 terms. Therefore, we run the algorithm presented above. This example may be verified with the `specific-identifiability.m2` script that is provided with the arXiv version of this paper. In the first step, the 495×153 matrix representing the span of M is constructed. Its rank is 153, as expected. The algorithm proceeds with the construction of the 45×45 matrix ϕ_2^p. Its rank is 17, so we can proceed with the third step. Both the kernel of ϕ_2^p and the complement of the image of ϕ_2^p consist of 28 equations. The linear space N is represented by the column span of a 495×784 matrix, thus containing a substantial number of redundancies. The rank of this matrix is 342, which, indeed, corresponds with $495 - 17 \cdot 9$. Then, a basis of the kernel of M is computed, containing 342 equations. Note that the dimension of N and the codimension of M should always be equal in the approach for certifying identifiability that was proposed in this section. Then, for each of the points, the 9×9 Hessian matrices are computed and stacked into a 9×3078 matrix H. The rank of H equals 8 for each of the points. In addition, the kernel of H^T consists of a single vector that must be a multiple of the coefficient vector of p_i; for example, the vector in the kernel of the stacked Hessian H corresponding to the last term in the Waring decomposition of p is a multiple of its coefficient vector $[\,1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 5\,]$. For each of the points, this is indeed the case. Finally, the algorithm positively concludes that p admits a unique Waring decomposition, i.e., p is identifiable.

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Luca Chiantini: Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Italy
E-mail address: `luca.chiantini@unisi.it`

Giorgio Ottaviani: Dipartimento di Matematica e Informatica "Ulisse Dini", Università di Firenze, Italy
E-mail address: `ottavian@math.unifi.it`

Nick Vannieuwenhoven: Department of Computer Science, KU Leuven, Belgium
E-mail address: `nick.vannieuwenhoven@cs.kuleuven.be`