Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Algebra 286 (2005) 421–455

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

The geometric interpretation of Fröberg–Iarrobino conjectures on infinitesimal neighbourhoods of points in projective space

Karen A. Chandler

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA

Received 20 September 2003

Communicated by Craig Huneke

Abstract

The study of infinitesimal deformations of a variety embedded in projective space requires, at ground level, that of deformation of a collection of points, as specified by a zero-dimensional scheme. Further, basic problems in infinitesimal interpolation correspond directly to the analysis of such schemes. An optimal Hilbert function of a collection of infinitesimal neighbourhoods of points in projective space is suggested by algebraic conjectures of R. Fröberg and A. Iarrobino. We discuss these conjectures from a geometric point of view. They give, for each such collection, a function (based on dimension, number of points, and order of each neighbourhood) which should serve as an upper bound to its Hilbert function (Weak Conjecture). The Strong Conjecture predicts when the upper bound is sharp, in the case of equal order throughout. In general we refer to the equality of the Hilbert function of a collection of infinitesimal neighbourhoods with that of the corresponding conjectural function as the Strong Hypothesis. We interpret these conjectures and hypotheses as accounting for the infinitesimal neighbourhoods of projective subspaces naturally occurring in the base locus of a linear system with prescribed singularities at fixed points. We develop techniques and insight toward the conjectures' verification and refinement. The main result gives an upper bound on the Hilbert function of a collection of infinitesimal neighbourhoods in $\mathbb{P}^n$ based on Hilbert functions of certain such subschemes of $\mathbb{P}^{n-1}$. Further, equality occurs exactly when the scheme has only the expected linear obstructions to the linear system at hand. It follows that an infinitesimal neighbourhood scheme obeys the Weak Conjecture provided that the schemes identified in codimension one

E-mail address: karen.a.chandler.6@nd.edu.

0021-8693/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2005.01.010

satisfy the Strong Hypothesis. This observation is then applied to show that the Weak Conjecture does hold valid in $\mathbb{P}^n$ for $n \leqslant 3$. The main feature here is that the result is obtained although the Strong Hypothesis is not known to hold generally in $\mathbb{P}^2$ and, further, $\mathbb{P}^2$ presents special exceptional cases. Consequences of the main result in higher dimension are then examined. We note, then, that the full weight of the Strong Conjecture (and validity of the Strong Hypothesis) are not necessary toward using the main theorem in the next dimension. We end with the observation of how our viewpoint on the Strong Hypothesis pertains to extra algebraic information: namely, on the structure of the minimal free resolution of an ideal generated by linear forms.
© 2005 Elsevier Inc. All rights reserved.

1. Introduction.

Let $\mathcal{K}$ be an infinite field, and $\mathbb{P}^n = \mathbb{P}^n_{\mathcal{K}}$.

The Hilbert function of a scheme $\mathcal{Z}$ embedded in projective space evaluates in each degree the codimension of the graded piece of the ideal of $\mathcal{Z}$ with respect to the relevant homogeneous coordinate ring. One seeks in general to establish how the information provided by the Hilbert function describes the geometry of the scheme and its embedding.

An *infinitesimal neighbourhood* of a variety X with respect to an embedding in $\mathbb{P}^n$ is a scheme defined by a power $\mathcal{I}_X^k$ of its ideal sheaf. We shall also refer to such a scheme as a (full) multiple subvariety of $\mathbb{P}^n$ of (overall) multiplicity k, and as a "fat subvariety" of $\mathbb{P}^n$. Each infinitesimal neighbourhood of a variety then refers to the extent of singularity of a hypersurface through the variety itself. We study here the Hilbert function of a collection of infinitesimal neighbourhoods of points in projective space.

An immediate motivation for this investigation is given by *infinitesimal interpolation*. The Hilbert function of a collection of infinitesimal neighbourhoods of points measures the number of linear conditions imposed on the linear system of hypersurfaces of each degree given by the requirement to vanish to specified order at each point. These data tell (in appropriate characteristic, say) the extent to which it is possible to interpolate the values of a polynomial of given degree, together with its partial derivatives up to specified orders, to a collection of points in affine space. (See [Ci,GS], for example. For "inappropriate characteristic" the same principle applies, subject to a modification of the notion of derivative; see [IK].)

Moreover, the study of such Hilbert functions is a basic starting point in that of an *infinitesimal deformation* of a (higher-dimensional!) variety X embedded in a projective space (see, e.g. [C1]). For example, to estimate (or evaluate) cohomologies of twists of the $(k-1)$th symmetric power of the conormal bundle of X in the projective space one may examine those of the scheme defined by $\mathcal{I}_X^k$. The standard method of hyperplane slicing gives cohomological data on this scheme from those of a lower-dimensional one. However, in low degree of twisting (the most interesting!) the standard approach is far too crude. We shall focus here on this phenomenon and work toward refining the technique (see also [C1, C2,C3,C4,C5,C8]), and thereby advance the theory of infinitesimal deformation.

Special attention is paid here to the situation of a generic collection of multiple points (i.e., the support is a generic subset of projective space). Some motivation for this restriction is evident: we do have the conjectures defined below as guidance. Surely the problem of finding the Hilbert function is made easier by having the freedom to choose generic points, and any upper bound obtained on the function for generic points gives automatically a bound for an arbitrary collection of points. But we also proceed here with an eye toward developing tools applicable to the Hilbert function of any collection of multiple points (or multiple varieties); such as identifying which such schemes have the maximal possible Hilbert function. For example, we find in [C8] that a variation of the technique introduced here applies well to a collection of multiple points lying on a rational normal curve (which ought, according to conjectures of Catalisano and Gimigliano [CEG], to give the "worst" Hilbert function amongst sets of points in linearly general position).

A zero-dimensional subscheme $\mathcal{Z} \subset \mathbb{P}^n$ is said to have the *maximal rank property* if its Hilbert function is as simple as possible: for each degree m, either $\mathcal{Z}$ does not lie on an m-ic hypersurface or $\mathcal{Z}$ imposes $\deg \mathcal{Z}$ (i.e. *independent*) conditions on the linear system of m-ics. One is led to consider, then, which $\mathcal{Z}$ *do not* enjoy this property? One expects, at least conjecturally, that for a generic such scheme $\mathcal{Z}$ the failure of maximal rank in a given degree should occur when the base locus of the linear system of m-ics through $\mathcal{Z}$ is forced to contain a positive dimensional scheme whose intersection with $\mathcal{Z}$ itself cannot impose the "expected number" of conditions on the system.

For a generic collection $\mathcal{Z}$ of multiple points in projective space, the inductive procedure (méthode d'Horace différentielle) of J. Alexander and A. Hirschowitz allows one to deduce maximal rank in a given degree from maximal rank conditions in lower degree and in lower dimension. This idea is used in [AH4] to obtain asymptotic results on maximal rank.

But in low degree m (compared to order of vanishing) such a scheme $\mathcal{Z}$ cannot impose independent conditions on m-ics, due to visible linear obstructions. Specifically, suppose that $\mathcal{Z}$ contains two points, of multiplicities j, k, and take $m \leqslant j + k - 2$. The line L between the two points meets $\mathcal{Z}$ in a subscheme of degree $j + k > m + 1$ which then cannot impose independent conditions on m-ics (i.e., L itself imposes only $m + 1$ conditions) and hence neither does $\mathcal{Z}$.

Therefore, a key issue on obtaining information on the Hilbert function of such a scheme, such as finding explicit (better yet, sharp) conditions for maximal rank, is to study cases in which maximal rank is obstructed by linear subspaces spanned by subsets of the set of points in the scheme.

Conjectures of R. Fröberg and A. Iarrobino give a proposed value (Strong Conjecture) or upper bound (Weak Conjecture) for such a Hilbert function. These conjectures arise indirectly from an algebraic conjecture of Fröberg [F]. He studies an ideal generated by a generic collection of forms, and asserts that its behaviour may be quantified (or at least estimated) by a natural generalisation of the formula for complete intersection ideals. Iarrobino further asserts [I] that, up to an identifiable region of cases, the conjecture of Fröberg should apply to an ideal generated by a generic collection of powers of linear forms of equal degree. (Of course, one must certainly exclude situations such as pth powers in characteristic p, by virtue of the "Freshman's Dream Theorem"!) An application of Macaulay duality, given by Emsalem and Iarrobino, to the case of generic powers of linear forms yields the conjectures on multiple points [EI].

We present here a direct geometric interpretation of these conjectures. Namely, the conjectural Hilbert function of multiple points reflects circumstances under which (multiple) planes spanned by subsets must appear in the base locus at issue, according to

Lagrange–Hermite, say. In particular, we argue that the Strong Conjecture for multiple points corresponds to situations in which the *only* obstructions to the scheme's imposing independent conditions are the "obvious linear ones", and that the Weak Conjecture amounts to *counting* such linear obstructions.

Let us recall the following definitions.

Definition 1. For a subscheme $\mathcal{Z} \subset \mathbb{P}^n$ with ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ the *Hilbert function* of $\mathcal{Z}$ (as a function of m) is given by

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) := \dim H^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)\big) - \dim H^0\big(\mathbb{P}^n, \mathcal{I}_{\mathcal{Z}}(m)\big).$$

Definition 2. Take a variety $X \subset \mathbb{P}^n$ and $k \in \mathbb{N}$. The $(k-1)$*th infinitesimal neighbourhood* of X (with respect to $\mathbb{P}^n$) is the scheme given by $\mathcal{I}^k$, where $\mathcal{I}$ is the ideal sheaf of X. We shall denote this scheme by $X^k \subset \mathbb{P}^n$.

So, for example, $X^0 = \emptyset$, and $X^1 = X$.

Note that for $p \in \mathbb{P}^n$ the degree of $\{p\}^k$ is $\binom{n+k-1}{n}$. Hence, for an r-dimensional variety $X \subset \mathbb{P}^n$, $\deg X^k = \binom{n+k-r-1}{n-r} \deg X$.

For brevity, when the ambient projective space of embedding is clear, we shall refer to $X^k \subset \mathbb{P}^n$ as X^k, a k-uple subscheme (or a "fat variety" of multiplicity k).

Definition 3. Given $A = (k_1, \ldots, k_d) \in \mathbb{N}^d$ define an *A-subscheme of* $\mathbb{P}^n$ as a union of $\{p_1\}^{k_1} \cup \cdots \cup \{p_d\}^{k_d}$ where $\{p_1, \ldots, p_d\}$ is a set of d points in $\mathbb{P}^n$. We shall say that an A-scheme is *homogeneous* (respectively, *quasihomogeneous*) if $k_1 = \cdots = k_d$ (respectively, after perhaps reordering, $k_2 = \cdots = k_d$). (For emphasis, we may refer to a scheme as having *mixed multiplicities* if it is not necessarily homogeneous.)

The Fröberg–Iarrobino Conjectures (see Section 4) refer to a function $G(d, A, n+1)_m$ and its correspondence with the Hilbert function of an A-subscheme of $\mathbb{P}^n$ (supported on d points). The Weak Conjecture (Conjecture 4.7) asserts that for each d-uple $A \in \mathbb{N}^d$ and each A-subscheme $\mathcal{Z} \subset \mathbb{P}^n$ we have

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant G(d, A, n+1)_m$$

for each degree m; while the Strong Conjecture (Conjecture 4.8) gives numerical conditions under which equality is predicted to hold between the two functions in the case of homogeneous schemes with generic support. Here we provide techniques for analysing the Hilbert function of a collection of multiple points and compare with the properties of the proposed function G.

The Strong Fröberg–Iarrobino Conjecture deals only with homogeneous schemes. We aim, further, to find the Hilbert function of a scheme of mixed multiplicities. Moreover, we shall exhibit (infinitely many) counterexamples to the Strong Conjecture [C6]; that is, classes of generic homogeneous subschemes of $\mathbb{P}^n$ whose Hilbert functions do not agree with the conjectured values. One seeks to: refine the Strong Fröberg–Iarrobino Conjecture on homogeneous schemes, and then to extend to those of mixed multiplicities. According to the evidence presented here, this requires the identification of nonlinear positive dimensional varieties lying in the base locus of a linear system of hypersurfaces through a multiple point scheme. In general, we refer to the *Strong Fröberg–Iarrobino Hypothesis* (on a given case) as the supposition that a generic A-subscheme of $\mathbb{P}^n$ has Hilbert function that agrees with the corresponding Fröberg–Iarrobino function (in a given degree). We shall also say that the Strong Hypothesis *applies* to a given A-scheme provided that its Hilbert function is equal to the conjectured value.

We obtain the following result (see Section 6).

Theorem 1.1. *Let $n, m, d \in \mathbb{N}$ and $A = (k_1, \ldots, k_d) \in \mathbb{N}^d$. Let us take $C_{ji} = ((k_1 + i - m)^+, \ldots, (k_j + i - m)^+)$ for each $j = 1, \ldots, d-1$, and $i = 0, \ldots, k_j - 1$. (Recall the notation: $a^+ = \max(a, 0)$.)*

Suppose that the Strong Fröberg–Iarrobino Hypothesis in $\mathbb{P}^{n-1}$ is verified by each generic C_{ji}-subscheme of $\mathbb{P}^{n-1}$ in degree i, for $j = 1, \ldots, d-1$ and $i = 0, \ldots, k_j - 1$.

Then:

(a) *Each A-subscheme of $\mathbb{P}^n$ satisfies the Weak Conjecture in degree m.*
(b) *A generic A-subscheme of $\mathbb{P}^n$ satisfies the Strong Hypothesis in degree m if and only if it displays only the expected linear obstructions in degree m (see Section 5, Definition 9).*
(c) *If $\mathcal{Z} \subset \mathbb{P}^n$ is any A-subscheme of $\mathbb{P}^n$ then $\mathcal{Z}$ verifies the Strong Hypothesis in degree m provided that it admits only the expected linear obstructions in degree m.*

We shall observe in Proposition 6.5 that we may "homogenise" this result to deal with the Strong Conjecture itself, obtaining more explicit conclusions.

The notion of expected linear obstructions, given in Definition 9 (Section 5), is based simply on the prediction of Bézout on how a line (and whence multiple lines as well as higher-dimensional linear subspaces) must appear in the base locus of a linear system if its intersection with that base locus has sufficient degree.

Theorem 1.1 implies that one may determine whether a given A-subscheme of $\mathbb{P}^n$ satisfies the Weak Conjecture by finding analogous subschemes of $\mathbb{P}^{n-1}$ for which the Strong Hypothesis applies. In particular, to verify the Weak Conjecture in $\mathbb{P}^n$, it suffices that the Strong Hypothesis applies to sufficiently many (and identifiable) subschemes of $\mathbb{P}^{n-1}$.

For example, in $\mathbb{P}^2$ the Strong Conjecture displays "extra exceptions" to the expected maximal rank of a (homogeneous) multiple subscheme. These are extended to conjectures on subschemes of $\mathbb{P}^2$ of mixed multiplicities. Progress on these conjectures on $\mathbb{P}^2$ has been made recently (see Section 3), but the main problem remains open, even for homogeneous schemes. Nevertheless, we employ Theorem 1.1 to obtain the following result (see Section 7).

Theorem 1.2. *The Weak Fröberg–Iarrobino conjecture holds valid in $\mathbb{P}^n$ for $n \leqslant 3$.*

Hence we see that the "full strength" of the Strong Hypothesis is not necessary toward verifying the Weak Conjecture in the next dimension from Theorem 1.1.

The main tool in finding an upper bound for the Hilbert function of a given collection of infinitesimal neighbourhoods (homogeneous or otherwise) is presented in Lemma 6.3. Here an inductive strategy on comparing the Hilbert function of a scheme, say $\mathcal{Z} \cup \{p\}^{k+1}$, with that of $\mathcal{Z} \cup \{p\}^k$ from explicit identification of the expected linear obstruction schemes is presented. Particularly, after intersecting such a scheme with a hyperplane H we produce a collection $\mathcal{W}$ of multiple points of H so that

$$h_{\mathbb{P}^n}\left(\mathcal{Z} \cup \{p\}^{k+1}, m\right) - h_{\mathbb{P}^n}\left(\mathcal{Z} \cup \{p\}^{k}, m\right) \leqslant \binom{n+k-1}{n-1} - h_H(\mathcal{W}, k).$$

Further, equality occurs exactly when the scheme $\mathcal{Z} \cup \{p\}^{k+1}$ has only the expected linear obstructions given by $\mathcal{Z} \cup \{p\}^k$ in the relevant degree m, as in Definition 9.

From Lemma 6.3 we obtain the proof of the main theorem (Section 6) on describing the Hilbert function of a collection of multiple points. Namely, an upper bound on the Hilbert function of a collection of multiple points of $\mathbb{P}^n$ is obtained from evaluation of the Hilbert function of collections of fat points in $\mathbb{P}^{n-1}$, as follows.

Theorem 1.3 (Main Theorem). *Let $n, m, d \in \mathbb{N}$ and $A = (k_1, \ldots, k_d) \in \mathbb{N}^d$. Take $C_{ji} = (k_1, \ldots, k_j) + \overline{i-m}$ for each $j = 1, \ldots, d-1$, and $i = 0, \ldots, k_j - 1$. For each A-subscheme $\mathcal{Z} \subset \mathbb{P}^n$ there are naturally induced C_{ji}-subschemes of $\mathbb{P}^{n-1}$, W_{ji}, so that*

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant \deg \mathcal{Z} - \sum_{j=1}^{d} \sum_{i=0}^{k_j-1} h_{\mathbb{P}^n}(W_{j,i}, i).$$

Equality holds if and only if only the expected linear obstructions to Z occur in degree m.

This result has immediate implications toward comparing the Hilbert function of a fat point scheme with the function $G(d, A, n)_m$ proposed by Iarrobino, as seen by the basic properties of the function G. From this we obtain the conclusion of Theorem 1.1.

In Section 8 we examine general applications of Theorem 1.3 toward $\mathbb{P}^n$. We restrict attention mainly to cases in which we are "one step away" from the expectation of maximal rank: only (multiple) lines are predicted to appear as the positive-dimensional schemes in the base locus of the given linear systems, in the formulation of the conjectural function. In Corollary 8.8 we obtain, for example, a geometric analogue of an algebraic result of Iarrobino (Section 4).

Finally, we remark on the algebraic conjectures of Fröberg, equipped with the extra information from the geometric viewpoint. We describe how Conjecture 5.4 implies the "Koszulness" of the minimal resolution of an ideal generated by powers of linear forms, and whence, general forms.

In sum, the geometric evidence presented here gives structure to the Fröberg–Iarrobino conjectures. Indeed, the Weak Conjecture appears tractable technically. Furthermore, one need not regard the Weak Conjecture as the "second best" result to obtain along these lines, but as a first step toward verifying the Strong Conjecture (and evaluating exceptional cases). Namely, from the Weak Conjecture it would follow that equality of the Hilbert function of an A-subscheme of $\mathbb{P}^n$ and $G(d, A, n+1)$ is an open condition (on $(\mathbb{P}^n)^d$), and hence may be verified by *producing* a scheme exhibiting such equality. Particularly, a usual strategy for verifying upper bounds on the Hilbert function of a general scheme is to construct a scheme that satisfies these conditions. In the situation of maximal rank, this always suffices; but it is necessary here to have a lower bound. Moreover, as we see in Theorem 1.1, the Weak Conjecture allows us to characterise schemes that do obey the Strong Hypothesis (including the Strong Conjecture).

The structure of the paper is as follows. We fix notation in Section 2. Next we consider the context of the problem at hand. We begin in Section 3 by describing basic results and techniques that may be used to predict maximal rank. Then in Section 4 we present the conjectures of Fröberg and Iarrobino, which imply in particular how maximal rank cannot always be achieved. From basic observations on intersection multiplicity we obtain in Section 5 the geometric interpretation of these conjectures along with refinements of the conjectures. In Section 6 we prove the main theorem and present its connection to the conjectures. Then, in Section 7, we validate the Weak Conjecture in $\mathbb{P}^3$. Further consequences in terms of verifying cases of the Weak Conjecture in $\mathbb{P}^n$ are given in Section 8. We end in Section 9 with discussion and speculation on the algebraic versions of the strong conjecture.

2. Basic bookkeeping

Given a projective subscheme $\mathcal{Z} \subset \mathbb{P}^n$ we write $\mathcal{I}_{\mathcal{Z},\mathbb{P}^n}$ (or $\mathcal{I}_{\mathcal{Z}}$) for its ideal sheaf.

Given a reduced subvariety $X \subset \mathbb{P}^n$ and $a \in \mathbb{N}$, define X^a as the subscheme $\mathcal{Z} \subset \mathbb{P}^n$ defined by $\mathcal{I}_X^a$. Let us extend this to $a \in \mathbb{Z}$ so that in case $a \leqslant 0$ we have $X^a = \emptyset$.

Since we shall consider collections of fat points of various multiplicities, let us keep some of the bookkeeping straight as follows.

Given $A = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ define an *A-subscheme of* $\mathbb{P}^n$ as a union $\mathcal{Z}$ of $\{p_1\}^{a_1} \cup \cdots \cup \{p_d\}^{a_d}$ where $\{p_1, \ldots, p_d\}$ is a set of d points in $\mathbb{P}^n$. In case $A = (a, \ldots, a)$ (and the number of points d is made clear) we shall write $A = \bar{a}$ and refer to an A-scheme as a *homogeneous* subscheme. We say that $\mathcal{Z}$ is a *generic A-subscheme* (or *general*) if the set $\{p_1, \ldots, p_d\}$ is generic.

As in [I] we define the following.

Definition 4. Denote by $HPTS(d, (k_1, \ldots, k_d), n+1)_m$ the value of Hilbert function in degree m of the generic union of d points in $\mathbb{P}^n$ of multiplicities $k_1, \ldots, k_d$.

Let us extend this definition in the obvious manner to the situation of perhaps having negative entries in the uple A:

$$HPTS(d, A, n+1)_m := HPTS\big(d, \big(\max(a_1, 0), \ldots, \max(a_d, 0)\big), n+1\big).$$

For such A we write $|A| = \sum_{i=1}^d \max(a_i, 0)$ and $\ell(A) = \#\{1 \leqslant i \leqslant d\colon a_i > 0\}$.

Given A, B both uples, we say that A and B are equivalent, or that *A may be written as B* if an A-scheme is a B-scheme. This says that $\ell(A) = \ell(B)$ and if $a_1, \ldots, a_d$

and $b_1, \dots, b_d$ are the positive entries of A and B, respectively ($d = \ell(A)$), then there exists a permutation σ on d letters so that $a_i = b_{\sigma(i)}$ for $i = 1, \dots, d$.

Now take $A = (a_1, \dots, a_d) \in \mathbb{N}^d$ so that $a_i > 0$ for $i = 1, \dots, d$. Given $B \in \mathbb{N}^r$ we shall say that $B \subseteq A$ if $B = (a_{i_1}, \dots, a_{i_r})$ with $1 \leqslant i_1 < \cdots < i_r \leqslant d$ (so B does respect ordering). If $B \in \mathbb{Z}^r$ we say that $B \leqslant A$ if B may be written as $(b_1, \dots, b_d) \in \mathbb{N}^d$ with each $b_i \leqslant a_i$, and that $B < A$ if $B \leqslant A$ but B may not be written as A.

3. General context

We pay particular attention throughout this paper to the Hilbert function of a generic collection of multiple points in projective space as a start toward the study of arbitrary collections of fat points. In each degree m the Hilbert function $HPTS(d, A, n+1)_m$ of a (general) A-scheme is bounded above by the degree of the scheme, with equality for m sufficiently large. Let us describe here some of the circumstances under which the two quantities are known (or conjectured) to agree. We refer the reader as well to the very readable and detailed accounts in the surveys of Ciliberto [Ci] and Harbourne [Ha4], particularly for thorough descriptions of work on $\mathbb{P}^2$.

Let $A = (k_1, \dots, k_d) \in \mathbb{N}^d$ and consider the Hilbert function of an A-subscheme of $\mathbb{P}^n$. For convenience, let us assume that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d$, for now.

When $d = 1$ we have

$$HPTS(1, \bar{k}, n+1)_m = \min\left(\binom{n+m}{m}, \binom{n+k-1}{k-1}\right);$$

that is, the scheme has the maximal rank property.

Now consider $d \geqslant 2$. For each $m \leqslant k_1 + k_2 - 2$ we have

$$HPTS\big(2, (k_1, k_2), n+1\big)_m < \binom{n+k_1-1}{2} + \binom{n+k_2-1}{2}.$$

Indeed, if $m \leqslant k_1 + k_2 - 2$, look at a (k_1, k_2)-scheme $\mathcal{Z}$ and take the line L spanned by the two reduced points on $\mathcal{Z}$. Then $\deg L \cap \mathcal{Z} = k_1 + k_2$. However, L imposes only $m+1$ conditions on the linear system of m-ics. Hence $\mathcal{Z} \cap L$ cannot impose independent conditions on m-ics, so neither does $\mathcal{Z}$. We regard this as an *expected linear obstruction*. (We shall make this notion precise in Definition 9.)

Likewise, extending to $A = (k_1, \dots, k_d)$ we have that an A-scheme cannot have maximal rank unless $HPTS(d, A, n+1)_m = \binom{n+m}{n}$ for $m = k_1 + k_2 - 2$ (that is, the degree of an A-scheme is large enough).

In the case of degree $m \geqslant k_1 + k_2 - 1$, the méthode d'Horace of [H1] does (essentially) apply toward verifying inductively a given case of maximal rank from ones occurring in lower degree and in lower dimension. This led to the following asymptotic result of J. Alexander and A. Hirschowitz.

Theorem 3.1 [AH4]. *Given $n, k \in \mathbb{N}$, there is a quantity $d(n,k)$ so that for all $d \geqslant d(n,k)$,*

$$HPTS(d, \bar{k}, n+1)_m = \min\left(d\binom{n+k-1}{n}, \binom{n+m}{m}\right).$$

One would like to sharpen this to an actual upper bound for $d(n,k)$; indeed, one that should be independent of the multiplicity k. Equivalently, the theorem predicts that there is a value $c(n,k)$ so that whenever $m \geqslant c(n,k)$ we have equality between *HPTS* and the desired quantity. Again one should like a bound on such a value, independent of n. This asks for results in cases $m \leqslant 2k-2$ (in which maximal rank cannot be achieved) to obtain a starting point in the induction process. Unfortunately, the méthode différentielle does not apply well to the low degree cases ($m \leqslant 2k-2$).

On the bright side (as used, e.g., in [A,C3,C7]) to proceed by induction on degree one does not require that *every* collection of multiple points has maximal rank. For example, to show that in degree $m \geqslant 3k$ a generic collection of kth order points exhibits maximal rank in degree m, it suffices to concoct a fat point scheme of large enough degree imposing independent conditions in degree $m-k+1$. But the degree required is strictly (and significantly) less than the boundary value of $\binom{n+m-k+1}{n}$. With any luck, then, down-to-earth methods in low degrees (e.g. [C2]) should yield k-schemes of maximal rank in degree $2k-1$, say, and of large enough degree to proceed inductively.

We assert the following statement.

Conjecture 3.2. *Let $\mathcal{Z} \subset \mathbb{P}^n$ be a generic collection of fat points of multiplicities $(k_1, \ldots, k_d)$, with $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$. Then for each m with $m \geqslant k_{d-2} + k_{d-1} + k_d$ we have*

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) = \min\left(\deg \mathcal{Z}, \binom{n+m}{m}\right).$$

Rewriting in terms of the number of points, it is easy to see that this implies:

Conjecture 3.3. *A generic homogeneous collection of d fat points of $\mathbb{P}^n$ has maximal rank if $d \geqslant 3^n$.*

We note that these conjectures are (necessarily) weaker than the Strong Iarrobino Conjecture described below, and agree with the Nagata conjecture and the Segre–Harbourne–Hirschowitz conjecture in the case of $\mathbb{P}^2$, as we shall now describe.

The case of the projective plane has seen a good deal of progress in recent years (see [Ha4] for an overview). First, Nagata's use [N] of Cremona transformations led to his conjecture that a general collection of multiple points in $\mathbb{P}^2$ fails to achieve maximal rank only if it contains a subcollection supported on at most eight points that displays "visible" obstructions to maximal rank. He evaluated the Hilbert function of a general collection of eight multiple points. This includes the following statement.

Theorem 3.4 [N]. *Let $d, m \in \mathbb{N}$, with $d \leqslant 6$. Let $A = (k_1, \dots, k_d) \in \mathbb{N}^d$ so that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d$ and $\sum_{i=1}^{5} k_i \leqslant 2m+1$. Then a generic A-subscheme of $\mathbb{P}^2$ has maximal rank in degree m.*

Notice that the assumption on $\sum_{i=1}^{5} k_i \leqslant 2m+1$ "prevents" the conic through the first five points from interfering with maximal rank. Quite generally, efforts of Segre on this theme yield the following conjecture.

Conjecture 3.5 [S]. *If a collection $\mathcal{Z}$ of multiple points in $\mathbb{P}^2$ does not display maximal rank in degree m then there is a curve C in the base locus of the system of m-ics through $\mathcal{Z}$ for which*

$$\deg \mathcal{Z} \cap C > h_{\mathbb{P}^2}(C, m).$$

Harbourne [Ha1] and Hirschowitz [H2] each presented explicit conjectures on the Hilbert function of a collection of multiple points in the plane, referring to the induced linear systems on the blow-up of $\mathbb{P}^2$ with respect to the points. Each noted that: if, indeed, the only exceptions to maximal rank are "due to" obstructions on schemes supported on eight or fewer points, the main information on such cases should be gleaned from the study of linear systems on a Hirzebruch surface. More recently, Ciliberto and Miranda [CM3] showed that the conjectures of Segre, Harbourne, and Hirschowitz are all equivalent; whence those of [Ha1,H2] precisely identify cases in which a curve obstructs the Hilbert function of a planar fat point scheme from achieving maximal rank.

Hirschowitz used his "méthode d'Horace" [H2] to evaluate the Hilbert function of a generic union of double points and likewise for triple points of $\mathbb{P}^2$. Harbourne has made considerable refinements on the predictions of Nagata (e.g., [Ha2]) on determining exceptional cases to maximal rank, with support on eight or fewer points; including an algorithm for computation of the conjectural Hilbert function of fat points in $\mathbb{P}^2$ (see [Ha3], along with the computer programme running on his web page).

Ciliberto and Miranda [CM1,CM2] have made further significant progress toward this problem. Their method involves passing between $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F}_1$. This led to a complete analysis of the cases $HPTS(d, (k_1, \dots, k_d), 3)$, $k_i \leqslant 3$ and $HPTS(d, \bar{k}, 3)$, $k \leqslant 12$. They, along with Orecchia, applied the technique later to extend to fat points of (equal) multiplicity up to 20 [CCMO].

Meanwhile, up in higher dimensions we do have the results of Alexander and Hirschowitz [H1,A,AH1,AH2,AH3] on the Hilbert function of generic double points in projective space using (nontrivial!) variations on the méthode d'Horace. (To complete that story, the author evaluated the one missing case [C5]!)

Next, the author shows in [C7] that a generic union of double and triple points in $\mathbb{P}^n$ does exhibit maximal rank in degree at least 7, and exhibits schemes that do not satisfy the Strong Hypothesis in lower degrees. This is done by expanding on the simplified version of "Horace" given in [C3] for a "brief proof" of the Alexander–Hirschowitz result. In the latter paper, this appears as an organisational means toward the result at issue, whereas in multiplicity at least three its use is critical, not only in bookkeeping but as an anchoring mechanism.

4. Fröberg–Iarrobino conjectures

The conjectures of Fröberg and Iarrobino apply to situations in which the Hilbert function of a collection of multiple points *should not* attain maximal rank in a given degree; namely, when that degree is small compared to multiplicities.

We present here a background on these conjectures, together with pertinent known results. First, the conjectures of Fröberg on the behaviour of an ideal generated by a generic set of forms in a (graded) polynomial ring are described. Next, we look at the “specialisations” made by Iarrobino to an ideal generated by powers of linear forms, chosen generically. We see then the interpretation of Emsalem and Iarrobino of the latter problem to the Hilbert function of multiple points.

4.1. Fröberg conjectures on the ideal of general forms

Let us take $R = K[Y_0, \ldots, Y_n]$, as a graded ring. Fröberg [F] considers an ideal J generated by a generic collection of forms of specified degrees. His conjectures suggest that the minimal free resolution of J in “looks like” that of a complete intersection until the degree is, by numerics, expected to be large enough that the ideal should contain all forms of that degree.

Definition 5. Let $A = (j_1, \ldots, j_d) \subset \mathbb{N}^d$. Let J be an ideal generated by a general set of d forms in $n+1$ variables, of degrees $j_1, \ldots, j_d$. We write

$$HGEN(d, A, n+1)_m = \dim R_m - \dim J_m.$$

Example. $d \leqslant n+1$. Then J is a complete intersection ideal with minimal free resolution:

$$0 \to R(-j_1 - \cdots - j_d) \to \cdots \to \bigoplus_{1 \leqslant i < k \leqslant d} R(-j_i - j_k) \to \bigoplus_{i=1}^{d} R(-j_i) \to J \to 0.$$

Hence

$$\dim R_m - \dim J_m = \sum_{B \subseteq A} (-1)^{\ell(B)} \binom{n+m-|B|}{n}.$$

For example, in the case $j = j_1 = \cdots = j_d$ we have

$$\dim R_m - \dim J_m = \sum_{t=0}^{n} (-1)^t \binom{d}{t} \binom{n+m-tj}{n}.$$

Fröberg’s conjectures extend this by asserting that if J is generated by $d \geqslant n+2$ general forms, then codim J_m should act as if the only relations between generators are Koszul for as long as possible, namely, until the first (numerical) opportunity for $J_m = R_m$.

Definition 6. Define a function F as follows. Given $n, d \in \mathbb{N}$, $A \in \mathbb{N}^d$, and $m \in \mathbb{N}$, set

$$F'(d, A, n+1)_m = \sum_{B \subseteq A} (-1)^{\ell(B)} \binom{n+m-|B|}{n},$$

and

$$F(d, A, n+1)_m = \begin{cases} 0, & \text{if } F'(d', A', n+1)_m \leqslant 0 \\ & \text{for some } d' \leqslant d \text{ and } A' \subseteq A, \\ F'(d, A, n+1)_m, & \text{otherwise.} \end{cases}$$

The conjectures of Fröberg are as follows.

Conjecture 4.1 (Strong Fröberg Conjecture (SF)).

$$HGEN(d, A, n+1)_m = F(d, A, n+1)_m.$$

Conjecture 4.2 (Weak Fröberg Conjecture (WF)).

$$HGEN(d, A, n+1)_m \geqslant F(d, A, n+1)_m.$$

Under the following hypotheses the Strong Conjecture is known to hold (see [I] for more details): $d \leqslant n+1$ (as we have just observed), $d = n+2$ (Stanley [St]), $n = 1$ (Fröberg [F]), and $n = 2$ (Anick [An]). In the case of equal degree; that is, $A = (j, \ldots, j)$, the Strong Conjecture has been verified in the following cases: $m = j+1$ (Hochster and Laksov [HL]), $n \leqslant 10$ and $j \leqslant 2$, along with $n \leqslant 7$ and $j \leqslant 3$ (Fröberg and Hollman [FH]); and the more detailed hypotheses of Aubry [Au], that m is sufficiently close to j.

Iarrobino shows that in the "first Koszul interval" for a specified case, the Weak Fröberg Conjecture may be verified by a lower degree case of the Strong Fröberg Conjecture.

Theorem 4.3 [I]. *Take $A = (j, j_1, \ldots, j_d)$ and $C = (j_1, \ldots, j_d)$. Assume that $j \leqslant \min\{j_i\}$ and $2j \leqslant m \leqslant 3j$. If*

$$HGEN(d, C, n+1)_{m-j} = F(d, C, n+1)_{m-j}$$

then

$$HGEN(d, A, n+1)_m \geqslant F(d, A, n+1)_m.$$

We shall obtain in Corollary 8.8 a geometric analogue of this result via Theorem 1.3.

4.2. *Iarrobino's conjectures on general linear forms*

Iarrobino extends the Fröberg Conjecture to powers of linear forms in appropriate characteristic.

Definition 7. Let $A = (a_1, \ldots, a_d)$. Let J be an ideal generated by a generic set of d powers of linear forms in $n+1$ variables, of degrees $a_1, \ldots, a_d$. Write

$$HPOWLIN(d, A, n+1)_m = \dim R_m - \dim J_m.$$

Of course, by upper-semicontinuity,

$$HPOWLIN(d, A, n+1)_m \geqslant HGEN(d, A, n+1)_m.$$

This yields:

Conjecture 4.4 (Strong Algebraic Fröberg–Iarrobino). *Let $n, m, d, a \in \mathbb{N}$. Then*

$$HPOWLIN(d, \bar{a}, n+1)_m \geqslant F(d, \bar{a}, n+1)_m.$$

Further, let p be the characteristic of $\mathcal{K}$. If $p = 0$ or $m > p$, we have

$$HPOWLIN(d, \bar{a}, n+1)_m = F(d, \bar{a}, n+1)_m,$$

except in the following circumstances: $d = n+3$, $d = n+4$, $n = 2$ *and* $d = 7$ *or* 8; $n = 3$ *and* $d = 9$.

Conjecture 4.5 (Weak Algebraic Fröberg–Iarrobino).

$$HPOWLIN(d, A, n+1)_m \geqslant F(d, A, n+1)_m.$$

Remark. We shall see more explicitly how the exceptional values arise from the interpretation to the postulation of multiple points described next.

4.3. Conjectures on multiple points

Emsalem and Iarrobino deduced from Macaulay duality that *HPOWLIN* is related to the Hilbert function of multiple points in appropriate characteristic.

Theorem 4.6 [EI]. *Let $A = (k_1, \ldots, k_d)$ and $A' = (m+1-k_1, \ldots, m+1-k_d)$. If $\operatorname{char} \mathcal{K} = 0$ or $\operatorname{char} \mathcal{K} > \max(m, k_1, \ldots, k_d)$ then*

$$HPTS(d, A, n+1)_m = \dim R_m - HPOWLIN(d, A', n+1)_m.$$

To see the main idea of this theorem, let us look at the case $\operatorname{char} \mathcal{K} = 0$. Let $S = \mathcal{K}[X_0, \ldots, X_n]$ and $R = \mathcal{K}[\partial/\partial X_0, \ldots, \partial/\partial X_n]$ (with R regarded as a polynomial ring in the "dummy variables" $\partial/\partial X_i$).

Macaulay duality refers to the obvious perfect pairing $\Phi_m : R_m \times S_m \to \mathcal{K}$. (Of course, it is not quite natural in that it depends on the coordinate choice.) Given an ideal I of S we obtain, in each degree m,

$$I_m^{\perp} = \pi_R(\ker \Phi|_{R_m \times I_m}) \subset R_m,$$

so that $\dim I_m^\perp = \dim S_m - \dim I_m$. Certainly for two ideals I, J we have $(I \cap J)_m^\perp = I_m^\perp + J_m^\perp$.

The case in point given by Emsalem and Iarrobino starts with the ideal of a multiple point, say $I = (X_1, \dots, X_n)^k \subset S$. Then $I_m = (X_1, \dots, X_n)^k S_{m-k}$ so that $I_m^\perp = \partial/\partial X_0 R_{k-1}$. (Each monomial, say, M in I_m satisfies $\partial/\partial {X_0}^{m-k+1} M = 0$.) Whence the dual to the mth graded piece of the ideal of a $(k_1, \dots, k_d)$-scheme is the mth piece of an ideal generated by d powers of linear forms, the powers being $m+1-k_1, \dots, m+1-k_d$.

So, Iarrobino's conjectures on powers of linear forms "translate" to candidates for values of the function *HPTS* evaluating the Hilbert function of fat points. Moreover, note that in the latter setting the characteristic of the field need not play a rôle.

Definition 8. Let

$$G'\bigl(d, (k_1, \dots, k_d), n+1\bigr)_m = \dim R_m - F'\bigl(d, (m+1-k_1, \dots, m+1-k_d), n+1\bigr)_m,$$

and

$$G\bigl(d, (k_1, \dots, k_d), n+1\bigr)_m = \dim R_m - F\bigl(d, (m+1-k_1, \dots, m+1-k_d), n+1\bigr)_m.$$

The geometric versions of the Fröberg–Iarrobino conjectures are the following (stronger) conjectures.

Conjecture 4.7 (Weak Fröberg–Iarrobino (WFI)).

$$HPTS(d, A, n+1)_m \leqslant G(d, A, n+1)_m.$$

Conjecture 4.8 (Strong Fröberg–Iarrobino (SFI: homogeneous)). *For each $n, d, m \in \mathbb{N}$ we have $HPTS(d, \bar{k}, n+1)_m \leqslant G(d, \bar{k}, n+1)_m$. Further, we have $HPTS(d, \bar{k}, n+1)_m = G(d, \bar{k}, n+1)_m$, except perhaps when one of the following conditions holds*: $d = n+3$, $d = n+4$, $n = 2$ *and* $d = 7$ *or* 8; $n = 3$, $d = 9$, $m = 2k$; *or* $n = 4$, $d = 14$, $m = 2k$ *and* $k = 2$ *or* 3.

Hence in each *homogeneous* case $d \geqslant n+5$ we have

$$\text{SFI} \implies \text{SF} \implies \text{WF} \implies \text{WFI}.$$

As previously stated, we shall say a generic A-subscheme of $\mathbb{P}^n$ (possibly of mixed multiplicities) satisfies the Strong Fröberg–Iarrobino Hypothesis if its Hilbert function agrees with the conjectural value $G(d, A, n+1)$. (Similarly, we say that it satisfies the Strong Hypothesis in a given degree if the value of the Hilbert function in this degree is equal to the specified value. Further, we refer to an arbitrary scheme as satisfying the Strong Hypothesis if its Hilbert function agrees with the value of the corresponding function G.)

It is straightforward to verify the Strong Conjecture in the case of $d \leqslant n+1$ fat points in $\mathbb{P}^n$ by examining the intersection ideal. For $d = n+2$ the conjecture holds valid as well, again by [St]. On the other hand, the exclusion of cases $d = n+3, n+4$, and so forth in

Conjecture 4.4 and then Conjecture 4.8 do arise from the expectation of special varieties through points that inhibit the Hilbert function.

We began a direct geometric analysis of the conjectures on homogeneous $\bar{k}$-subschemes of $\mathbb{P}^n$ in [C2]. Since the cases of degree $m \leqslant k$ are trivial to verify (at most $n+1$ points are involved) the focus there is finding criteria determining the inequality:

$$HPTS(d,k,n+1)_{k+1} \geqslant G(d,k,n+1)_{k+1}.$$

The result invokes the identification of neighbourhoods of planes of each dimension lying in the base locus of a $(k+1)$-ic linear system through a $\bar{k}$-scheme (made precise in Section 5).

Here we expand on this method for use in the general setting.

5. Interpreting conjectures geometrically

We describe the correspondence of the Fröberg–Iarrobino conjectures with the issue of linear obstructions to the maximal rank of a collection of fat points. In particular, we make precise the notion of a scheme that displays only the expected linear obstructions (as in Theorem 1.3), which we "expect" to satisfy the Strong Hypothesis. We start with a simple identification of multiple planes (of each dimension) that must appear in the base locus of a linear system through a given collection of multiple points. We compare this information to the conjectures at hand by means of intersection degrees. This leads naturally to extensions and refinements of these conjectures.

Basic Observation. Let $p, q \in \mathbb{P}^n$ and $L = \operatorname{span}\{p, q\}$. By Bézout (or by Lagrange–Hermite!), any m-ic form vanishing on $\{p\}^k \cup \{q\}^j$ vanishes on L if $m \leqslant k + j - 1$. Further, such an m-ic must vanish on L^{k+j-m}. This is easy to see in characteristic 0, simply by taking derivatives. Generally, take $I = (X_0, X_1, \dots, X_n)^k \cap (X_0, X_2, \dots, X_n)^j$. Suppose that $F \in I_m$ and write $F = X_0^r X_1^s G$ so that neither X_0 nor X_1 divides G. Let us write $G = N + G_1$ so that N is a monomial (divisible by neither X_0 nor X_1) and G_1 is a sum of fewer monomial terms than G. We have that $X_1^s G \in (X_1, \dots, X_n)^j$, so that $\deg G \geqslant j - s$, and hence $r \leqslant m - j$. Likewise, $s \leqslant m - k$, so $\deg G \leqslant k + j - m$. In particular $N \in (X_2, \dots, X_n)^{k+j-m}$, so that we may replace G by G_1 and repeat the procedure, reducing the number of terms at each step. Whence G vanishes on L^{k+j-m} and so does F.

Consequently, we obtain the following result.

Lemma 5.1. *Let $P, Q \subset \mathbb{P}^n$ be projective subspaces. Then every m-ic vanishing on $P^k \cup Q^j$ must vanish on* $\operatorname{span}(P \cup Q)^{k+j-m}$.

Proof. Suppose that $F \in I(P^k \cup Q^j)$. Let $t \in \operatorname{span}(P \cup Q)$. Then $t \in L$ where L is the line between two points, say, $p, q \in P \cup Q$. As we have just seen, $F \in I(t)^{k+j-m}$ since $F \in I(p)^k \cap I(q)^j$. □

This yields the following generalisation of our initial note.

Corollary 5.2. *Let $n, m, d \in \mathbb{N}$ and $(k_1, \ldots, k_d) \in \mathbb{N}^d$. Suppose that $P_1, \ldots, P_d$ are projective subspaces of $\mathbb{P}^n$. Then every m-ic through $P_1^{k_1} \cup \cdots \cup P_d^{k_d}$ must vanish on* $\operatorname{span}(P_1 \cup \cdots \cup P_d)^r$ *for $r = k_1 + \cdots + k_d - (d-1)m$.*

Proof. Apply induction together with Lemma 5.1 to the union of $P_d^{k_d}$ and $\operatorname{span}(P_1 \cup \cdots \cup P_{d-1})^j$, where $j = k_1 + \cdots + k_{d-1} - (d-2)m$. $\square$

This motivates the following description.

Definition 9. Let $n, m, d, k \in \mathbb{N}$, and $(k_1, \ldots, k_d) \in \mathbb{N}^d$. Let $\mathcal{Z} = P_1^{k_1} \cup \cdots \cup P_d^{k_d}$ and $P \subset \mathbb{P}^n$, where $P_1, \ldots, P_d$, and P are projective subspaces of $\mathbb{P}^n$ (such as points). Let us refer to the *expected linear obstruction scheme on P^k* induced by m-ics through $P^{k-1} \cup \mathcal{Z}$ as the subscheme of P^k predicted by Corollary 5.2 namely,

$$\left(P^{k-1} \cup \bigcup \operatorname{span}(P \cup P_{i_1} \cup \cdots \cup P_{i_r})^{j(i_1,\ldots,i_r)}\right) \cap P^k,$$

where the union ranges over sets of indices $(i_1, \ldots, i_r)$ with $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant d$, and $j(i_1, \ldots, i_r) = k + k_{i_1} + \cdots + k_{i_r} - rm - 1$.

We shall say that $P^k \cup \mathcal{Z}$ is *only linearly obstructed* by $P^{k-1} \cup \mathcal{Z}$ in degree m if the base locus of the linear system of m-ics through $P^{k-1} \cup \mathcal{Z}$ meets P^k in precisely the expected linear obstruction scheme.

We say that $P^k \cup \mathcal{Z}$ is *only linearly obstructed* by $\mathcal{Z}$ in degree m if $P^\ell \cup \mathcal{Z}$ is only linearly obstructed by $P^{\ell-1} \cup \mathcal{Z}$ for each ℓ with $1 \leqslant \ell \leqslant k$.

Finally, we say $\mathcal{Z}$ has *only the expected linear obstructions* in degree m if for each $1 \leqslant j \leqslant d$ we have that $P_1^{k_1} \cup \cdots \cup P_j^{k_j} \cup \cdots \cup P_d^{k_d}$ is only linearly obstructed by $P_1^{k_1} \cup \cdots \cup P_j^0 \cup \cdots \cup P_d^{k_d}$.

Note that, in the context of the definition, if the set of planes is generic, the planes are of equal dimension, and $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d$, then $\mathcal{Z}$ is nonlinearly obstructed if and only if $\mathcal{Z}$ is nonlinearly obstructed by $P_2^{k_2} \cup \cdots \cup P_d^{k_d}$.

Let us now make an obvious and (hence!) useful simplification to Definition 9.

Lemma 5.3. *Let $n, m, k \in \mathbb{N}$ and $(k_1, \ldots, k_d) \in \mathbb{N}^d$. Take projective subspaces $P, P_1, \ldots, P_d$ of $\mathbb{P}^n$ and let $\mathcal{Z} = P_1^{k_1} \cup \cdots \cup P_d^{k_d}$. Take Υ as the expected linear obstruction subscheme of P^k given by m-ics through $P^{k-1} \cup \mathcal{Z}$ and Υ_1 as the scheme $\Upsilon_1 = (P^{k-1} \cup \bigcup_{i=1}^d \operatorname{span}(P \cup P_i)^{k+k_i-m-1}) \cap P^k$. Then $\Upsilon = \Upsilon_1$.*

Proof. We do have $\Upsilon_1 \subseteq \Upsilon$. After intersection with a hyperplane we reduce, inductively, to the case of $P = \{p\}$, $p \in \mathbb{P}^n$.

By symmetry, it suffices to see that for each $r \leqslant d$ and $j = k + k_1 + \cdots + k_r - rm - 1$ we have $(\{p\}^{k-1} \cup \operatorname{span}(p \cup P_1 \cup \cdots \cup P_r)^j) \cap \{p\}^k \subseteq \Upsilon_1$. Let us fix such an r, then choose an open affine subspace of $\mathbb{P}^n$ containing p (viewed as an origin point) and take $\mathfrak{m}$ as the maximal ideal of p. In the associated projective space the quotient $\mathfrak{m}^{k-1}/\mathfrak{m}^k$ identifies forms of degree $k-1$. Then the desired inclusion follows straight from Corollary 5.2. $\square$

Now let us relate this to $HPTS(d, A, n+1)_m$. When $G(d, A, n+1)_m < \binom{n+m}{m}$, we have

$$G(d, A, n+1)_m = \sum_{t=1}^{\infty} (-1)^{t-1} \sum_{\ell(B)=t} \binom{n + |B| - t - (t-1)m}{n}, \tag{1}$$

where the inner sum is over subindices $\bar{0} < B \subseteq A$. We may interpret each term as follows:

$$t = 1: \quad \sum_{i=1}^{d} \binom{n + k_i - 1}{n},$$

which is the degree of the scheme;

$$t = 2: \quad \sum_{\{1 \leqslant i < j \leqslant d\}} \binom{n + k_i + k_j - 2 - m}{n},$$

counts (and subtracts) obstructions due to lines between pairs of points;

$$t = 3: \quad \sum_{\{1 \leqslant i_1 < i_2 < i_3 \leqslant d\}} \binom{n + k_{i_1} + k_{i_2} + k_{i_3} - 3 - 2m}{n},$$

counts obstructions *to* lines between pairs of points given by planes between threesomes, as a correction to the term $t = 2$; and so on. That is, the tth term in (1) accounts for each of the $(t-1)$-planes occurring in the base locus of an A-scheme in degree m.

From the case of $d \leqslant n+1$ where the SFI hypothesis does hold valid we see that the function G exhibits the intersection numbers with regard to the (multiple) planes determined by Corollary 5.2. Whence, we may view the Strong Fröberg–Iarrobino Conjecture as asserting that a generic A-scheme has only linear obstructions, and the Weak Conjecture as enumerating the linear obstructions occurring. We shall exhibit this phenomenon in Section 6.

These observations suggest the following:

Conjecture 5.4. *Let $n, m, d \in \mathbb{N}$, and $A \in \mathbb{N}^d$. Suppose that $\mathcal{Z}$ is an A-subscheme of $\mathbb{P}^n$. Then*

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant G(d, A, n+1)_m.$$

Furthermore, assume that

$$G(d, A, n+1)_m < \binom{n+m}{m}.$$

We have equality of $h_{\mathbb{P}^n}(\mathcal{Z}, m)$ *and* $G(d, A, n+1)_m$ *precisely when* $\mathcal{Z}$ *exhibits only the expected linear obstructions in degree* m.

By definition, if a multiple scheme has only the expected linear obstructions in a given degree, none of its multiple subschemes can be nonlinearly obstructed in this degree. Under the viewpoint that $G(d, A, n+1)_m - HPTS(d, A, n+1)_m$ does count nonlinear obstructions to an A-scheme in degree m (as we shall justify in Section 6), let us observe the following.

Corollary 5.5. *Suppose that Conjecture* 5.4 *does hold valid. Let* $n, m, d \in \mathbb{N}$ *and* $A \in \mathbb{N}^d$. *Assume that*

$$G(d, A, n+1)_m < \binom{n+m}{m}.$$

Suppose that $\mathcal{Z} \subset \mathbb{P}^n$ *is an* A*-subscheme. If*

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) = G(d, A, n+1)_m$$

then for every $A' \leqslant A$ *and each* A'*-subscheme* $\mathcal{Z}'$ *of* $\mathcal{Z}$ (*respecting ordering*) *we have*

$$h_{\mathbb{P}^n}(\mathcal{Z}', m) = G(d, A', n+1)_m.$$

This suggests, more generally, that the difference between the Hilbert function of a collection of multiple points and the conjectural value keeps track of nonlinear obstructions, in the following sense.

Conjecture 5.6. *Let* $n, m, d \in \mathbb{N}$ *and* $A \in \mathbb{N}^d$, *for which*

$$G(d, A, n+1)_m < \binom{n+m}{m}.$$

Take $A' \in \mathbb{N}^d$ *for which* $A' \leqslant A$. *Suppose that* $\mathcal{Z} \subset \mathbb{P}^n$ *is an* A*-subscheme and that* $\mathcal{Z}' \subset \mathcal{Z}$ *is an* A'*-subscheme. Suppose that for some* $\alpha \in \mathbb{N}$ *we have*

$$h_{\mathbb{P}^n}(\mathcal{Z}', m) = G(d, A', n+1)_m - \alpha.$$

Then

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant G(d, A, n+1)_m - \alpha.$$

Of course, the conclusion of each conjecture is obvious in case the function G predicts maximal rank for a given scheme! We ask the reader to check that this is not obvious in general!

Note also that Conjecture 5.6 is stronger than the Weak Fröberg–Iarrobino Conjecture: given a generic A-scheme $\mathcal{Z}$ satisfying the hypothesis of the conjecture, we do have

$$h_{\mathbb{P}^n}(\mathcal{Z}_{\mathrm{red}}, m) = \deg \mathcal{Z},$$

so the conjecture predicts that $h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant G(d, A, n+1)_m$. However, experimental evidence on construction (!) points towards the statement's being simpler to verify inductively than the Weak Conjecture itself.

6. Main theorem

The main theorem (Theorem 1.3) gives an upper bound on the Hilbert function of any collection of infinitesimal neighbourhoods of points in $\mathbb{P}^n$ based on Hilbert functions of certain such subschemes of $\mathbb{P}^{n-1}$. Particularly, the scheme of interest is shown in Theorem 1.1 to verify the bound given by the Weak Fröberg–Iarrobino Conjecture when each of the specified ("smaller") ones satisfy the Strong Hypothesis. Moreover, we obtain equality if and only if these schemes in lower dimension have only the expected linear obstructions, as the conjectural function has been shown to compute.

The proof is attained, inductively, by the comparison in Lemma 6.3 of the Hilbert function of a given A-scheme with that of a B-scheme, where $B = A - (0, \ldots, 0, 1)$. To relate these we evaluate in Lemma 6.1 the degree of the linear obstruction scheme occurring between A and B, which may naturally be seen in terms of a Hilbert function of fat points in codimension one. Further, we see from Lemma 6.2 that equality in the estimate of Lemma 6.3 arises precisely when only the expected linear obstructions occur.

Lemma 6.1. *Let $n, d, a \in \mathbb{N}$, and $(j_1, \ldots, j_d) \in \mathbb{N}^d$. Let $p \in \mathbb{A}^n$ and let $L_1, \ldots, L_d$ be distinct lines of $\mathbb{A}^n$ through p. Let $\rho \subset \mathbb{A}^n$ be the scheme $(p^a \cup L_1^{j_1} \cup \cdots \cup L_d^{j_d}) \cap p^{a+1}$. In the projective space $\mathbb{P}^{n-1}$ of lines through p take the $(j_1, \ldots, j_d)$-scheme $\mathcal{W} \subset \mathbb{P}^{n-1}$ given by $L_1^{j_1} \cup \cdots \cup L_d^{j_d}$. Then*

$$\deg \rho = \deg p^a + h_{\mathbb{P}^{n-1}}(\mathcal{W}, a).$$

Proof. Call $\mathfrak{m}$ the maximal ideal of p in the affine coordinate ring of $\mathbb{A}^n$ and $I_1, \ldots, I_d$ the ideals of the d lines. Identify $\mathfrak{m}^a/\mathfrak{m}^{a+1}$ with the vector space of forms of degree a in the prescribed projective space of lines through p and $(\mathfrak{m}^a \cap I_1^{j_1} \cap \cdots \cap I_d^{j_d} + \mathfrak{m}^{a+1})/\mathfrak{m}^{a+1}$ with forms of degree a vanishing on the subscheme $\mathcal{W} \subset \mathbb{P}^{n-1}$. $\square$

Lemma 6.2. *Let $n, m, k \in \mathbb{N}$. Suppose that $\mathcal{Z} \subset \mathbb{P}^n$ is any subscheme and $p \in \mathbb{P}^n$ so that $p \notin \mathcal{Z}$. Take γ as the intersection of p^k with the base locus of m-ics through $\mathcal{Z} \cup \{p\}^{k-1}$ (so that $h_{\mathbb{P}^n}(\mathcal{Z} \cup \{p\}^{k-1}, m) = h_{\mathbb{P}^n}(\mathcal{Z} \cup \gamma, m)$). Then*

$$h_{\mathbb{P}^n}\left(\mathcal{Z} \cup \{p\}^k, m\right) = \min\left(h_{\mathbb{P}^n}\left(\mathcal{Z} \cup p^{k-1}, m\right) + \deg p^k - \deg \gamma, \binom{n+m}{m}\right).$$

Remark. Quite generally, consider the base locus of (say) the linear system of m-ics through a subscheme $\mathcal{Z} \subset \mathbb{P}^n$. We may of course determine the base locus scheme $\mathcal{Y} \subset \mathbb{P}^n$ from the mth graded piece $I(\mathcal{Z})_m$ of the ideal of $\mathcal{Z}$. The subtlety is that there is no guarantee that $I(\mathcal{Y})_m = I(\mathcal{Z})_m$. Indeed, this issue may be viewed as the crux of the challenge in verifying the Segre conjecture (along with desired analogues in higher dimension).

The point of Lemma 6.2 is simply that we may locate such subschemes in a relative sense: comparing $I(\mathcal{Z} \cup \{p\}^{k-1})_m \supseteq I(\mathcal{Z} \cup \{p\}^k)_m$ for $p \in \mathbb{P}^n$. Whence we obtain a scheme γ with $\{p\}^{k-1} \subset \gamma \subset \{p\}^k$ accounting for the base locus with respect to $\{p\}^{k-1}$. So, in the context of Lemma 6.3 we may compare such a scheme γ with an expected linear obstruction scheme.

Proof. We may assume that

$$h_{\mathbb{P}^n}\big(\mathcal{Z} \cup \{p\}^k, m\big) < \binom{n+m}{n};$$

in particular, that $m \geqslant k$. Choose a linear form L on $\mathbb{P}^n$ for which $L \notin \mathfrak{m}$, where $\mathfrak{m} = I(p)$. Set $V = I(\mathcal{Z} \cup \{p\}^{k-1})_m$, so that $\mathfrak{m}^k_m \subseteq \mathfrak{m}^k_m + V \subseteq \mathfrak{m}^{k-1}_m$. Call $r = \dim(\mathfrak{m}^k_m + V) - \dim \mathfrak{m}^k_m$. We have the direct sum of vector spaces:

$$\mathfrak{m}^{k-1}_m = \mathfrak{m}^k_m + L^{m-k+1}\mathfrak{m}^{k-1}_{k-1}$$

and likewise:

$$\mathfrak{m}^k_m + V = \mathfrak{m}^k_m + L^{m-k+1}\langle F_1, \dots, F_r\rangle, \quad \text{with } F_1, \dots, F_r \in \mathfrak{m}^{k-1}_{k-1}.$$

So the scheme γ is given by the ideal $\mathfrak{m}^k + (F_1, \dots, F_r)$, and $\deg \gamma = \deg\{p\}^k - r$. We see then that $I(\gamma)_m = I(\mathcal{Z} \cup \{p\}^{k-1})_m + \mathfrak{m}^k_m$ and whence:

$$\begin{aligned} h_{\mathbb{P}^n}\big(\mathcal{Z} \cup \{p\}^k, m\big) &= h_{\mathbb{P}^n}\big(\mathcal{Z} \cup \{p\}^{k-1}, m\big) + r \\ &= h_{\mathbb{P}^n}\big(\mathcal{Z} \cup \{p\}^{k-1}, m\big) + \deg\{p\}^k - \deg \gamma. \qquad \square \end{aligned}$$

We apply these results toward linear obstruction schemes.

Lemma 6.3. *Let $n, m, d \in \mathbb{N}$, and $A = (k_1, \dots, k_d, k) \in \mathbb{N}^{d+1}$. Let $\{p_1, \dots, p_d, p\} \subset \mathbb{P}^n$ and $\mathcal{Z} = \bigcup_{i=1}^d \{p_i\}^{k_i}$. Choose a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ for which $\Gamma \cap \mathbb{P}^{n-1} = \emptyset$. Take*

$$C = (c_1, \dots, c_d) := (k_1, \dots, k_d) - \overline{m+1-k},$$

and let

$$\mathcal{W} = \bigcup_{i=1}^d \big(\operatorname{span}(p_i, p) \cap \mathbb{P}^{n-1}\big)^{c_i}.$$

Then

$$h_{\mathbb{P}^n}\big(\mathcal{Z}\cup\{p\}^k,m\big)\leqslant h_{\mathbb{P}^n}\big(\mathcal{Z}\cup\{p\}^{k-1},m\big)+\binom{n+k-2}{n-1}-h_{\mathbb{P}^{n-1}}\big(\mathcal{W}\cap\mathbb{P}^{n-1},k-1\big).$$

Equality occurs precisely if $\mathcal{Z}\cup\{p\}^k$ is only linearly obstructed by $\mathcal{Z}\cup\{p\}^{k-1}$. In particular, for a generic scheme, take $B=(k_1,\dots,k_d,k-1)$. Then

$$HPTS(d+1,A,n+1)_m\leqslant HPTS(d+1,B,n+1)_m+\binom{n+k-2}{n-1}-HPTS(d,C,n)_{k-1}.$$

Equality occurs exactly when a generic A-subscheme of $\mathbb{P}^n$ is only linearly obstructed by a B-subscheme.

Proof. Let us put together the relevant information from our previous observations. Take γ as the intersection of $\{p\}^k$ with the base locus of the linear system of m-ics through $\mathcal{Z}\cup\{p\}^{k-1}$. So γ contains the linear obstruction scheme ρ which is (by Lemma 5.3) given by

$$\rho=\big(p^{k-1}\cup\mathcal{W}\big)\cap p^k.$$

By Lemma 6.2 we have

$$h_{\mathbb{P}^n}\big(\mathcal{Z}\cup\{p\}^k,m\big)=h_{\mathbb{P}^n}\big(\mathcal{Z}\cup\{p\}^{k-1},m\big)+\deg\{p\}^k-\deg\gamma,$$

so that

$$h_{\mathbb{P}^n}\big(\mathcal{Z}\cup\{p\}^k,m\big)\leqslant h_{\mathbb{P}^n}\big(\mathcal{Z}\cup\{p\}^{k-1},m\big)+\deg\{p\}^k-\deg\rho,$$

with equality occurring exactly when $\mathcal{Z}\cup\{p\}^k$ is only linearly obstructed by $\mathcal{Z}\cup\{p\}^{k-1}$.

From Lemma 6.1 we may now plug in $\deg\rho=\deg\{p\}^{k-1}+h_{\mathbb{P}^{n-1}}(\mathcal{W}\cap\mathbb{P}^{n-1},k-1)$ to obtain the desired conclusion. □

Altogether we have:

Proof of Theorem 1.3. Let $n,m,d\in\mathbb{N}$ and $A=(k_1,\dots,k_d)\in\mathbb{N}^d$. Let us take C_{ji} as stated; that is, $C_{ji}=(k_1,\dots,k_j)+\overline{i-m}$, for $j=1,\dots,d-1$, and $i=0,\dots,k_j-1$.

Given an A-subscheme $\mathcal{Z}\subset\mathbb{P}^n$ we wish to produce C_{ji}-subschemes $\mathcal{W}_{ji}\subset\mathbb{P}^{n-1}$ from $\mathcal{Z}$ for which

$$h_{\mathbb{P}^n}(\mathcal{Z},m)\leqslant\deg\mathcal{Z}-\sum_{j=1}^{d-1}\sum_{i=1}^{k_j-1}h_{\mathbb{P}^{n-1}}(\mathcal{W}_{ji},i);\tag{2}$$

equality holding just when $\mathcal{Z}$ displays only the expected linear obstructions in degree m. Namely, taking $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ as a hyperplane that does not meet the support of $\mathcal{Z}$ we have

$$\mathcal{W}_{ji} = \left(\bigcup_{r=1}^{j} \operatorname{span}(p_{j+1}, p_r)^{k_r+i-m} \right) \cap \mathbb{P}^{n-1}.$$

From Lemma 6.3 we obtain this by double induction on d (starting with $d = 0$) and then on k_d (from the initial value $k_d = 0$).

We find that equality holds in (2) exactly when $\mathcal{Z}$ exhibits only the expected linear obstructions, again from Lemma 6.3. □

Let us compare the description of the Hilbert function of an A-subscheme in $\mathbb{P}^n$ used in Theorem 1.3 with the behaviour of the conjectural function of Iarrobino.

Lemma 6.4. *Let $n, m, d \in \mathbb{N}$, $A = (k_1, \dots, k_d)$, and $C_{ji} = (k_1, \dots, k_j) + \overline{i-m}$, for $j = 1, \dots, d-1$, $i = 0, \dots, k_j - 1$. If*

$$G(d, A, n+1) - m < \binom{n+m}{m}$$

then

$$G(d, A, n+1)_m = \sum_{j=1}^{d-1} \sum_{i=1}^{k_j-1} G(j, C_{ji}, n)_i.$$

(*More technically, it is enough to assume that*

$$G'\big(\ell(A'), A', n+1\big) \leqslant \binom{n+m}{m}$$

for each $A' \leqslant A$.)

Proof. Compute directly from Eq. (1). □

Hence, according to Theorem 1.3 the function G may be viewed directly as keeping track of the expected linear obstruction schemes identified in Definition 9.

Proof of Theorem 1.1. Let us take $n, m, d \in \mathbb{N}$, $A = (k_1, \dots, k_d) \in \mathbb{N}^d$, and $C_{ji} = (k_1, \dots, k_j) + \overline{i-m}$ for each $j = 1, \dots, d-1$ and $i = 0, \dots, k_j - 1$.

Suppose, as indicated, that

$$HPTS(j, C_{ji}, n)_i = G(j, C_{ji}, n)_i,$$

for each $j = 1, \dots, d-1$ and $i = 0, \dots, k_j - 1$.

If $G(d, A, n+1)_m = \binom{n+m}{n}$ then the Weak Conjecture holds trivially for an A-scheme. Otherwise, take $\mathcal{Z} \subset \mathbb{P}^n$ as a generic A-subscheme. By Theorem 1.3 we have

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant \sum_{j=1}^{d} \binom{n+k_j-1}{n} - \sum\sum G(j, C_{ji}, n)_i = G(d, A, n+1)_m, \qquad (3)$$

and equality holds exactly when $\mathcal{Z}$ has only the expected linear obstructions in degree m. Hence, by upper-semicontinuity, the inequality applies to every A-subscheme of $\mathbb{P}^n$ and the Weak Conjecture is satisfied by each A-scheme in degree m.

Let us now take $\mathcal{Z}_0$ as an arbitrary A-subscheme. of $\mathbb{P}^n$, and take $\mathcal{W}_{ji}$ as the $\mathcal{C}_{ji}$-subschemes of $\mathbb{P}^{n-1}$ identified in Theorem 1.3. Note that $h_{\mathbb{P}^{n-1}}(\mathcal{W}_{ji}, i) \leqslant G(j, C_{ji}, n)_i$ for each pair j, i.

Suppose that $\mathcal{Z}_0$ has only the expected linear obstructions in degree d. Then

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) = \deg Z - \sum\sum h_{\mathbb{P}^{n-1}}(\mathcal{W}_{ji}, i) \geqslant G(d, A, n+1)_m.$$

According to the Weak Conjecture, as we've just verified in this case, we must have equality here. □

Now let us examine the special case of quasihomogeneous schemes (including homogeneous schemes, as in the Strong Conjecture).

Proposition 6.5. *Let $n \in \mathbb{N}$. Suppose that the Strong Fröberg–Iarrobino Conjecture holds in $\mathbb{P}^{n-1}$. Assume further that the Weak Fröberg–Iarrobino Conjecture applies to each quasihomogeneous scheme of fat points in $\mathbb{P}^n$ with support on $n+4$ points. Then*:

(a) *The Weak Fröberg–Iarrobino Conjecture holds valid in $\mathbb{P}^n$ for every quasihomogeneous fat point scheme in $\mathbb{P}^n$.*
(b) *A generic quasihomogeneous collection of infinitesimal neighbourhoods of points in $\mathbb{P}^n$ satisfies the Strong Hypothesis if and only if it exhibits only the expected linear obstructions in each degree* (*Definition* 9).
(c) *For any quasihomogeneous collection of infinitesimal neighbourhoods in $\mathbb{P}^n$, its Hilbert function agrees with the function given by the Strong Hypothesis provided that it has only the expected linear obstructions.*

Hence to verify the Weak Conjecture in this setting, along with examining the Strong Conjecture itself, it suffices to examine schemes supported on $n+4$ points of $\mathbb{P}^n$.

Proof. We verify in Theorem 1.2 (Section 7) that the Weak Conjecture holds valid (indeed, for schemes of mixed multiplicities) in $\mathbb{P}^n$ for each $n \leqslant 3$. Further, according to results from [C8] along with the given hypothesis, we have that each fat point subscheme of $\mathbb{P}^n$ supported on at most $n+3$ points satisfies the Weak Conjecture.

Note, next, that in order to apply Theorem 1.3 to a quasihomogenous subscheme of $\mathbb{P}^n$, it suffices to see that the Strong Conjecture (on homogeneous schemes) applies to corresponding subschemes of $\mathbb{P}^{n-1}$.

According to the given assumptions along with the conditions given by the Strong Conjecture, it remains to deal with $\mathbb{P}^n$ for $n = 4$ and $n = 5$.

Let us make the following observations:

- Fix $n, m, \ell \in \mathbb{N}$. The function $G(d, (\ell, \ell, \ldots, \ell, k), n + 1)_m$ is strictly increasing in k until it reaches its maximum value $\binom{n+m}{m}$.
- Consider the "extra" exceptions predicted by the Strong Conjecture occurring for $n \leqslant 4$ (given by homogeneous schemes supported on at least $n + 5$ points). For $n = 3$, the additional cases (i.e., with $d \geqslant 8$) have $d = 9$ and $m = 2k$, and the extra cases for $n = 4$ ($d \geqslant 9$) are: $d = 14$, $m = 2k$, $k = 2$ or 3. In each of these, we have

$$G(d, k, n + 1)_m = \binom{n + m}{m}.$$

Hence, each time such a case arises in the application of Theorem 1.1 toward a quasi-homogeneous A-scheme, in a given degree m, we have already that

$$G(d, A, n + 1)_m = \binom{n + m}{m}.$$

More precisely, take $n = 4$. According to Theorem 1.1 we must show that

$$HPTS\big(10, (\ell, \ldots, \ell, k), 5\big)_m \leqslant G\big(10, (\ell, \ldots, \ell, k), 5\big)_m,$$

whenever $2(\ell + k - 1 - m) = k - 1$. Note that for each $r \leqslant k - 2$ we have $2(\ell + r - m) < r$, so we do have

$$HPTS\big(10, (\ell, \ldots, \ell, k - 1), 5\big)_m \leqslant G\big(10, (\ell, \ldots, \ell, k - 1), 5\big)_m.$$

Further, by the previous observations,

$$G\big(10, (\ell, \ldots, \ell, k - 1), 5\big)_m = \binom{m + 4}{4},$$

and hence the same holds for $G(10, (\ell, \ldots, \ell, k), 5)_m$, and we are done.

Similar verification is valid for $n = 5$. $\square$

Let us remark on the comparison between determining when the Strong Hypothesis applies to an arbitrary scheme and finding obstructions to that scheme.

Proposition 6.6. *Suppose that the Weak Fröberg–Iarrobino Conjecture holds valid*; *that is, the Hilbert function of any collection of multiple points is bounded above by the conjectural value. Let $n, m, d \in \mathbb{N}$ and $A \in \mathbb{N}^d$. Suppose that $\mathcal{Z} \subset \mathbb{P}^n$ is any A-subscheme. Then $\mathcal{Z}$ satisfies the Strong Fröberg–Iarrobino Hypothesis* (*respectively, in a given degree*) *provided that $\mathcal{Z}$ presents only the expected* (*linear*) *obstructions* (*respectively, in that degree*).

Proof. It suffices to prove the result in a given degree m. We may assume without loss of generality that there is an m-ic vanishing on $\mathcal{Z}$ and that $n \geqslant 2$. Take $\mathcal{Z} \subset \mathbb{P}^n$ as prescribed and $\mathcal{W}_{j,i} \subset \mathbb{P}^{n-1}$ as in Theorem 1.3. So $\mathcal{Z}$ has only the expected linear obstructions in degree m if and only if

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) = \deg \mathcal{Z} - \sum \sum h_{\mathbb{P}^{n-1}}(\mathcal{W}_{ji}, i).$$

Now, according to the Weak Conjecture applied to $\mathbb{P}^{n-1}$, we have that

$$\sum \sum h_{\mathbb{P}^{n-1}}(\mathcal{W}_{ji}, i) \leqslant \sum \sum G(j, C_{ji}, n)_i. \tag{4}$$

Hence (from the Weak Conjecture applied to $\mathbb{P}^n$ along with Lemma 6.4) if $\mathcal{Z}$ has only the expected linear obstructions in degree m we have equality in (4), so that

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) = G(d, A, n+1)_m.$$

Notice, it follows from our assumptions that if $\mathcal{Z}$ has the only expected linear obstructions, so must each of the schemes $\mathcal{W}_{ji}$. □

Let us compare our results here with the conjectures overall. Take an A-subscheme of $\mathbb{P}^n$. Then Theorem 1.3 provides the conclusion of the Weak Conjecture when each of the derived C_{ji}-schemes in $(n-1)$-space do satisfy the Strong Fröberg–Iarrobino Hypothesis. Otherwise, each C_{ji}-scheme suffering from lack of SFI must have nonlinear obstructions. Conjecture 5.4 demands that each such obstruction is then carried over to the A-scheme.

7. The Weak Conjecture holds valid in $\mathbb{P}^3$

We illustrate here the use of Theorem 1.3 in verifying the Weak Fröberg–Iarrobino Conjecture in a given dimension without the full requirement of the Strong Hypothesis in each lower dimensional case. We prove Theorem 1.2, that the Weak Fröberg conjecture does hold in $\mathbb{P}^n$ for $n \leqslant 3$. Of course, since the Strong Conjecture of Fröberg and Iarrobino holds in $\mathbb{P}^1$, regardless of multiplicities, then Weak Conjecture does in $\mathbb{P}^2$. However, as described in Section 3, the Strong Conjecture remains open in $\mathbb{P}^2$ and presents many exceptions to the Strong Hypothesis. We observe, though, that in the homogeneous situation the required results for the application of Theorem 1.3 follow easily from general results of Nagata on multiple points in $\mathbb{P}^2$. So, to deal with mixed multiplicities we make use of the following lemma.

Lemma 7.1 (Numerical observation). *Let $A = (k_1, \ldots, k_d) \in \mathbb{N}^d$, with $k_1 \geqslant k_2 + 2$. Take $B = (k_1 - 1, k_2 + 1, k_3, \ldots, k_d)$. Then $G(d, A, n+1)_m \geqslant G(d, B, n+1)_m$.*

Indeed, equality occurs if and only if

$$G(d, B, n+1)_m = \binom{n+m}{m}.$$

(For the main idea, note that for a fixed pair of integers n, c the maximal value of the quantity $\binom{n+a}{n} + \binom{n+c-a}{n}$ is obtained from $a = \lceil c/2 \rceil$, so $c - a = \lfloor c/2 \rfloor$.)

Proof of Theorem 1.2. Let $A = (k_1, \dots, k_d, \ell) \in \mathbb{N}^{d+1}$, with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d \geqslant \ell$. Fix $m \in \mathbb{N}$. We wish to show that

$$HPTS(d+1, A, 4)_m \leqslant G(d+1, A, 4)_m.$$

Let us assume inductively that such an inequality holds for each $B < A$; without loss of generality we may also assume that

$$G(d, A, 4)_m < \binom{m+3}{3}.$$

First, if $k_1 + \ell \leqslant m+1$ we may take $B = (k_1, \dots, k_d)$ and then

$$G(d+1, A, 4)_m = G(d, B, 4)_m + \binom{\ell+2}{3},$$

and we are done by the induction hypothesis with respect to B.

So let us assume that $k_1 + \ell \geqslant m+2$. We now take $B = (k_1, \dots, k_d, \ell - 1)$ and then $C = (c_1, \dots, c_s) \in \mathbb{N}^s$ by rewriting the d-tuple $(k_1, \dots, k_d) - \overline{m+1-\ell}$ so that only positive terms occur. (Namely, we take $s \leqslant d$ maximal with respect to the property that $k_s + k \geqslant m+2$. and then $C = (k_1, \dots, k_s) - \overline{m+1-k}$.) By Lemma 6.3 we have

$$HPTS(d+1, A, 4)_m \leqslant HPTS(d+1, B, 4)_m + \binom{\ell+1}{2} - HPTS(s, C, 3)_{\ell-1}.$$

By the induction hypothesis, we are done once we see that $HPTS(s, C, 3)_{\ell-1} = G(s, C, 3)_{\ell-1}$.

If $s \leqslant 4$ we are done (see Section 4). Next, let us observe that $s \leqslant 6$. Note, first, that for each $a \in \mathbb{N}$ and each $m \leqslant 2a - 1$,

$$G(8, \bar{a}, 4)_m = \binom{m+3}{3}. \tag{5}$$

(To see this it suffices to evaluate $G(8, \bar{a}, 4)_m$ for $m = 2a - 1$, where G computes the degree of the scheme.) Hence if $s \geqslant 7$, we would have

$$G(d, A, 4)_m \geqslant G\bigl(8, (k, \dots, k, \ell), 4\bigr)_m,$$

where $k = k_s$. It is easy to see, by the numerical observation lemma, that we may find an integer a for which the latter item is at least $G(8, \bar{a}, 4)_m$ and $2a \geqslant m+1$. So Eq. (5) contradicts our hypothesis on $G(d, A, 4)_m$.

Thus we are left with the cases $5 \leqslant s \leqslant 6$. From Nagata's results [N] (see Section 3) we have

$$HPTS(s, C, 3)_{\ell-1} = G(s, C, 3)_{\ell-1}$$

provided that

$$\sum_{i=1}^{5}(k_i + \ell - 1 - m) \leqslant 2(\ell - 1) + 1.$$

Let us claim, then, that we do have this inequality due to the hypotheses on A. That is, imagine that $\sum_{i=1}^{5}(k_i + \ell - 1 - m) \geqslant 2(\ell - 1) + 2$. Let us check that

$$G(d + 1, A, 4)_m = \binom{m+3}{3}.$$

Since $A \geqslant (k_1, \dots, k_5, \ell)$ it is enough to replace A by the latter.

As before we may find an integer a for which $G(d, A, 4)_m \geqslant G(6, \bar{a}, 4)_m$ and $5(2a - 1 - m) \geqslant 2(a - 1) - 3$, so $8a \geqslant 5m$. In particular $2a \geqslant m + 2$ but $3a < 2m$ and we have

$$G(6, \bar{a}, 4)_m = 6\binom{a+2}{3} - \binom{6}{2}\binom{2a+1-m}{3}.$$

It is easy to compute (substitute $m = \lfloor 8a/5 \rfloor$, say) that the latter quantity is at least $\binom{m+3}{3}$, as claimed. $\square$

We remark that the complete classification of maximal rank in $\mathbb{P}^2$ is not required in the above proof. This phenomenon extends to higher dimension: one may explicitly identify (according to Theorem 1.1 along with combinatorics) which instances of the Strong Hypothesis may be required to verify the Weak Conjecture in $\mathbb{P}^n$.

8. General consequences

Here we shall derive some immediate consequences of the results obtained in Section 6 toward $\mathbb{P}^n$. We focus on the "first order" cases of the Weak Fröberg–Iarrobino Conjecture; namely, where expected base loci do not include planes of dimension two. In the previous case of $\mathbb{P}^3$ these occurred as the only nontrivial situations to verify; we see here the extension of arguments to higher dimension. For this, we compare (via Lemma 8.6) the behaviour of the Hilbert function in a given degree m with that of each $(A - \bar{i})$-scheme in degree $m - i$, respectively. This provides a verification of Conjecture 5.4 in the first-order situation.

Let us start with the following four observations by applying Theorem 1.1 directly.

Corollary 8.1. *Let $A = (k_1, \ldots, k_d) \in \mathbb{N}^d$. Assume that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d$. Then*

$$HPTS(d, A, n+1)_m \leqslant G(d, A, n+1)_m$$

for $m \geqslant k_1 + k_2 - 3$.

Proof. If

$$G(d, A, n+1)_m = \binom{n+m}{m}$$

(such as in the case $m \leqslant k_1 - 1$) we are done.

Assume, by induction on the number of points and the orders, that the result holds for all $B < A$ (such as $|B| = 1$).

Call $B = (k_1 - 1, k_2, \ldots, k_d)$, and $C = (c_2, \ldots, c_d)$ where $c_i = k_1 + k_i - 1 - m$. Then $c_i \leqslant 2$ for $i = 2, \ldots, d$.

We have

$$G(d, A, n+1)_m = G(d, B, n+1)_m + \binom{n+k-2}{n-1} - G(d-1, C, n)_{k-1}$$

and

$$G(d, A, n+1)_m > G(d, B, n+1)_m,$$

so that

$$G(d-1, C, n)_{k-1} < \binom{n+k-2}{n-1}.$$

By the Alexander–Hirschowitz theorem [H1,A,AH1,AH2,AH3] we have

$$HPTS(d-1, C, n)_{k-1} = G(d-1, C, n)_{k-1}.$$

Whence by Lemma 6.3 we find

$$HPTS(d, A, n+1)_m \leqslant HPTS(d, B, n+1)_m + \binom{n+k-2}{n-1} - G(d-1, C, n)_{k-1},$$

which by the induction hypothesis is at most:

$$G(d, B, n+1)_m + \binom{n+k-2}{n-1} - G(d-1, C, n)_{k-1} = G(d, A, n+1)_m. \qquad \square$$

Corollary 8.2. *Let $n, d, k \in \mathbb{N}$. Then*

$$HPTS(d, \bar{k}, n+1)_{2k-2} \leqslant \binom{n+k-1}{n} d - \binom{d}{2}.$$

Corollary 8.3. *Let $n, d, k \in \mathbb{N}$. Assume that $k \geqslant 4$. Then*

$$HPTS(d, \bar{k}, n+1)_{2k-3} \leqslant \binom{n+k-1}{n} d - (n+1)\binom{d}{2}.$$

Corollary 8.4. *Let $n, d, \in \mathbb{N}$ and $A = (k_1, \ldots, k_d) \in \mathbb{N}^d$. If $\max\{k_i\} \leqslant 4$ then for each degree m we have*

$$HPTS(d, A, n+1)_m \leqslant G(d, A, n+1)_m.$$

Corollary 8.5. *Let $n, d, k \in \mathbb{N}$. Assume that for each $d_1 \leqslant d$, $k_1 \leqslant k-1$, and $m_1 \geqslant 2k_1 - 1$ a generic union of d_1 k_1-tuple points of $\mathbb{P}^{n-1}$ has maximal rank with respect to $|\mathcal{O}_{\mathbb{P}^{n-1}}(m_1)|$ for every degree $m_1 \geqslant 2k_1 - 1$.*

Then in each degree m with $2m \geqslant 3k - 2$ we have

$$HPTS(d, k, n+1)_m \leqslant G(d, k, n+1)_m.$$

One obtains an analogous conclusion to that of Corollary 8.5 on an A-scheme of mixed multiplicities, $A = (k_1, \ldots, k_d)$ where $k_1 \geqslant \cdots \geqslant k_d$ and the degree m under consideration satisfies $2m \geqslant k_1 + 2k_2 - 2$. Further, we shall see in Corollary 8.7 how to simplify the hypotheses of the above result and apply toward A-schemes, homogeneous or otherwise.

Let us continue the examination of situations in which the base locus of a system of m-ics through a general collection of fat points is expected to contain lines but not planes. We aim toward simplifying the use of Theorem 1.3 under such a circumstance. In Corollary 8.8 this gives a result directly comparable to that of Iarrobino in the setting of the Fröberg conjectures.

The main instrument is the following.

Lemma 8.6. *Let $n, m, d \in \mathbb{N}$ and $A \in \mathbb{Z}^d$. Given a generic A-subscheme $\mathcal{Z} \subset \mathbb{P}^n$, let $\mathcal{Z}^{-r}$ denote the corresponding $(A - \bar{r})$-subscheme of $\mathcal{Z}$ for each $r \in \mathbb{N}$.*

If

$$h_{\mathbb{P}^n}\big(\mathcal{Z}^{-1}, m-1\big) = \deg \mathcal{Z}^{-1} - \alpha$$

then

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) \leqslant \deg \mathcal{Z} - \alpha.$$

Therefore, if

$$h_{\mathbb{P}^n}(\mathcal{Z}, m) = \deg \mathcal{Z},$$

we have

$$h_{\mathbb{P}^n}(\mathcal{Z}^{-r}, m-r) = \deg \mathcal{Z}^{-r}$$

for each $r = 0, \ldots, m$.

Proof. Let $\Gamma = \mathcal{Z}_{\mathrm{red}}$, and write $A = (k_1, \ldots, k_d)$. We may assume without loss of generality that $k_i \geqslant 2$ for each i from 1 to d.

Assume that $h_{\mathbb{P}^n}(\mathcal{Z}^{-1}, m-1) = \deg \mathcal{Z}^{-1} - \alpha$.

Take the homogeneous coordinate ring $S = \mathcal{K}[X_0, \ldots, X_n]$ of $\mathbb{P}^n$ with coordinates chosen so that none of the points $[1:0:\cdots:0]$, up to $[0:\cdots:0:1]$ lie on $\mathcal{Z}$. View $R = \mathcal{K}[X_1, \ldots, X_n]$ as the coordinate ring of the hyperplane $\mathbb{P}^{n-1}$ described by the form X_0, and take π_p as the projection from the points $p = [1:0:\cdots:0]$ onto $\mathbb{P}^{n-1}$.

We obtain then that the ideal $I(\pi_p(\Gamma)) = I(\Gamma) \cap R$ has $I(\Gamma) \cap R_m = d$. Consider the exact sequences in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R_m \arrow[r] & S_m \arrow[r, "\partial/\partial X_0"] & S_{m-1} \arrow[r] & 0 \\
0 \arrow[r] & I(\Gamma)_m \cap R_m \arrow[r] \arrow[u, hook] & I(\Gamma)_m \arrow[r, "\partial/\partial X_0"] \arrow[u, hook] & S_{m-1} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Let us filter:

$$S_m \supseteq I(\Gamma)_m = V_0 \supset V_1 \supset \cdots \supset V_n = I(\mathcal{Z})_m,$$

where

$$V_j = \left\{ F \in V_0 \colon \frac{\partial F}{\partial X_i} \in I\bigl(\mathcal{Z}^{-1}\bigr),\ i = 1, \ldots, j \right\}.$$

From the diagram

$$\begin{tikzcd}
0 \arrow[r] & V_0 \cap R_m \arrow[r] & V_0 \arrow[r, "\partial/\partial X_0"] & S_{m-1} \arrow[r] & 0 \\
0 \arrow[r] & V_1 \cap R_m \arrow[r] \arrow[u, hook] & V_1 \arrow[r, "\partial/\partial X_0"] \arrow[u, hook] & I(\mathcal{Z}^{-1})_m \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

we see that $\dim V_0/V_1 \leqslant \deg \mathcal{Z}^{-1} - \alpha$. Routinely we obtain

$$\dim V_j/V_{j+1} \leqslant \sum_{i=1}^{d} \binom{n + k_i - 1 - j}{n - j}$$

for $j = 1, \ldots, n-1$ (namely, the degree of an $(A - \bar{1})$-subscheme of $\mathbb{P}^{n-j}$. In sum we then obtain the desired inequality. $\square$

Remark. The argument applies equally to an A-scheme with arbitrary support $\Gamma \subset \mathbb{P}^n$ provided that the projection $\pi_p(\Gamma) \subset \mathbb{P}^{n-1}$, as in the preceding proof, does have $h_{\mathbb{P}^{n-1}}(\pi_p(\Gamma), m) = d$.

Corollary 8.7. *Let $n, m, d \in \mathbb{N}$, and $A = (k_1, \dots, k_d) \in \mathbb{N}^d$. Suppose that*

$$k_1 \leqslant \dots \leqslant k_d \quad \text{and} \quad k_{d-2} + k_{d-1} + k_d \leqslant 2m + 2.$$

Let

$$C = (c_1, \dots, c_{d-1}) := (k_1, \dots, k_{d-1}) - \overline{m + 1 - k_d}.$$

Suppose that

$$HPTS(d-1, C, n)_{k_d - 1} = G(d-1, C, n)_{k_d - 1}.$$

Then

$$HPTS(d, A, n+1)_m \leqslant G(d, A, n+1)_m.$$

Again, equality applies to a generic A-scheme if and only if no obstructions occur other than the expected linear ones.

Remark. We may likewise extend the preceding corollary to an arbitrary A-subscheme of $\mathbb{P}^n$ under the hypothesis that the scheme $\mathcal{W} \subset \mathbb{P}^n$ identified in Lemma 6.3 achieves the value of G in degree $k_d - 1$, and the projection of $\mathcal{Z}_{\mathrm{red}}$ to $\mathbb{P}^{n-1}$ attains maximal rank in degree m.

Proof. We may assume that

$$G(d, A, n+1)_m < \binom{n+m}{m}.$$

Take $C_{j,i} = (k_1, \dots, k_{j-1}) - \overline{m - i}$, for each $j = 2, \dots, d$ and $i = 0, \dots, k_j - 1$. By Theorem 1.3 we are done once we see that

$$HPTS(j-1, C_{j,i}, n)_i = G(j-1, C_{j,i}, n)_i$$

for all j, i. By hypothesis, equality holds for $(j, i) = (d, k_d - 1)$. So by Lemma 8.6 we obtain equality as well in the case of $(d-1, i)$ for all $i \leqslant k_d - 1$. According to our numerical hypothesis this says that for each $i \leqslant k_d - 1$, a $C_{d,i}$-scheme imposes independent conditions on i-ics. Hence its subschemes, notably, the $C_{j,i}$-schemes for $j = 1, \dots, k_d - 1$ do as well. □

Remark. Now let us compare the result of Corollary 8.7 with Theorem 4.3 of Iarrobino (see Section 4). Take

$$A = (k_1, \dots, k_d, k), \quad k \geqslant \max k_i,$$

and

$$C = (k_1, \dots, k_d) - \overline{m+1-k},$$

as in the statement of Corollary 8.7. Then the Macaulay dual of A in degree m is described by the $(d+1)$-tuple

$$A^{\perp} = (j_1, \dots, j_d, j)$$

where $j_i = m - k_i + 1$ and $j = m + 1 - k$, so $j \leqslant \min\{j_i\}$. The dual of C in degree $k-1$ $(= m - j)$ is given by

$$C^{\perp} = (j_1, \dots, j_d).$$

The hypothesis that $2m \geqslant k_{d-1} + k_d + k - 2$ (along with $m \leqslant k_d + k - 1$, to make things interesting, say) gives that $j_d + j \leqslant m \leqslant j_{d-1} + j_d + j - 1$; i.e., we are in the range described by Iarrobino. From his statement that

$$HGEN\big(d+1, A^{\perp}, n+1\big)_m \geqslant F\big(d+1, A^{\perp}, n+1\big)_m$$

if

$$HGEN\big(d, C^{\perp}, n+1\big)_{m-j} = F\big(d, C^{\perp}, n+1\big)_{m-j},$$

one may expect to obtain the upper bound $HPTS(d+1, A, n+1)_m \leqslant G(d+1, A, n+1)_m$ from information on a C-scheme living in $\mathbb{P}^n$.

By Corollary 8.7 we obtain the following statement.

Corollary 8.8. *Let $n, m, d, k \in \mathbb{N}$. Suppose that $A \in \mathbb{N}^{d+1}$ and $C \in \mathbb{Z}^d$ satisfy the numerical hypotheses of Corollary* 8.7*. Assume that $\mathcal{Z} \subset \mathbb{P}^n$ is a C-subscheme supported on a generic subset of a hyperplane $\mathbb{P}^{n-1}$. If*

$$h_{\mathbb{P}^n}(\mathcal{Z}, k-1) = G(d, C, n+1)_{k-1}$$

then

$$HPTS(d+1, A, n+1)_m \leqslant G(d+1, A, n+1)_m.$$

Proof. One should only notice from the sequence

$$0 \to \mathcal{I}_{\tilde{\mathcal{Z}}}(k-2) \to \mathcal{I}_{\mathcal{Z}}(k-1) \to \mathcal{I}_{\mathcal{Z}\cap\mathbb{P}^{n-1},\mathbb{P}^{n-1}}(k-1) \to 0$$

we have $H^1(\mathbb{P}^{n-1}, \mathcal{I}_{\mathcal{Z}\cap\mathbb{P}^{n-1},\mathbb{P}^{n-1}}(k-1)) = 0$, so that $HPTS(d, C, n)_{k-1} = G(d, C, n)_{k-1}$. By Lemma 8.6 we obtain that

$$HPTS(d, C - \bar{i}, n)_{k-1-i} = G(d, C - \bar{i}, n)_{k-1-i}$$

for each i from 0 to $k-1$, so

$$\sum_{i=0}^{k-1} H(d, C-\bar{i}, n)_{k-1-i} = G(d, C, n+1)_{k-1}.$$

Taking $B = (k_1, \ldots, k_d)$ we find that

$$HPTS(d+1, A, n+1)_m \leqslant HPTS(d, B, n+1)_m + \binom{n+k-1}{n} - G(d, C, n+1)_{k-1}.$$

Assuming inductively that $HPTS(d, B, n+1)_m \leqslant G(d, B, n+1)_m$ we see that, as advertised,

$$HPTS(d+1, A, n+1)_m \leqslant G(d+1, A, n+1)_m. \qquad \square$$

9. The algebraic reinterpretation

Let us look back to the algebraic conjectures of Fröberg and Iarrobino. Particularly, we shall stick to the cases of appropriate characteristic and number of points.

As remarked earlier, the Strong Fröberg Conjecture derives from the expectation that the minimal free resolution of an ideal I generated by general forms should exhibit only Koszul relations "as much as possible" with respect to degrees. But, of course, $\dim I_m$ need not itself predict the resolution in degree m.

However, in the case of the Strong Algebraic Fröberg–Iarrobino conjecture we obtain the extra information from Theorem 1.3. Namely, when we dualise to the study of multiple points, the equality of *HPTS* with the function G should predict (according to Conjecture 5.4) only linear obstructions. Redualising and interpreting linear obtstructions by Macaulay methods, such an equality yields the information that the corresponding syzygies are exactly as predicted. Whence the Strong Conjecture of Fröberg–Iarrobino not only implies that of Fröberg but gives desired conclusions on the resolution.

Proposition 9.1. *Assume that Conjecture* 5.4 *holds. Suppose that $I \subset S = \mathcal{K}[X_0, \ldots, X_n]$ is an ideal generated by powers of linear forms, and that I satisfies the Strong Algebraic Fröberg–Iarrobino Conjecture. Take M maximal for which $I_M \neq S_M$ (as determined by the conjecture). Then for each $m \leqslant M$ the mth graded piece of the minimal free resolution is Koszul.*

Notice how Proposition 10.1 compares with a recent result of [MMR]. There an ideal $(F_1, \ldots, F_d)$ given by a general collection of forms is considered under the hypothesis that each of the ideals $(F_1, \ldots, F_j)$ satisfies the Strong Fröberg Conjecture for $j = 1, \ldots, d$. The conclusion on resolution is then just as in the proposition above.

So our assumptions in the proposition are partly stronger, partly weaker than those in [MMR], but with the bonus of geometric insight.

Acknowledgments

I am very grateful to Tony Iarrobino for his inspiring my work along this direction and for discussions, and to Brian Harbourne for his useful remarks. Thanks also to Joe Harris for sharing his comments.

References

[A] J. Alexander, Singularités imposables en position générale aux hypersurfaces de $\mathbb{P}^n$, Compositio Math. 68 (3) (1988) 305–354.

[AH1] J. Alexander, A. Hirschowitz, Un lemme d'Horace différentiel: application aux singularités hyperquartiques de $\mathbb{P}^5$, J. Algebraic Geom. 1 (3) (1992) 411–426.

[AH2] J. Alexander, A. Hirschowitz, La méthode d'Horace éclatée: application à l'interpolation en degré quatre, Invent. Math. 107 (3) (1992) 585–602.

[AH3] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, J. Algebraic Geom. 4 (2) (1995) 201–222.

[AH4] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, Invent. Math. 140 (2) (2000) 303–325.

[An] D. Anick, Thin algebras of embedding dimension three, J. Algebra 100 (1) (1986) 235–259.

[Au] M. Aubry, Série de Hilbert d'une algèbre de polynômes quotient, J. Algebra 176 (2) (1995) 392–416.

[CEG] M.V. Catalisano, P. Ellia, A. Gimigliano, Fat points on rational normal curves, J. Algebra 216 (2) (1999) 600–619.

[C1] K. Chandler, Geometry of dots and ropes, Trans. Amer. Math. Soc. 347 (3) (1995) 767–784.

[C2] K. Chandler, Higher infinitesimal neighbourhoods, J. Algebra 205 (2) (1998) 460–479.

[C3] K. Chandler, A brief proof of a maximal rank theorem for generic double points in projective space, Trans. Amer. Math. Soc. 353 (5) (2000) 1907–1920.

[C4] K. Chandler, Linear systems of cubics singular at general points of projective space, Compositio Math. 134 (3) (2002) 269–282.

[C5] K. Chandler, Uniqueness of exceptional singular quartic, Proc. Amer. Math. Soc. 132 (2) (2004) 347–352.

[C6] K. Chandler, Classes of examples and counterexamples to the Strong Fröberg–Iarrobino Conjectures, preprint.

[C7] K. Chandler, Triple points in projective space, preprint.

[C8] K. Chandler, Multiple points on a rational normal curve, in preparation.

[Ci] C. Ciliberto, Geometric aspects of polynomial interpolation in more variables and of Waring's problem, in: European Congress of Mathematics, vol. I, Barcelona, 2000, in: Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 289–316.

[CM1] C. Ciliberto, R. Miranda, Degenerations of planar linear systems, J. Reine Angew. Math. 501 (1998) 191–220.

[CM2] C. Ciliberto, R. Miranda, Linear systems of plane curves with base points of equal multiplicity, Trans. Amer. Math. Soc. 352 (9) (2000) 4037–4050.

[CM3] C. Ciliberto, R. Miranda, The Segre and Harbourne–Hirschowitz conjectures, in: Applications of Algebraic Geometry to Coding Theory, Physics and Computation, Eilat, 2001, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 36, Kluwer Academic, Dordrecht, 2001, pp. 37–51.

[CCMO] C. Ciliberto, F. Cioffi, R. Miranda, F. Orecchia, Bivariate Hermite interpolation and linear systems of plane curves with base fat points, in: Lecture Notes Ser. Comput., vol. 10, World Scientific, 2003, pp. 87–102.

[EI] J. Emsalem, A. Iarrobino, Inverse system of a symbolic power, I, J. Algebra 174 (3) (1995) 1080–1090.

[F] R. Fröberg, An inequality for Hilbert series of graded algebras, Math. Scand. 56 (2) (1985) 117–144.

[FH] R. Fröberg, J. Hollmann, Hilbert series for ideals generated by generic forms, J. Symbolic Comput. 17 (2) (1994) 149–157.

[GS] M. Gasca, T. Sauer, Polynomial interpolation in several variables, Adv. Symbolic Comput. 12 (2000) 377–410.

[Ha1] B. Harbourne, The geometry of rational surfaces and Hilbert functions of points in the plane, in: Canad. Math. Soc. Conf. Prof., vol. 6, Amer. Math. Soc., Providence, RI, 1986, pp. 95–111.

[Ha2] B. Harbourne, Points in good position in $\mathbb{P}^2$, in: Zero-Dimensional Schemes, Ravello, 1992, de Gruyter, Berlin, 1994, pp. 213–229.

[Ha3] B. Harbourne, An algorithm for fat points on $\mathbf{P}^2$, Canad. J. Math. 52 (1) (2000) 123–140.

[Ha4] B. Harbourne, Problems and progress: A survey on fat points in $\mathbb{P}^2$, in: Zero-Dimensional Schemes and Applications, Naples, 2000, in: Queen's Papers in Pure and Appl. Math., vol. 123, Queen's Univ., Kingston, ON, 2002, pp. 85–132.

[H1] A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusieurs variables, Manuscripta Math. 50 (1985) 337–388.

[H2] A. Hirschowitz, Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles génériques, J. Reine Angew. Math. 397 (1989) 208–213.

[HL] M. Hochster, D. Laksov, The linear syzygies of generic forms, Comm. Algebra 15 (1–2) (1987) 227–239.

[I] A. Iarrobino, Inverse system of a symbolic power III: thin algebras and fat points, Compositio Math. 108 (3) (1997) 319–356.

[IK] A. Iarrobino, V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Lecture Notes in Math., vol. 1721, Springer-Verlag, Berlin, 1999.

[MMR] J. Migliore, R. Miro-Róig, Ideals of general forms and the ubiquity of the Weak Lefschetz Property, J. Pure Appl. Algebra, in press.

[N] M. Nagata, On rational surfaces, II, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 33 (1960) 271–293.

[S] B. Segre, Alcune questioni su insiemi finiti di punti in geometria algebrica, Atti Convegno Intern. Geom. Algebr. Torino, 1962, pp. 15–33.

[St] R. Stanley, Hilbert functions of graded algebras, Adv. Math. 28 (1) (1978) 57–83.