MONOMIALS AS SUMS OF k^{th}-POWERS OF FORMS

ENRICO CARLINI AND ALESSANDRO ONETO

ABSTRACT. Motivated by recent results on the Waring problem for polynomial rings [FOS12] and representation of monomial as sum of powers of linear forms [CCG12], we consider the problem of presenting monomials of degree kd as sums of k^{th}-powers of forms of degree d. We produce a general bound on the number of summands for any number of variables which we refine in the two variables case. We completely solve the $k = 3$ case for monomials in two and three variables.

1. INTRODUCTION

Let $S := \bigoplus_{i\in\mathbb{N}} S_i = \mathbb{C}[x_0, \ldots, x_n]$ be the ring of polynomials in $n+1$ variables with complex coefficients and with the standard gradation. Given a homogeneous polynomial, or form, $F \in S_k$ of degree $k \geq 2$, we can ask what is the minimal number of linear forms needed to write F as sum of their k^{th}-power. The problems concerning this *additive decomposition of forms* are called *Waring problems for polynomials* and such minimal number is usually called *Waring rank*, or simply rank, of F.

In the last decades, this kind of problems attracted a great deal of work. In 1995, J.Alexander and A.Hirschowitz determined the rank of the generic form [AH95]. However, given an explicit form F, to compute the Waring rank of F is more difficult and we know the answer only in a few cases. One of these cases is the monomial case.

In [CCG12], E. Carlini, M.V. Catalisano and A.V. Geramita gave an explicit formula to compute the Waring rank of a given monomial in any number of variables and any degree.

In [FOS12], R. Fröberg, G. Ottaviani and B. Shapiro considered a more general Waring problem. Given a form $F \in \mathbb{C}[x_0, \ldots, x_n]$ of degree kd, one can ask what is the minimal number of forms of degree d needed to write F as sum of their k^{th}-powers.

Definition 1.1. Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a form of degree kd with $k \geq 2$, we set

$$\#_k(F) := \min\{s \mid F = g_1^k + \ldots + g_s^k\}$$

where g_i's are forms of degree d. We call $\#_k(F)$ the k^{th}-*Waring rank of* F, or simply the k^{th}-rank of F.

Clearly, the $d = 1$ case is the "standard" Waring problem. In [FOS12], the authors considered the $d \geq 2$ cases and they proved that *any* generic form of degree kd can be written as sum of k^n k^{th}-powers.

Motivated by these recent results about the Waring rank of monomials and about the mentioned generalization of the Waring problem, we started to investigate the k^{th}-Waring rank for monomials of degree kd.

In Section 3.1, we prove Theorem 3.2 stating that the k^{th}-rank of a monomial of degree kd is less or equal than 2^{k-1}, for any d and any number of variables. Then we focus on some special cases: the binary case in Section 3.2, and the case with three or more variables in Section 3.3. In the binary ($n = 1$) case we produce a general bound on $\#_k(M)$. While, for $n \geq 2$, we give a complete description of the $k = 3$ case.

The authors wish to thank R.Froeberg and B.Shapiro for their many helpful suggestions and wonderful ideas. The first author was a guest of the Department of Mathematics of the University of Stockholm when this work was started. The first author received financial support also from the KTH and the Fondo Giovani Ricercatori of the Politecnico di Torino.

2. Basic Facts

First we recall the result about the Waring rank of monomials mentioned above.

Theorem 2.1. [CCG12] *Given a monomial $M = x_0^{a_0} \ldots x_n^{a_n}$ of degree k such that $1 \leq a_0 \leq \ldots \leq a_n$, then*

$$\#_k(M) = \frac{1}{a_0 + 1} \prod_{i=0}^{n} (a_i + 1). \tag{1}$$

We now introduce some elementary tools to study the k^{th}-rank of monomials.

Remark 2.2. Consider a monomial M of degree kd in the variables $\{x_0, \ldots, x_n\}$. We say that a monomial M' of degree kd' in the variables $\{X_0, \ldots, X_m\}$ is a *grouping* of M if there exists a positive integer l such that $d = ld'$ and M can be obtained from M' by substituting each variable X_i with a monomial of degree l in the x's, i.e. $X_i = N_i(x_0, \ldots, x_n)$ for each $i = 1, \ldots, m$ with $\deg(N_i) = l$. The relation between the k^{th}-rank of M and M' is given by

$$\#_k(M') \geq \#_k(M).$$

Indeed, given a decomposition of M' as sum of k^{th}-powers, i.e.

$$M' = \sum_{i=1}^{r} F_i(X_0, \ldots, X_n)^k, \text{ with } \deg(F_i) = d',$$

we can write a decomposition for M by using the substitution given above, i.e.

$$M = \sum_{i=1}^{r} F_i(N_0(x_0, \ldots, x_n), \ldots, N_m(x_0, \ldots, x_n))^k.$$

Remark 2.3. Consider a monomial M of degree kd in the variables $\{x_0, \ldots, x_n\}$. We say that a monomial M' of the same degree is a *specialization* of M if M' can be found from M after a certain number of identifications of the type $x_i = x_j$. Again, it makes sense to compare the two k^{th}-ranks and we get

$$\#_k(M) \geq \#_k(M').$$

Indeed, given a decomposition of M as sum of r k^{th}-powers, we can write a decomposition for M' with the same number of summands applying the identifications between variables to each addend.

Remark 2.4. Consider a monomial M of degree kd_1 and N a monomial of degree d_2. We can look at the monomial $M' = MN^k$. Clearly the degree of M' is also divisible by k; again, it makes sense to compare the k^{th}-rank of M and M'. The relation is

$$\#_k(M) \geq \#_k(M').$$

Indeed, given a decomposition as sum of k^{th}-powers for M, e.g. $M = \sum_{i=1}^{r} F_i^k$ with F_i's forms of degree d_1, we can easily find a decomposition for M' with the same number of summands, i.e. $M' = MN^k = \sum_{i=1}^{r}(F_iN)^k$.

The inequality on the k^{th}-rank in Remark 2.4 can be strict in general as we can see in the following example.

Example 2.5. Consider $k = 3$ and the monomials $M = x_1x_2x_3$ and $M' = (x_0^2)^3M = x_0^6x_1x_2x_3$. By Theorem 2.1, we know that $\#_3(M) = \#_3(x_1x_2x_3) = 4$, but we can consider a *grouping* of the monomial M', i.e. $M' = (x_0^3)^2(x_1x_2x_3) = X_0^2X_1$. By Remark 2.2 and Theorem 2.1, we have $\#_3(M') \leq \#_3(X_0^2X_1) = 3$.

As a straightforward application of these remarks we get the following lemma which is useful to reduce the number of cases to consider once k and n are fixed.

Lemma 2.6. *Given a monomial* $M = x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}$ *of degree* kd*, then*

$$\#_k(M) \leq \#_k([M]),$$

where $[M] := x_0^{[a_0]_k}x_1^{[a_1]_k}\cdots x_n^{[a_n]_k}$*, where the* $[a_i]_k$*'s are the remainders of the* a_i*'s modulo* k*.*

Proof. We can write $a_i = k\alpha_i + [a_i]_k$ for each $i = 0, \dots, n$. Hence, we get that $M = N^k[M]$, where $N = x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Obviously, $k|\deg([M])$ and by Remark 2.4, we are done. □

Remark 2.7. With the above notations and numerical assumptions, we have that $[a_0]_k + \cdots + [a_n]_k$ is a multiple of k and also it has to be at most $(k-1)(n+1) = kn - n + k - 1$. Hence, fixed the number of variables $n+1$ and the integer k, we will have to consider only a few cases with respect to the remainders of the exponents modulo k.

3. Results on the k^{th}-rank for monomials

In this section we collect our results on the k-th rank of monomials.

3.1. **The general case.**

Here we present some general results on the k-th Waring rank for monomials.

Remark 3.1. Using the idea of grouping variables, we can easily get a complete description of the $k = 2$ case. Given a monomial M of degree $2d$ which is not a square, we have $\#_2(M) = 2$. Indeed,

$$M = XY = \left[\frac{1}{2}(X+Y)\right]^2 + \left[\frac{i}{2}(X-Y)\right]^2,$$

where X and Y are two monomials of degree d.

In the next result, we see that case $k = 2$ is the unique in which the k^{th}-rank of a monomial can be equal to two.

Theorem 3.2. *If M is a monomial of degree kd, then $\#_k(M) \leq 2^{k-1}$. Moreover, $\#_k(M) = 2$ if and only if $k = 2$ and M is not a square.*

Proof. Any monomial $M \in S_{kd}$ is a specialization of the monomial $x_1 \cdot \ldots \cdot x_{kd}$. Now, we can consider the grouping given by

$$X_1 = x_1 \cdot \ldots \cdot x_d, \ldots, X_k = x_{(k-1)d+1} \cdot \cdots \cdot x_{kd}.$$

Thus, by Remark 2.3, Remark 2.2 and Theorem 2.1, we get the bound

$$\#_k(M) \leq \#_k(X_1 \cdot \ldots \cdot X_k) = 2^{k-1}.$$

Now suppose that $k > 2$ and $\#_k(M) = 2$. Hence, we can write $M = A^k - B^k$ for suitable $A, B \in S_d$. Factoring we get

$$M = \prod_{i=1}^{k} (A - \xi_i B),$$

where the ξ_i are the k^{th}-roots of 1. In particular, the forms $A - \xi_i B$ are monomials. If M is not a k^{th}-power, using $A - \xi_1 B$ and $A - \xi_2 B$ we get that A and B are not trivial binomials. Hence a contradiction as $A - \xi_3 B$ cannot be a monomial. To conclude the proof we use the $k = 2$ case seen in Remark 3.1. □

Remark 3.3. For $n \geq 2$ and k small enough, we may observe that our result gives a better upper bound for the k^{th}-rank of monomials of degree kd than the general result of [FOS12]. Indeed, if we look for which k the inequality $2^{k-1} \leq k^n$ holds, for $n = 2$ we have $k \leq 6$ and, for $n = 3$, $k \leq 9$. Increasing n, we can find even better results, e.g. for $n = 10$ our Theorem 3.2 gives a better upperbound (for monomials) for any $k \leq 59$.

3.2. **Two variables case** $(n = 1)$**.** In the case of binary monomials, we can improve the upper bound given in Theorem 3.2.

Proposition 3.4. *Let $M = x_0^{a_0} x_1^{a_1}$ be a binary monomial of degree kd. Then,*

$$\#_k(M) \leq \max\{[a_0]_k, [a_1]_k\} + 1.$$

Proof. By Lemma 2.6, we know that $\#_k(M) \leq \#_k([M])$; hence, we consider the monomial $[M] = x_0^{[a_0]_k} x_1^{[a_1]_k}$. Now, we observe that, as we said in Remark 2.7, the degree of $[M]$ is a multiple of k and also $\leq 2k - 2$; hence, $\deg([M])$ is either equal to 0, i.e. $[M] = 1$, or k. In the first case, it means that M was a pure k^{th}-power, and the k^{th}-rank is

$$\#_k(M) = 1 = \max\{[a_0]_k, [a_1]_k\} + 1.$$

If $\deg([M]) = k$, we can apply Theorem 2.1 to $[M]$ and we get

$$\#_k(M) \leq \#_k([M]) = \max\{[a_0]_k, [a_1]_k\} + 1.$$

□

Remark 3.5. As a consequence of Proposition 3.4, for binary monomials we have that $\#_k(M) \leq k$. Actually, this upper bound can be directly derived from the main result in [FOS12]. We observe that this upperbound is sharp by considering $\#_k(x_0 x_1^{k-1}) = k$.

As a consequence of Theorem 3.2, we are able to easily give a solution for the $k = 3$ case for binary monomials.

Corollary 3.6. *Given a binary monomial M of degree $3d$, we have*

(1) $\#_3(M) = 1$ *if M is a pure cube;*
(2) $\#_3(M) = 3$ *otherwise.*

Proof. By Remark 3.5, we have that the 3^{rd}-rank can be at most 3; on the other hand, by Theorem 3.2, we have that, M is not a pure cube, the rank has to be at least 3. □

For $k \geq 4$ the situation is not so easily described and even in the case $k = 4$ we have only partial results.

Remark 3.7. The first new step is to consider the $k = 4$ case for binary monomials. In such case we can only have rank $1, 3$ or 4.

Let $M = x_0^{a_0} x_1^{a_1}$ be a binary monomial of degree $4d$. By Remark 2.6, we can consider the monomial $[M]$ obtained by considering the exponents modulo 4. Since $[M]$ has degree divisible by 4 and less or equal to 6, we have to consider only three cases with respect the remainders of the exponents modulo 4, i.e.

$$([a_0]_4, [a_1]_4) \in \{(0,0), (1,3), (2,2)\}.$$

The $(0,0)$ case corresponds to pure fourth powers, i.e. monomials with 4^{th}-rank equal to 1. In the $(2,2)$ case we have

$$\#_4(M) \leq \#_4(x_0^2 x_1^2) = 3;$$

since the 4^{th}-rank cannot be two, we have that binary monomials in the $(2,2)$ class have 4^{th}-rank equal to three.

Unfortunately, we can not conclude in the same way the $(1,3)$ case. Since $\#_4(x_0 x_1^3) = 4$, a monomial in the $(1,3)$ class could still have rank equal to 4. Indeed, for example, by using the computer algebra system CoCoA, we have computed $\#_4(x_0 x_1^7) = \#_4(x_0^3 x_1^5) = 4$.

A similar analysis can be performed for $k \geq 5$, but we can only obtain partial results.

3.3. $k = 3$ **case in three and more variables.** In this section we consider the case $k = 3$ with more than two variables. By Theorem 3.2, we have that, also in this case, we can only have 3^{rd}-rank equal to $1, 3$ or 4.

This lack of space allows us to give a complete solution for monomials in three variables and degree $3d$.

Proposition 3.8. *Given a monomial $M = x_0^{a_0} x_1^{a_1} x_2^{a_2}$ of degree $3d$, we have that*

(1) $\#_3(M) = 1$ *if M is a pure cube;*
(2) $\#_3(M) = 4$ *if $M = x_0 x_1 x_2$;*
(3) $\#_3(M) = 3$ *otherwise.*

Proof. By Lemma 2.6, we consider the monomials $[M]$ with degree divisible by 3 and less or equal than 6. Hence, we have only four possible cases, i.e.

$$([a_0]_3, [a_1]_3, [a_2]_3) \in \{(0,0,0), (0,1,2), (1,1,1), (2,2,2)\}.$$

The $(0,0,0)$ case corresponds to pure cubes and then to monomials with 3^{rd}-rank equal to one. In the $(0,1,2)$ case we have, by Theorem 2.1,

$$\#_3(M) \leq \#_3(x_1 x_2^2) = 3;$$

since, by Theorem 3.2, the rank of monomials which are not pure cubes is at least 3, we get the equality. Similarly, we conclude that we have rank three also for monomials in the $(2,2,2)$ class. Indeed, by using grouping and Theorem 2.1, we have

$$\#_3(M) \leq \#_3(x_0^2x_1^2x_2^2) \leq \#_3(XY^2) = 3.$$

Now, we just need to consider the $(1,1,1)$ class.

By Theorem 2.1, we have $\#_3(x_0x_1x_2) = 4$. Hence, we can consider monomials $M = x_0^{a_0}x_1^{a_1}x_2^{a_2}$ with $a_0 = 3\alpha+1,\ a_1 = 3\beta+1,\ a_2 = 3\gamma+1$ and where at least one of α,β,γ is at least one, say $\alpha > 0$. By Remark 2.4, we have

$$\#_3(M) = \#_3((x_0^{\alpha-1}x_1^{\beta}x_2^{\gamma})^3x_0^4x_1x_2) \leq \#_3(x_0^4x_1x_2).$$

Now, to conclude the proof, it is enough to show that $\#_3(x_0^4x_1x_2) = 3$. Indeed, we can write

$$x_0^4x_1x_2 = \left[\sqrt{\frac{1}{6}}x_0^2 + x_1x_2\right]^3 + \left[-\frac{1}{6}x_0^2 + x_1x_2\right]^3 + \left[\sqrt[3]{-2}x_1x_2\right]^3,$$

and thus we are done. □

Using the same ideas, we can produce partial results in the four and five variables cases with $k = 3$.

Remark 3.9. Given a monomial $M = x_0^{a_0}x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with degree $3d$, we consider the monomial $[M]$ which has degree divisible by 4 and less or equal than 8. Hence, we need to consider only the following classes with respect to the remainders of the exponents modulo 3

$$([a_0]_3,[a_1]_3,[a_2]_3,[a_3]_3) \in \{(0,0,0,0),(0,0,1,2),(0,1,1,1),(0,2,2,2),(1,1,2,2)\}.$$

The $(0,0,0,0)$ case corresponds to pure cubes and we have rank equal to one. Now, we use again Lemma 2.6, grouping and Theorem 2.1.

In the $(0,0,1,2)$ class, we have

$$\#_3(M) \leq \#_3(x_2x_3^2) = 3;$$

in the $(0,2,2,2)$ class, we have

$$\#_3(M) \leq \#_3(x_1^2x_2^2x_3^2) \leq \#_3(XY^2) = 3;$$

in the $(1,1,2,2)$ class, we have

$$\#_3(M) \leq \#_3(x_0x_1x_2^2x_3^2) \leq \#_3((x_0x_1)(x_2x_3)^2) = \#_3(XY^2) = 3.$$

Again, since the 3^{rd}-rank has to be at least three by Theorem 3.2, we conclude that in these classes the 3^{rd}-rank is equal to three.

The $(0,1,1,1)$ class is a unique missing case because the upper bound with $\#_3(x_1x_2x_3) = 4$ is clearly useless. Another idea would be to compute the 3^{rd}-rank of $x_0^3x_1x_2x_3$. Indeed, each monomial in four variables and degree $3d$ is of the type $N^k(x_0^3x_1x_2x_3)$, hence, by Remark 2.4, we have

$$\#_3(M) \leq \#_3(x_0^3x_1x_2x_3).$$

Finding $\#_3(x_0^3x_1x_2x_3) = 3$, we would be done.

Remark 3.10. Given a monomial $M = x_0^{a_0}x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}$ with degree $3d$, we consider the monomial $[M]$ which has degree divisible by 4 and less or equal to 10. Hence, we need to consider only the following classes with respect to the remainders of the exponents modulo 3.

The $(0,0,0,0,0)$ class corresponds to pure cubes and 3^{rd}-rank equal to one. By using Lemma 2.6, grouping, previous results in three or four variables and Theorem 2.1, we get the following results.

In the $(0,0,0,1,2)$ case, we have

$$\#_3(M) \leq \#_3(x_3x_4^2) = 3;$$

in the $(0,0,2,2,2)$ case, we have

$$\#_3(M) \leq \#_3(x_1^2x_2^2x_3^2) = 3;$$

in the $(0,1,1,2,2)$ case, we have

$$\#_3(M) \leq \#_3(x_0x_1x_2^2x_3^2) = 3;$$

in the $(1,2,2,2,2)$ case, we have

$$\#_3(M) \leq \#_3(x_0x_1^2x_2^2x_3^2x_4^2) = \#_3((x_0x_1^2)(x_2x_3x_4)^2) \leq \#_3(XY^2) = 3.$$

Hence, by Theorem 3.2, in these cases we have 3^{rd}-rank equal to three.

There are only two missing cases: the $(0,0,1,1,1)$ case, which can be reduced to the unique missing case in four variables seen above; the $(1,1,1,1,2)$ case, for which it would be enough to show that $\#_3(x_0x_1x_2x_3x_4^2) = 3$.

4. Final remarks

We conclude with some final remarks which suggest some projects for the future.

Remark 4.1. In this paper we work over the field of complex numbers. However, for a monomial $M \in S_{kd}$ it is reasonable to look for a *real* Waring decomposition, i.e. $M = \sum F_i^k$ where each F_i has real coefficients. Even if Remarks 2.2,2.3, and 2.4 still hold over the reals, this is not longer true for Theorem 2.1. This is the main obstacle to extend our results over $\mathbb{R}$. However, in [BCG11] it is shown that the degree d monomial x^ay^b is the sum of $a+b$, and no fewer, d-th powers of real linear forms. Thus, we can easily prove the analogue of Proposition 3.4. Let $\#_k(M, \mathbb{R})$ be the *real* k-th rank, then

$$\#_k({x_0}^{a_0}{x_1}^{a_1}, \mathbb{R}) \leq [a_0]_d + [a_1]_d$$

and the bound is sharp. Notice that Corollary 3.6 cannot be extend to the real case as we cannot use Theorem 3.2.

Remark 4.2. In [CCG12] it is proved that monomials in three variables produce example of forms having (standard) Waring rank higher than the generic form. This is not longer true, in general, for the k-th rank. For example, in the $k = 3$ case and in three variables, the 3-rd rank of a monomial is at most 4. While, for $d >> 0$ the 3-rd rank of the generic form of degree $3d$ is 9, see in [FOS12].

References

[]AH95 James Alexander and André Hirschowitz. Polynomial interpolation in several variables. *Journal of Algebraic Geometry*, 4(2):201–222, 1995.

[]BCG11 Mats Boij, Enrico Carlini, and Anthony V. Geramita. Monomials as sums of powers: the real binary case. *Proc. Amer. Math. Soc.*, 139(9):3039–3043, 2011.

[]CCG12 Enrico Carlini, Maria Virginia Catalisano, and Anthony V Geramita. The solution to the waring problem for monomials and the sum of coprime monomials. *Journal of Algebra*, 370:5–14, 2012.

[]FOS12 Ralf Fröberg, Giorgio Ottaviani, and Boris Shapiro. On the waring problem for polynomial rings. *Proceedings of the National Academy of Sciences*, 109(15):5600–5602, 2012.

(E. Carlini) DISMA- Department of Mathematical Sciences, Politecnico di Torino, Turin, Italy

E-mail address: enrico.carlini@polito.it

(A. Oneto) Department of Mathematics, Stockholm University, SE-106 91, Stockholm, Sweden

E-mail address: oneto@math.su.se