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On the maximum rank of a real binary form

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Abstract

We show that a real binary form f of degree $n \geq 3$ has n distinct real roots if and only if for any $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}$ all the forms $\alpha f_x + \beta f_y$ have $n-1$ distinct real roots. This answers to a question of P. Comon and G. Ottaviani in [1], and allows to complete their argument to show that f has symmetric rank n if and only if it has n distinct real roots.

1 Introduction

This paper deals with the following problem: Given a degree n polynomial $f \in \mathbb{K}[x_1, \ldots, x_m]$ find the rank (or Waring rank) of f i.e. the minimum number of summands which achieve the following decomposition:

$$f = \lambda_1 l_1^n + \cdots + \lambda_r l_r^n \quad \text{with } \lambda_i \in \mathbb{K} \text{ and } l_i \text{ linear forms.}$$

For $\mathbb{K} = \mathbb{C}$ and f generic the answer has been given (see [2, 3]), nevertheless some questions remain unsolved, e.g. it's not yet known which is the stratification of the set of complex polynomials by the rank. However one can see [4] for an answer in the binary case.
In the real case, i.e. $\mathbb{K} = \mathbb{R}$, the situation becomes more complicated. In contrast to the complex case which has a generic rank, in the real case the generic rank is substituted by the concept of typical rank. A rank k is said *typical* for a given degree n if there exists a euclidean open set in the space of real degree n polynomials such that any f in such open et has rank k. We will prove the following theorem, posed as a question in [1].

Theorem 1. *Let $f(x, y)$ be a real homogeneous polynomial of degree $n \geq 3$ without multiple roots in $\mathbb{C}$. Then f has all real roots if and only if for any $(0,0) \neq (\alpha, \beta) \in \mathbb{R}^2$ the polynomial $\alpha f_x + \beta f_y$ has $n-1$ distinct real roots.*

Notice that the "only if" part of the theorem is easy. Indeed given any $(\alpha, \beta) \neq (0,0)$ one may consider a new coordinate system l, m on the projective line, such that $x = \alpha l + \alpha' m$, and $y = \beta l + \beta' m$, so that $\partial_l = \alpha \partial_x + \beta \partial_y$. Writing f as a function of l, m and de-homogenizing by setting $m = 1$ one sees that f_l has $n-1$ distinct roots by the theorem of Rolle.

In [1] the result above has been considered in connection with the problem of determining the *rank* of a real binary form, that is the minimum number r such that $f(x, y) = \lambda_1 l_1^n + \cdots + \lambda_r l_r^n$, with $\lambda_i \in \mathbb{R}$ and $l_i = \alpha_i x + \beta_i y \in \mathbb{R}[x, y]$

for $i = 1, \dots r$. Using the arguments already given in [1] and applying Theorem 1, one gets the following result.

Corollary 1. *A real binary form $f(x, y)$ of degree $n \geq 3$ without multiple roots in $\mathbb{C}$ has rank n if and only if it has n distinct real roots.*

We leave the following question open for further investigations. Partial evidence for it has been given from the results in [1], where it has given a positive answer for $n \leq 5$, and where the reader can find references for the existing literature on rank problems for real tensors.

Question 1. *Are all the ranks $\lfloor n/2 \rfloor + 1 \leq k \leq n$ typical for real binary forms of degree n?*

2 Main Theorem

Let $f(x, y)$ be a real homogeneous polynomial of degree $n \geq 3$ without multiple roots in $\mathbb{C}$. Then $\nabla f(x, y) \neq (0, 0)$ for any $(x, y) \neq (0, 0)$ and one can define the maps $\bar{\phi} : S^1 \to S^1$ and $\bar{\psi} : S^1 \to S^1$ setting, for any (x, y) with $x^2 + y^2 = 1$, $\bar{\phi}(x, y) = |\nabla f|^{-1}(f_x, f_y)$ and $\bar{\psi}(x, y) = |\nabla f|^{-1}(xf_x + yf_y, -yf_x + xf_y)$, with $|\nabla f| = (f_x^2 + f_y^2)^{1/2}$. Setting $(x, y) = (\cos\theta, \sin\theta)$, one can also write $\bar{\phi}$ and $\bar{\psi}$ as functions of θ.

Notation. We denote $\partial_\theta = -y\partial_x + x\partial_y$ the basis tangent vector to S^1 at the point (x, y). Given any differentiable map $\phi : S^1 \to M$ to a differentiable manifold M, we denote $\phi_* : T_\theta S^1 \to T_{\phi(\theta)}M$ the associated tangent map. If $M = S^1$, and the map ϕ is defined in terms of angular coordinates by the function $\theta_1(\theta)$, we recall that the *degree*, or *winding number*, of ϕ is the number

$$\deg \phi = \frac{1}{2\pi} \int_0^{2\pi} \theta_1'(\theta) d\theta.$$

This is always an integer number, and for any $z \in S^1$ one has $\#\phi^{-1}(z) \geq |\deg \phi|$.

The following lemmas are straightforward calculations and their proofs are omitted.

Lemma 1. *Assume that $\theta_1'(\theta)$ never vanishes. Then $\#\phi^{-1}(z) = |\deg \phi|$ for any $z \in S^1$.*

We assume that for any $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{R})$ the polynomial $\alpha f_x + \beta f_y$ has $n - 1$ distinct roots in $\mathbb{R}$. Under this assumption, we want to show that the absolute value of the degree of $\bar{\psi}$ is n. Since $f(x, y) = 0$ iff $\bar{\psi}(x, y) = (0, \pm 1)$ this implies that f has all its roots in $\mathbb{R}$. Indeed $\bar{\psi}(-x, -y) = (-1)^n \bar{\psi}(x, y)$, henceforth when n is even $n/2$ real roots of $f(x, y) = 0$ are in $\bar{\psi}^{-1}(0, 1)$ and the other $n/2$ roots are in $\bar{\psi}^{-1}(0, -1)$; otherwise when n is odd one gets $\bar{\psi}^{-1}(0, 1) = \bar{\psi}^{-1}(0, -1)$, hence $\bar{\psi}^{-1}(0, 1)$ is the set of the n real roots of $f(x, y) = 0$.

Lemma 2. *Let $F : S^1 \to \mathbb{R}^2$ be a differentiable function defined by $F(x,y) = (F_1(x,y), F_2(x,y)) = (a(\theta), b(\theta))$. Then $F_*(\partial_\theta) = A\partial_x + B\partial_y$ with*

$$\begin{aligned} A &= -yF_{1x} + xF_{1y} = a'(\theta) \\ B &= -yF_{2x} + xF_{2y} = b'(\theta). \end{aligned}$$

Notation. Given a map $f : S^1 \to \mathbb{R}^2$, which one can write $f(\theta) = (a(\theta), b(\theta))$, we denote with (f, f_θ) the matrix

$$\begin{pmatrix} a(\theta) & b(\theta) \\ a'(\theta) & b'(\theta) \end{pmatrix}.$$

Notice that the sign of the determinant of this matrix expresses
if f_* is orientation-preserving at the point $f(\theta)$.

Lemma 3. *Let $g : S^1 \to \mathbb{R}^2$ and $\rho : S^1 \to \mathbb{R}_+$ be differentiable functions. Then $\det(g, g_\theta) = \rho^{-2} \det(\rho g, (\rho g)_\theta)$.*

Notice that if $\bar{g} : S^1 \to S^1$ is the map $\bar{\phi}(x,y) = |\nabla f|^{-1}(f_x, f_y)$ then one may calculate the sign of $\det(\bar{\phi}, \bar{\phi}_\theta)$ by reducing to the simpler map $\phi = (f_x, f_y) : S^1 \to \mathbb{R}^2$.

Notation. We denote by $H(f) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$, the *hessian* of f.

Proposition 1. *Let $\phi : S^1 \to \mathbb{R}^2$ be the map defined above. Then*

$$\det(\phi, \phi_\theta) = (n-1)^{-1} H(f).$$

Proof. We have $\phi_*(\partial_\theta) = A\partial_x + B\partial_y$ with A and B determined as in Lemma 2, hence

$$\begin{aligned} \det(\phi, \phi_\theta) &= \det \begin{pmatrix} f_x & f_y \\ -yf_{xx} + xf_{xy} & -yf_{yx} + xf_{yy} \end{pmatrix} \\ &= \frac{1}{n-1} \det \begin{pmatrix} xf_{xx} + yf_{xy} & xf_{yx} + yf_{yy} \\ -yf_{xx} + xf_{xy} & -yf_{yx} + xf_{yy} \end{pmatrix} \\ &= \frac{1}{n-1} \det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \\ &= \frac{1}{n-1} H(f) \end{aligned}$$

□

Proposition 2. *Let $\phi : S^1 \to \mathbb{R}^2 \cong \mathbb{C}$ defined by $\phi(\theta) = a(\theta) + ib(\theta)$ and $\psi : S^1 \to \mathbb{C}$ defined by $\psi(\theta) = e^{-i\theta}\phi(\theta)$. Then $\det(\psi, \psi_\theta) = \det(\phi, \phi_\theta) - a^2 - b^2$.*

Proof. We calculate $\psi'(\theta) = (a' + b + i(b' - a))e^{-i\theta}$. It follows that

$$\det(\psi, \psi_\theta) = \det \begin{pmatrix} a & b \\ a' + b & b' - a \end{pmatrix} = \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} - a^2 - b^2.$$

□ □

Notice that by Lemma 3, if $\phi : S^1 \to \mathbb{R}^2$ is given in polar coordinates by $\phi(\theta) = \rho(\theta)\bar{\phi}(\theta)$, with $\bar{\phi} : S^1 \to S^1$, then $\det(\phi, \phi_\theta) = \rho^2 \det(\bar{\phi}, \bar{\phi}_\theta)$. Moreover, expressing $\bar{\phi}$ in terms of angular coordinates by means of a function $\theta_1(\theta)$, one sees easily that $\det(\bar{\phi}, \bar{\phi}_\theta) = \theta_1'(\theta)$. We are interested in $\bar{\phi} = |\nabla f|^{-1}(f_x, f_y)$ and $\bar{\psi} = |\nabla f|^{-1}(xf_x + yf_y, -yf_x + xf_y)$. In this case we get the following result.

Corollary 2. *In the notations above, the following statements hold.*

1. $\det(\bar{\phi}, \bar{\phi}_\theta) = \theta_1'(\theta) = (n-1)^{-1}|\nabla f|^{-2}H(f)$.
2. $\deg \bar{\psi} = \deg \bar{\phi} - 1$.

Proof. The first statement follows form Proposition 1 and Lemma 3. The second one follows from Proposition 2 and Lemma 3 since

$$\deg \bar{\psi} = \frac{1}{2\pi}\int \det(\bar{\psi}, \bar{\psi}_\theta) = \frac{1}{2\pi}\int \det(\bar{\phi}, \bar{\phi}_\theta) - 1 = \deg \bar{\phi} - 1.$$

□

Now we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. Since for any $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}$ the polynomial $\alpha f_x + \beta f_y$ has $n-1$ distinct real roots, then the map $(f_x, f_y) : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$ has no ramification at any real point of $\mathbb{P}^1_{\mathbb{R}}$. Equivalently, the jacobian of $\bar{\phi}$ which is equal to the hessian $H(f)$ is always non-zero at the real points of $\mathbb{P}^1_{\mathbb{R}}$. We call the map $\bar{\phi} : S^1 \to S^1$ defined by $\bar{\phi} = |\nabla f|^{-1}(f_x, f_y)$ and we also express it as $\theta_1 = \theta_1(\theta)$ in angular coordinates. By the observation above and Corollary 2 it follows that the derivative $\theta_1'(\theta)$ is non vanishing at any $\theta \in S^1$. Hence $\theta_1'(\theta)$ is either always positive or always negative.
Claim: $\theta_1'(\theta) < 0$ for any θ.

The sign of $\theta_1'(\theta)$ is the same than the sign of $H(f)$. Since we already know that it is constant it will be sufficient to evaluate it at a single point $(x, y) \in S^1$. We choose to examine the point $(1, 0)$. One observes that for any binary form $g(x, y) = \binom{m}{0}a_0x^m + \binom{m}{1}a_1x^{m-1}y + \cdots + \binom{m}{m}a_my^m$ of degree $m \geq 3$, the Hessian $H(g)$ calculated at $(1, 0)$ is equal to

$$m(m-1)\det\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}.$$

Similarly the Hessian of its derivative g_x at $(1, 0)$ is given by

$$m(m-1)(m-2)\det\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}.$$

Therefore we find that

$$H(g)(1, 0) = (m-2)^{-1}H(g_x)(1, 0).$$

Applying this result to $g = f$, we are reduced to compute the sign of $H(f_x)$. We know that f_x has $n-1$ distinct real roots, so all of its derivatives $\partial_x^i(f_x)$ have all their roots real and distinct, up to $i = n-3$. The last of these derivatives is $h = \partial_x^{n-2} f$, and its Hessian is a constant equal to $-\Delta(h)$, hence $H(h) < 0$. Applying recursively the reduction step, we find that $H(f)(1,0) = (n-2)^{-1} H(f_x)(1,0) = ((n-2)!)^{-1} H(h) < 0$, proving the claim.

By Corollary 2(*1.*), Lemma 1 and applying the claim above, we get that $\deg \bar{\phi} < 0$ and $\#\phi^{-1}(z) = |\deg \phi|$ for any $z \in S^1$, hence $\deg \bar{\phi} = -n + 1$. Moreover, by Corollary 2(*2.*), we have $\deg \bar{\psi} = \deg \bar{\phi} - 1 = -n$, hence $\#\text{real roots}(f) \geq |\deg \bar{\psi}| = n$. This completes the proof of the Theorem. $\square$

We conclude giving a self-contained proof of the result on the rank of a real binary form mentioned in the introduction. The arguments given are all already in [1].

Proof of Corollary 1. The statement holds for $n = 3$, as shown in [1], Proposition 2.2. Assuming $n > 3$, suppose the statement holds in degree $n-1$. Assume $\text{rank}(f) = r$, so one can write $f = \lambda_1 l_1^n + \cdots + \lambda_r l_r^n$, with r minimal. Then one can consider $l = l_1$ and $m = l_r$ and $g(t) = m^{-n} f$, with $t = l/m$. One sees that $m^{-n+1} f_l = g'(t)$ can be expressed as a sum of at most $r-1$ n-th powers of linear forms in t. If f has n distinct real roots then, by induction hypothesis f_l has $n-1$ distinct real roots and we find $r - 1 \geq n - 1$, i.e. $r \geq n$. Since the inequality $r \leq n$ always holds, as shown in [1] Proposition 2.1, we have $r = n$. Conversely, if the rank of f is n then take $r = n$ and consider any derivative $\alpha f_x + \beta f_y = f_l$, after defining a suitable coordinate system l, m, as explained in the introduction. We can consider the polynomial $g'(t) = m^{-n+1} f_l$. If it has rank $< n-1$ then by indefinite integration over t one sees easily that f has rank$< n$, contrary to the assumption. So $\text{rank}(f_l) = n-1$ and, by induction hypothesis, it also holds that f_l has $n-1$ distinct real roots. By the arbitrariness of l and by Theorem 1, we conclude that f has n distinct roots. $\square$

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