# SOME EXAMPLES OF FORMS OF HIGH RANK

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Abstract. We describe some forms with greater Waring rank than previous examples. In 3 variables we give forms of odd degree with strictly greater rank than the ranks of monomials, the previously highest known rank. This narrows the possible range of values of the maximum Waring rank of forms in 3 variables. In 4 variables we give forms of odd degree with strictly greater than generic rank. In degrees greater than or equal to 5 these are the first examples showing that there exist forms with Waring rank strictly greater than the generic value.

## 1. Introduction

For a complex homogeneous form $F$ of degree $d$, the **Waring rank** $r(F)$ is the least $r$ such that there exist linear forms $\ell_1, \ldots, \ell_r$ and scalars $c_1, \ldots, c_r$ satisfying $F = c_1\ell_1^d + \cdots + c_r\ell_1^d$. For example,

$$xyz = \frac{1}{24}\Big\{(x+y+z)^3 - (x+y-z)^3 - (x-y+z)^3 - (-x+y+z)^3\Big\}$$

which shows $r(xyz) \leq 4$; and one can show in fact $r(xyz) = 4$. For extensive introductions to Waring rank, including several different proofs that $r(xyz) = 4$, and including discussions of the history and applications of Waring rank, see for example [18, 21, 26, 11, 16, 24].

By the Alexander–Hirschowitz theorem [1] a general form $F$ of degree $d > 1$ in $n$ variables has rank $r(F)$ equal to

$$\left\lceil \frac{1}{n}\binom{n+d-1}{n-1} \right\rceil,$$

except if $d = 2$ (then $r(F) = n$) or $(n, d) = (3, 4), (4, 4), (5, 4), (5, 3)$ (then $r(F)$ is 1 more than the above expression). This value is called the **generic rank**. We denote it $r_{\text{gen}}(n, d)$.

It is an open question what is the maximum Waring rank of forms of degree $d$ in $n$ variables for each $(n, d)$, known only in some small cases. We write $r_{\max}(n, d)$ for the maximum Waring rank. Of course the maximum rank must be greater than or equal to the rank of a general form: $r_{\max}(n, d) \geq r_{\text{gen}}(n, d)$. Several upper bounds are known, such as $r_{\max}(n, d) \leq 2r_{\text{gen}}(n, d)$ [5] (see also [4], [19], [2]). For $d = 2$ it is known that $r_{\max}(n, d) = r_{\text{gen}}(n, d) = n$. For $n = 2$, $d \geq 3$, it is known that $r_{\max}(n, d) = d > r_{\text{gen}}(n, d) = (d + 2)/2$. For larger values $n, d \geq 3$ much less is known. One might ask whether the difference between the maximum Waring rank and the generic rank is unbounded. But it is not even known whether this difference is positive, i.e., the maximum Waring rank is strictly greater than the generic

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rank. We focus on the latter question: for each $n, d \geq 3$ does there exist a form with rank strictly greater than the generic rank?

The answer is known for some small cases. For plane cubics $r_{\max}(3,3) = 5$ and $r_{\text{gen}}(3,3) = 4$, see for example [25, §96], [12], [22, §8]. For plane quartics $r_{\max}(3,4) = 7$ and $r_{\text{gen}}(3,4) = 6$, see [20, 13]. For cubic surfaces $r_{\max}(4,3) = 7$ while $r_{\text{gen}}(4,3) = 5$, see [25, §97]. (See [17] for the form $F = x_1 x_2^2 + x_3 x_4^2$ of degree $d = 3$ in $n = 4$ variables which has rank 6.) To our knowledge, the maximum Waring rank is not known up to now for any other values of $(n, d)$.

For $n = 3$ and $d \geq 5$, while the maximum Waring rank is not yet known, it is known that there exist forms with strictly greater than the generic rank. The greatest Waring rank of a form in 3 variables previously known is attained by monomials, see [10]. Explicitly, if $d$ is odd, the monomial $xy^{(d-1)/2}z^{(d-1)/2}$ has rank $r(xy^{(d-1)/2}z^{(d-1)/2}) = ((d+1)/2)^2$; if $d$ is even, the monomial $xy^{(d-2)/2}z^{d/2}$ has rank $r(xy^{(d-2)/2}z^{d/2}) = d(d+2)/4$. For $d \geq 5$ these are the greatest known ranks of forms in 3 variables, until now. In particular, for $d \geq 5$ their ranks are strictly greater than generic ranks. See Table 1.

As far as we know, these monomials in 3 variables are the only forms in $n \geq 3$ variables known to have greater than the generic rank, except in the cases $(n, d) = (3,3), (3,4), (4,3)$ discussed above.

We give a lower bound for Waring rank and some new examples of forms whose Waring ranks are strictly greater than previously known examples.

Forms in 3 variables

| degree | | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Generic rank | | 4 | 6 | 7 | 10 | 12 | 15 | 19 | 22 | 26 | 31 |
| Greatest rank of monomial | | 4 | 6 | 9 | 12 | 16 | 20 | 25 | 30 | 36 | 42 |
| Maximum rank | lower bound | 5 | 7 | 10 | 12 | 17 | 20 | 26 | 30 | 37 | 42 |
| | upper bound | | | | 18 | 18 | 30 | 32 | 44 | 50 | 62 |

Table 1. Generic, maximum, and monomial ranks in $n = 3$ variables. The upper bound on maximum rank is provided by [2, 5, 14], see also Remark 3. The lower bound on maximum rank is mostly provided by monomials (even degrees $d \geq 6$), and Theorem 1 (odd degrees $d \geq 5$).

**Theorem 1.** *Let $d \geq 3$ be odd. There exist forms of degree $d$ in $n = 3$ variables of rank strictly greater than $((d+1)/2)^2$, the maximum rank of a monomial: $r_{\max}(3,d) > ((d+1)/2)^2$.*

In particular, De Paris had previously shown that for forms of degree $d = 5$ in $n = 3$ variables the maximum Waring rank is either 9 or 10, see [14]. The monomial $xy^2z^2$ has $r(xy^2z^2) = 9$, and De Paris shows the upper bound $r_{\max}(3,5) \leq 10$. We show that $r_{\max}(3,5) > 9$, i.e., there exists a form of rank 10, so the maximum rank is 10. Explicitly we show that $F = xyz^3 + y^4 z$ has $r(F) = 10$.

And we show the following:

**Theorem 2.** *Let $d \geq 3$ be odd. There exist forms of degree $d$ in $n = 4$ variables of rank strictly greater than the generic rank: $r_{\max}(4,d) > r_{\text{gen}}(4,d)$.*

These are the first cases with $n \geq 4$, except for $(n, d) = (4,3)$ mentioned previously.

Forms in 4 variables

| degree | | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| Generic rank | | 5 | 10 | 14 | 21 | 30 | 42 | 55 | 72 |
| Greatest rank of monomial | | 4 | 8 | 12 | 18 | 27 | 36 | 48 | 64 |
| Maximum rank | lower bound | 7 | 10 | 15 | 21 | 31 | 42 | 56 | 72 |
| | upper bound | | 17 | 28 | 42 | 60 | 84 | 110 | 144 |

Table 2. Generic, maximum, and monomial ranks in $n = 4$ variables. The upper bound on maximum rank is provided by [2, 5]. The lower bound on maximum rank is provided by generic rank (even degrees), and Theorem 2 (odd degrees).

The key idea for the lower bound that we use has been observed independently by Carlini, Catalisano, Chiantini, Geramita, and Woo, and applied by them to show new cases of the Strassen Additivity Conjecture [8], [9].

*Remark* 3. By [2, Prop. 3.9] we have $r_{\max}(3,6) \leq 19$ and by [2, Prop. 4.2], we have $r_{\max}(3,7) \leq 18$. But of course the function $d \mapsto r_{\max}(n,d)$ (for $n$ fixed) is nondecreasing (and so is the function $n \mapsto r_{\max}(n,d)$, for $d$ fixed). It follows that $r_{\max}(3,6) \leq r_{\max}(3,7) \leq 18$.

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## 2. Preliminaries

We work over the complex numbers $\mathbb{C}$. Fix $S = \mathbb{C}[x_1, \dots, x_n]$ and the dual ring $T = \mathbb{C}[\alpha_1, \dots, \alpha_n]$ acting on $S$ by letting each $\alpha_i$ act as $\partial/\partial x_i$; this is called the **apolarity** action. We denote it by the symbol $\lrcorner$, as in $\alpha_i \lrcorner x_i^k = k x_i^{k-1}$. In small dimensions we may take variables $x, y, z$ and dual variables $\alpha, \beta, \gamma$. In any case, elements of $S$ are denoted by Roman letters and elements of $T$ are denoted by Greek letters. For readability we may omit the symbol $\lrcorner$, so that any juxtaposition of Greek letters with Roman letters denotes the apolarity action, as in $\alpha^2\beta^3x^4y^5 = \alpha^2\beta^3 \lrcorner x^4y^5 = 240x^2y^3$.

For a vector space $V$ we denote by $\mathbb{P}V$ the projective space of lines in $V$. For a nonzero vector $v \in V$ we write $[v] \in \mathbb{P}V$ for the line in $V$ spanned by $v$. For an ideal $I$ or form $F$ we write $V(I)$ or $V(F)$ for the affine scheme or variety defined by $I$ or $F$. When $I$ or $F$ is homogeneous we write $\mathbb{P}V(I)$ or $\mathbb{P}V(F)$ for the corresponding projective scheme or variety.

For $F \in S$ let $F^\perp = \{\Theta \in T \mid \Theta \lrcorner F = 0\}$, the **apolar** or **annihilating ideal** of $F$. Recall the Apolarity Lemma, that for a scheme $Z \subset \mathbb{P}^{n-1}$ with saturated homogeneous ideal $I$, $[F]$ lies in the linear span of the Veronese image $v_d(Z)$ if and only if $I \subset F^\perp$; see for example [18, Lemma 1.15]. When $Z = \{[\ell_1], \dots, [\ell_r]\}$ is reduced, this says there are scalars $c_i$ such that $F = \sum c_i \ell_i^d$ if and only if $I \subset F^\perp$. We may replace each $\ell_i$ by $c_i^{1/d}\ell_i$ and write simply $F = \sum \ell_i^d$; so $F = \sum \ell_i^d$, up to scaling, if and only if $I = I(\{[\ell_1], \dots, [\ell_r]\}) \subset F^\perp$. Hence the Waring rank $r(F)$ is the least length of a reduced saturated homogeneous one-dimensional ideal $I \subset F^\perp$. A scheme $Z$ or ideal $I$ is called **apolar to** $F$ if $I \subset F^\perp$, equivalently if $[F]$ lies in the span of the $d$'th Veronese image of $Z$; so the Waring rank of $F$ is equal to the least length of a zero-dimensional reduced apolar scheme to $F$. A typical approach to giving lower bounds for $r(F)$ is to analyze reduced apolar schemes to $F$. This is the approach we take here.

Some related notions are worth mentioning. The **cactus rank** $cr(F)$, or **scheme length**, of $F$ is the least length of a saturated homogeneous one-dimensional ideal $I \subset F^\perp$ (not necessarily reduced). The **smoothable rank** $sr(F)$ is the least length of a smoothable zero-dimensional apolar scheme (recall that a scheme is smoothable if it lies in an irreducible family whose general member is smooth). The $r$'th **secant variety** of the Veronese variety is the Zariski closure of the locus of forms of Waring rank $r$. The **border rank** $br(F)$ is the least $r$ such that $[F]$ lies in the $r$'th secant variety, that is, $F$ is a limit of forms of rank $r$. Evidently $cr(F) \leq sr(F) \leq r(F)$ and $br(F) \leq r(F)$. In fact $br(F) \leq sr(F)$. All these inequalities may be strict, or may be equalities. For examples with $cr(F) < br(F)$, see for instance [3]. For examples with $cr(F) > br(F)$, see [6].

Let $A^F = T/F^\perp$, the **apolar algebra** of $F$. Let $\mathrm{Diff}(F) = T \lrcorner F = \{\Theta \lrcorner F \mid \Theta \in T\} \subset S$; note $\mathrm{Diff}(F) \cong A^F$ as $\mathbb{C}$-vector spaces. Recall (see for example [15]) that for any $F \in S$, $\Theta \in T$ we have $F^\perp : \Theta = (\Theta \lrcorner F)^\perp$ and we have the short exact sequence

$$0 \to T/(F^\perp : \Theta) \xrightarrow{\Theta} T/F^\perp \to T/(F^\perp + \Theta) \to 0.$$

In particular $\mathrm{length}(T/(F^\perp + \Theta)) = \dim \mathrm{Diff}(F) - \dim \mathrm{Diff}(\Theta F)$. Let $\mathrm{al}(F) = \mathrm{length}(A^F) = \dim \mathrm{Diff}(F)$, the **apolar length** of $F$, so

$$\mathrm{length}(T/(F^\perp + \Theta)) = \mathrm{al}(F) - \mathrm{al}(\Theta F) = \mathrm{length}(A^F/\Theta A^F).$$

The following was essentially observed in [15].

**Proposition 4.** *Let $F \in S$ be a homogeneous form of degree $d$, let $\alpha \in T_1$ be a linear form, and let $I \subseteq F^\perp$ be a saturated homogeneous one-dimensional apolar ideal. Suppose that the zero-dimensional scheme $\mathbb{P}V(I)$ has no point of support on the hyperplane $\mathbb{P}V(\alpha)$; equivalently, $I = I : \alpha$. Then* $\deg I \geq \mathrm{al}(F) - \mathrm{al}(\alpha F)$.

*Proof.* From $I + \alpha \subseteq F^\perp + \alpha$ we get $\mathrm{Spec}(T/(F^\perp + \alpha)) \subseteq V(I) \cap V(\alpha)$, a proper intersection by hypothesis, having length equal to $\deg I$. Thus $\deg I \geq \mathrm{length}(T/(F^\perp + \alpha)) = \mathrm{al}(F) - \mathrm{al}(\alpha F)$. □

This does not require $I$ to be reduced, so it leads to a bound for cactus rank $cr(F)$. The hypothesis that $\mathbb{P}V(I)$ has no point of support on $\mathbb{P}V(\alpha)$ can be realized by, for example, taking $\alpha$ general: for $\alpha$ general, $cr(F) \geq \mathrm{al}(F) - \mathrm{al}(\alpha F)$; this is Theorem 3.1 of [15].

Here are the new observations which form the starting point for this paper.

**Proposition 5.** *Let $F \in S$ be a homogeneous form of degree $d$, let $\alpha \in T_1$ be a linear form, and let $I \subseteq F^\perp$ be a reduced saturated homogeneous one-dimensional apolar ideal. Then* $\deg(I : \alpha) \geq \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$. *In particular $Z = \mathbb{P}V(I)$ has at least* $\mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$ *points of support off of the hyperplane* $\mathbb{P}V(\alpha)$.

*Proof.* Note $I : \alpha$ is a saturated homogeneous ideal and $I : \alpha \subset F^\perp : \alpha = (\alpha F)^\perp$. And $\mathbb{P}V(I : \alpha)$ has no point of support on $\mathbb{P}V(\alpha)$. The result follows by Proposition 4. □

*Remark* 6. If $Z$ is a zero-dimensional scheme with multiplicity at most $k$ at each support point in $\mathbb{P}V(\alpha)$ then $Z - (Z \cap \mathbb{P}V(\alpha))$ has length at least $\mathrm{al}(\alpha^k F) - \mathrm{al}(\alpha^{k+1} F)$.

*Remark* 7. In particular this ignores multiplicities (or reducedness) of $Z$ outside of $\mathbb{P}V(\alpha)$. At this time we do not know how to exploit reducedness of $Z$ outside of $\mathbb{P}V(\alpha)$ to give an improved bound.

A somewhat more general version of the next statement was observed independently by Carlini, Catalisano, Chiantini, Geramita, and Woo [8].

**Theorem 8.** *Let $F \in S$ be a homogeneous form of degree $d$ and let $\alpha \in T_1$ be a linear form. Then $r(F) \geq \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$.*

*Proof.* Let $I \subset F^\perp$ be a reduced apolar ideal of degree $\deg I = r(F)$. Then $r(F) = \deg I \geq \deg(I : \alpha) \geq \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$ by Proposition 5. □

The next statement does not seem to have been previously observed, to our knowledge.

**Corollary 9.** *If $\mathrm{al}(F) - \mathrm{al}(\alpha F) > \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$ then $r(F) > \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$.*

*Proof.* Let $I \subset F^\perp$ be a reduced apolar ideal of degree $\deg I = r(F)$. By Proposition 5 $\mathbb{P}V(I)$ has at least $\mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F)$ points off of $\mathbb{P}V(\alpha)$. If $r(F) = \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F) = \deg(I)$ then $\mathbb{P}V(I)$ has no support on $\mathbb{P}V(\alpha)$. In this case Proposition 4 yields $r(F) = \deg I \geq \mathrm{al}(F) - \mathrm{al}(\alpha F)$, as claimed. □

*Example* 10. Let $F = G(x)H(y) + K(y)$, where $x$ and $y$ denote tuples of independent variables, and suppose $\alpha \in T_1$ is differentiation by one of the $x$ variables, so that $\alpha H = \alpha K = 0$. Then $\alpha F = (\alpha G)H$ and $\mathrm{Diff}(\alpha F) \cong \mathrm{Diff}(\alpha G) \otimes \mathrm{Diff}(H)$; similarly $\alpha^2 F = (\alpha^2 G)H$ and $\mathrm{Diff}(\alpha^2 F) \cong \mathrm{Diff}(\alpha^2 G) \otimes \mathrm{Diff}(H)$. Then

$$r(F) \geq (\dim \mathrm{Diff}(\alpha G) - \dim \mathrm{Diff}(\alpha^2 G))(\dim \mathrm{Diff}(H)) = (\mathrm{al}(\alpha G) - \mathrm{al}(\alpha^2 G))\,\mathrm{al}(H).$$

In particular $r(x^a H(y) + K(y)) \geq \mathrm{al}(H)$.

Especially, let $F = x_1^{a_1} \cdots x_n^{a_n}$, $1 \leq a_1 \leq \cdots \leq a_n$, $\alpha = \alpha_1$. Then we obtain $r(F) \geq \mathrm{al}(x_2^{a_2} \cdots x_n^{a_n}) = (a_2 + 1) \cdots (a_n + 1)$. This recovers the theorem of Carlini–Catalisano–Geramita on Waring ranks of monomials [10], see also [23, 7]. (In fact the proof given by Carlini–Catalisano–Geramita is quite close to the idea of Theorem 8.)

*Example* 11. It is shown in [23] that if $F^\perp$ is a complete intersection generated in degrees $d_1 \leq \cdots \leq d_n$ then $cr(F) = d_1 \cdots d_{n-1} \leq r(F) \leq d_2 \cdots d_n$.

Suppose $F^\perp = (\phi_1, \ldots, \phi_n)$ is a complete intersection with $\deg \phi_i = d_i$ for each $i$, where $d_1 \leq \cdots \leq d_n$, and suppose $\alpha \in T_1$ is such that $\alpha^2 \mid \phi_1$. Note that $F^\perp : \alpha = (\phi_1/\alpha, \phi_2, \ldots, \phi_n)$ and $F^\perp : \alpha^2 = (\phi_1/\alpha^2, \phi_2, \ldots, \phi_n)$. Hence $r(F) \geq \mathrm{al}(\alpha F) - \mathrm{al}(\alpha^2 F) = d_2 \cdots d_n \geq r(F)$.

This generalizes the example of monomials. Compare Lemma 2.1 of [9].

## 3. Forms with higher than general rank

We adopt a slightly modified form of notation of [18]:

**Definition 12.** Fix integers $n, d, s$. Recall that $T = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$. Let $H(n, d, s)$ be the function $H(n, d, s)(i) = \min\{\dim_{\mathbb{C}} T_i, \dim_{\mathbb{C}} T_{d-i}, s\}$. Let $H(n, d)$ be the function $H(n, d)(i) = \min\{\dim_{\mathbb{C}} T_i, \dim_{\mathbb{C}} T_{d-i}\}$.

(In [18] these are written $H(s,d,n)$ and $H(d,n)$ respectively, although [18] uses $j$ in place of $d$ and $r$ in place of $n$.) As usual we may write these functions by writing their sequences of values for $i = 0, 1, \dots$: thus, for example, $H(3,6,8) = 1,3,6,8,6,3,1$, all subsequent values being zero.

Recall the following well-known facts.

**Proposition 13.** *The Hilbert function of apolar algebra behaves as follows.*

(1) ([18, Prop. 3.12]) *Fix integers $n$ and $d$. Let $G \in S_d$ be general. Then the Hilbert function of $A^G$ is $H(n,d)$.*
(2) ([18, Lemma 1.17]) *Fix integers $n, d, s$. Let $\ell_1, \dots, \ell_s \in S_1$ be general linear forms and $G = \ell_1^d + \dots + \ell_s^d$. Then the Hilbert function of $A^G$ is $H(n,d,s)$.*

In the first case the algebra $A^G$ is called **compressed** (see [18] for a more general notion of compressed algebras which are not necessarily Gorenstein or graded). These statements hold also in positive characteristic by taking $G$ to be a DP-form, see [18].

**Lemma 14.** (1) *Fix integers $n$ and $d$. Let $G \in S_d$ be any form such that $A^G$ has Hilbert function $H(n,d)$. Then the apolar length of $G$ is $\mathrm{al}(G) = \binom{n+\lfloor (d-1)/2 \rfloor}{n} + \binom{n+\lceil (d-1)/2 \rceil}{n}$.*
(2) *Fix integers $n, d, s$. Let $G \in S_d$ be any form such that $A^G$ has Hilbert function $H(n,d,s)$. Suppose $\dim T_i \leq s < \dim T_{i+1}$, where $i < d/2$. Then the apolar length of $G$ is $\mathrm{al}(G) = 2\binom{n+i}{i} + s(d-2i-1)$.*

The proof is an easy computation which we leave to the reader.

We write $\mathrm{al}_{\mathrm{gen}}(n,d)$ for the apolar length of a general form in $n$ variables of degree $d$; that is, $\mathrm{al}_{\mathrm{gen}}(n,d) = \binom{n+\lfloor (d-1)/2 \rfloor}{n} + \binom{n+\lceil (d-1)/2 \rceil}{n}$.

Before we produce forms with strictly greater rank than previously known examples, we carry out some preliminary computations that involve producing new forms with rank at least as great as previously known examples.

By Theorem 8 (or Example 10), $r(x_1 H(x_2, \dots, x_n) + K(x_2, \dots, x_n)) \geq \mathrm{al}(H)$, independent of the choice of $K$. In particular if $H$ is general this shows that

$$r_{\max}(n,d) \geq \mathrm{al}_{\mathrm{gen}}(n-1, d-1). \tag{1}$$

An easy computation shows for $d$ odd,

$$\mathrm{al}_{\mathrm{gen}}(3, d-1) = r_{\mathrm{gen}}(4,d). \tag{2}$$

It is also easy to see that $\mathrm{al}_{\mathrm{gen}}(3, d-1) < r_{\mathrm{gen}}(4,d)$ for $d$ even; $\mathrm{al}_{\mathrm{gen}}(n-1, d-1) < r_{\mathrm{gen}}(n,d)$ for $n \geq 5$ and $d \gg 0$; on the other hand $\mathrm{al}_{\mathrm{gen}}(2, d-1) > r_{\mathrm{gen}}(3,d)$ for $d \geq 5$.

*Example* 15. Let $H(y,z)$ be a general binary form of degree $d-1$ and let $K(y,z)$ be an arbitrary binary form of degree $d$. Then $r(xH+K) \geq \mathrm{al}(H)$. Since $H$ is general we compute $\mathrm{al}(H) = (d^2+2d)/4$ if $d$ is even, $(d+1)^2/4$ if $d$ is odd. In any case $\mathrm{al}(H) \approx d^2/4$. By the Alexander–Hirschowitz theorem the general rank of a form of degree $d$ in 3 variables is

$$\left\lceil \frac{1}{3}\binom{d+2}{2} \right\rceil = \left\lceil \frac{(d+2)(d+1)}{6} \right\rceil \approx \frac{d^2}{6},$$

or one more than this if $d = 4$. Thus the forms $xH + K$ have higher than general rank for $d$ large enough; $d \geq 5$ will do. Note that this is independent of the choice of $K$! The ternary monomials considered in [10] are given by $H = y^{\lfloor (d-1)/2 \rfloor} z^{\lceil (d-1)/2 \rceil}$, $K = 0$.

*Example* 16. In $n = 4$ variables, with $d \geq 3$ odd, for $H(x_2, x_3, x_4)$ general of degree $d-1$ and $K(x_2, x_3, x_4)$ arbitrary of degree $d$, $F = x_1 H + K$ has rank $r(F) \geq \mathrm{al}(H) = \mathrm{al}_{\mathrm{gen}}(3, d-1) = r_{\mathrm{gen}}(4, d)$. So these forms have rank at least as great as general rank.

So taking $H$ general, and $K$ arbitrary, shows explicitly that $F$ realizes the obvious inequality $r_{\max}(4, d) \geq r_{\mathrm{gen}}(4, d)$ for $d$ odd; and $r_{\max}(3, d)$ is greater than or equal to the maximum rank of a ternary monomial. Now the idea is that we can improve (1) by choosing $H$ to be not general, and $K$ meeting certain conditions.

**Lemma 17.** *For any $a \geq 0$ and any nonzero $\Psi \in T_b$ the linear map $D = D_{\Psi,a} : S_{a+b} \to S_a$, $F \mapsto \Psi F$, is surjective.*

*Proof.* Fix a monomial order $<$, such as lexicographic order. Let $\alpha^{m_0}$ be the $<$-last monomial in $\Psi$. For any monomial $x^m$ of degree $a$, $x^m$ is the leading monomial of $\Psi x^{m+m_0}$. This gives a triangular system of linear equations whose solution expresses each monomial $x^m$ as an element of the image of $D_{\Psi,a}$. □

**Theorem 18.** *Let $n \geq 3$ and $d = 2k+1 \geq 3$. There exists a form of degree $d$ in $n$ variables of rank strictly greater than the apolar length of a general form of degree $d-1$ in $n-1$ variables: $r_{\max}(n, d) > \mathrm{al}_{\mathrm{gen}}(n-1, d-1)$.*

*Proof.* We use the $n$ variables $x_1, x_2, \ldots, x_n$; for convenience we write $x = x_1$ and $\alpha = \alpha_1$. Let $s = \binom{n+k-2}{k} - 1$ and let $G(x_2, \ldots, x_n)$ be a general sum of $s$ $(d-1)$st powers of linear forms in variables $x_2, \ldots, x_n$. Let $F = xG + K(x_2, \ldots, x_n)$, with $K$ a form of degree $d$ to be determined later. Eventually, $K$ will be a general form of degree $d$ in $n-1$ variables, however, for the sake of argument, we do not assume anything on $K$ yet. Since $\alpha F = G$ and $\alpha^2 F = 0$ we get $r(F) \geq \mathrm{al}(G)$. By construction and Proposition 13 (2) $A^G$ has Hilbert function $H(n-1, d-1, s)$ and $\mathrm{al}(G) = \mathrm{al}_{\mathrm{gen}}(n-1, d-1) - 1$. That is,

$$\begin{aligned} r(F) \geq \mathrm{al}(G) &= \binom{n-1+\lfloor (d-2)/2 \rfloor}{n-1} + \binom{n-1+\lceil (d-2)/2 \rceil}{n-1} - 1 \\ &= \binom{n+k-2}{n-1} + \binom{n+k-1}{n-1} - 1. \end{aligned}$$

This holds regardless of the choice of $K$.

We have $G^\perp = F^\perp : \alpha$, so $F^\perp \subseteq G^\perp$. From the Hilbert function of $A^G$ we see that $G^\perp$ has the minimal generator $\alpha$, a single minimal generator in degree $k$, and all other minimal generators must be in degrees $k+1$ or higher. It follows that for degrees $2 \leq i \leq k-1$ we have $(F^\perp)_i \subseteq (G^\perp)_i = (\alpha)_i$. But if $\alpha\Theta \in F^\perp$ for some $\Theta \in T_{i-1}$ then $\Theta \in (G^\perp)_{i-1} = (\alpha)_{i-1}$, so $\alpha\Theta \in (\alpha^2)$. This shows $(F^\perp)_i = (\alpha^2)_i$ for $2 \leq i \leq k-1$.

Now, let $K$ be chosen so that $(F^\perp)_k = (\alpha^2)_k$. We will show later that there exists an open dense subset of such $K$, in fact satisfying an additional constraint that we will describe.

From this we can compute the apolar length of $F$:

$$\begin{aligned} \mathrm{al}(F) &= 2\left\{1 + n + \left(\tbinom{n+1}{2} - 1\right) + \left(\tbinom{n+2}{3} - n\right) + \cdots + \left(\tbinom{n+k-1}{k} - \tbinom{n+k-3}{k-2}\right)\right\} \\ &= 2\left\{\binom{n+k-2}{k-1} + \binom{n+k-1}{k}\right\} \\ &= 2\,\mathrm{al}(G) + 2 \\ &= 2\,\mathrm{al}_{\mathrm{gen}}(n-1, d-1). \end{aligned}$$

By Corollary 9 we get

$$r(F) \geq \binom{n+k-2}{n-1} + \binom{n+k-1}{n-1} = \mathrm{al}(G) + 1 = \mathrm{al}_{\mathrm{gen}}(n-1, d-1).$$

So far, this is the same value we would get by taking $G$ to be general. Now we will show that we can increase the bound on $r(F)$ by 1.

We claim that $r(F) \geq \mathrm{al}(G) + 2$. So, suppose to the contrary that $r(F) = r = \binom{n+k-2}{n-1} + \binom{n+k-1}{n-1} = \mathrm{al}(G) + 1$. Let $F = \ell_1^d + \dots + \ell_r^d$ and let $I = I(\{[\ell_1], \dots, [\ell_r]\})$. By Proposition 5, there must be at least $\mathrm{al}(G)$ points off of the hyperplane $\mathbb{P}V(\alpha)$. If all of the points $[\ell_i]$ are off of $\mathbb{P}V(\alpha)$ then by Proposition 4 we have in fact $r(F) \geq \mathrm{al}(F) - \mathrm{al}(\alpha F) = \mathrm{al}(F) - \mathrm{al}(G) = \mathrm{al}(G) + 2$, giving the claimed improvement. (Here the fact $\mathrm{al}(G)$ is 1 less than generic means $\mathrm{al}(F) - \mathrm{al}(G)$ is 1 more than we would have if $G$ were generic; this is where the non-genericity of $G$ gives an improvement in the bound for $r(F)$.) Otherwise there is exactly one $[\ell_i]$ lying on $\mathbb{P}V(\alpha)$. Without loss of generality $[\ell_r]$ lies on $\mathbb{P}V(\alpha)$ and the others lie off of it. That is, $\ell_r = \ell_r(x_2, \dots, x_n)$ does not depend on $x$. Let $F' = F - \ell_r^d = \ell_1^d + \dots + \ell_{r-1}^d = xG + (K - \ell_r^d)$. We will choose $K$ in such a way that $(F^\perp)_k = (F'^\perp)_k = (\alpha^2)_k$. Then the above arguments will apply to $F'$ and give us $r(F') \geq \mathrm{al}(G) + 1$. That is, $r - 1 \geq r(F') \geq \mathrm{al}(G) + 1$. Thus $r(F) \geq \mathrm{al}(G) + 2$, as claimed.

What is left is to show that there exists some $K$ such that $(F^\perp)_k = (\alpha^2)_k$ and for any linear form $\ell = \ell(x_2, \dots, x_n)$, $((F - \ell^d)^\perp)_k = (\alpha^2)_k$.

Let $\Psi \in (G^\perp)_k$ be the minimal generator of $G^\perp$ of degree $k$. Since $\alpha \in G^\perp$ we can take $\Psi$ to only involve $\alpha_2, \dots, \alpha_n$. Recall that $T_{k-1}G \subseteq S_{k+1}$ is the subspace consisting of $(k-1)$st derivatives of $G$; we have $\dim T_{k-1}G = \binom{n+k-2}{n-1}$ by the Hilbert function of $A^G$, but $\dim S_{k+1} = \binom{n+k}{n-1}$. So $T_{k-1}G \subsetneq S_{k+1}$.

Let $K \in \mathbb{C}[x_2, \dots, x_n]_d$ be any form so that $\Psi K \notin T_{k-1}G$. There exist a plethora of such forms by Lemma 17.

With such a choice we claim $(F^\perp)_k = (\alpha^2)_k$. Suppose $\Theta = \Theta(\alpha, \alpha_2, \dots, \alpha_n) \in (F^\perp)_k$. We may discard all terms containing $\alpha^2$, so we may write $\Theta = \alpha\phi + \psi$ where $\phi, \psi$ only involve $\alpha_2, \dots, \alpha_n$, $\phi \in T_{k-1}$, $\psi \in T_k$. Then $0 = \Theta F = x(\psi G) + \phi G + \psi K$, so $\psi G = \phi G + \psi K = 0$. Thus $\psi \in (G^\perp)_k$. Since $\psi$ only involves $\alpha_2, \dots, \alpha_n$, $\psi = c\Psi$ for some $c \in \mathbb{C}$. So $0 = \phi G + \psi K = \phi G + c\Psi K$. Since $\Psi K \notin T_{k-1}G$ it must be $c = 0$ and $\phi G = 0$, so $\Theta = \alpha\phi$ where $\phi \in (G^\perp)_{k-1} = (\alpha)_{k-1}$. Then $\Theta \in (\alpha^2)_k$.

Now the idea is to choose $K \in \mathbb{C}[x_2, \dots, x_n]_d$ a form so that not only $\Psi K \notin T_{k-1}G$, but in fact $\Psi(K - \ell^d) \notin T_{k-1}G$ for all $\ell = \ell(x_2, \dots, x_n)$. The linear map $D = D_{\Psi,k+1} : T_d \to S_{k+1}$ is surjective, so $D^{-1}(T_{k-1}G)$ has codimension equal to the codimension of $T_{k-1}G$, which is $\binom{n+k}{n-1} - \binom{n+k-2}{n-1}$. The projective Veronese variety $\{[\ell^d] : \ell = \ell(x_2, \dots, x_n)\}$ has dimension $n-2$. We have $\binom{n+k}{n-1} - \binom{n+k-2}{n-1} = \binom{n+k-2}{n-2} + \binom{n+k-1}{n-2} \geq \binom{n-1}{n-2} > n-2$. So a general translate of the Veronese variety is disjoint from $\mathbb{P}(D^{-1}(T_{k-1}G))$. This shows that for general $K$, $\Psi(K - \ell^d) \notin T_{k-1}G$ as claimed.

By the above calculation, $((F - \ell^d)^\perp)_k = (\alpha^2)_k$ so $r(F - \ell^d) \geq \mathrm{al}(G) + 1$ for all $\ell = \ell(x_2, \dots, x_n)$. As discussed above, then $r(F) \geq \mathrm{al}(G) + 2$. □

*Proof of Theorem 1.* The apolar length of a general binary form of degree $d - 1 = 2k$ is $(k+1)^2 = ((d+1)/2)^2$. So there exists a form of degree $d$ in 3 variables of rank strictly greater than $((d+1)/2)^2$, as claimed. □

*Proof of Theorem 2.* There exists a form in 4 variables of degree $d$ of rank strictly greater than the apolar length of a general form in 3 variables of degree $d-1=2k$, which is $\binom{k+2}{3}+\binom{k+3}{3}$, which is equal to the generic rank of a form in 4 variables of degree $d$. $\square$

The genericity conditions in the proof of Theorem 18 are very explicit and can be easily applied in practice. We illustrate this in the case of ternary quintics.

De Paris has shown that every ternary quintic (form of degree $d=5$ in $n=3$ variables) has Waring rank at most 10, see [14]. It is well known $r(xy^2z^2)=9$. But it is left open by De Paris whether the maximum rank of a ternary quintic is 9 or 10.

**Theorem 19.** *There exists a ternary quintic form of rank* 10. *Explicitly,* $F=xyz^3+y^4z$ *has* $r(F)=10$.

*Proof.* Here is an explicit expression showing $r(F)\le 10$: $F=(xyz^3-2y^2z^3-(1/5)z^5)+(y^4z+2y^2z^3+(1/5)z^5)$. Here $(y^4z+2y^2z^3+(1/5)z^5)^\perp=(\beta^2-\gamma^2,\beta\gamma^4)$, and since $\beta^2-\gamma^2$ has distinct roots, this binary form has $r(y^4z+2y^2z^3+(1/5)z^5)=2$. And compute $(xyz^3-2y^2z^3-(1/5)z^5)^\perp=(\alpha^2,4\alpha\beta+\beta^2,\beta^2\gamma^2-\gamma^4)$, which is a complete intersection generated in degrees $2,2,4$ with the first generator divisible by (equal to) the square of a linear form; by Example 11 $r(xyz^3-2y^2z^3-(1/5)z^5)=2\cdot 4=8$. Thus $r(F)\le 2+8=10$.

However the more important point is to show $r(F)\ge 10$. We compute:

$$\begin{aligned}
\alpha^2F&=0, & \beta^2F&=12y^2z,\\
\alpha\beta F&=z^3, & \beta\gamma F&=3xz^2+4y^3,\\
\alpha\gamma F&=3yz^2, & \gamma^2F&=6xyz.
\end{aligned}$$

Observe that the nonzero derivatives listed above are linearly independent: in fact no monomial appears in more than one of them. So $(F^\perp)_2$ is spanned by $\alpha^2$. This shows that the Hilbert function of $A^F$ is $1,3,5,5,3,1$. In particular $\mathrm{al}(F)=18$.

Observe also that $\alpha F=yz^3$, $\alpha^2F=0$. By Theorem 8, $r(F)\ge\dim\mathrm{Diff}(yz^3)=8$. If $r(F)=8$ then $r(F)\ge\dim\mathrm{Diff}(F)-\dim\mathrm{Diff}(yz^3)=18-8=10$. So $r(F)>8$.

Now we rule out the possibility $r(F)=9$. Suppose to the contrary $F=\ell_1^5+\cdots+\ell_9^5$. Proposition 5 shows at least 8 of the $[\ell_i]$ lie off of the hyperplane $\mathbb{P}V(\alpha)$; but if all 9 lie off of the hyperplane, then by Proposition 4 $r(F)\ge\dim\mathrm{Diff}(F)-\dim\mathrm{Diff}(\alpha F)=10$. So say $\ell_1,\dots,\ell_8$ lie off of $V(\alpha)$ and $\ell_9=ay+bz$ lies on $V(\alpha)$. Let $G=F-(ay+bz)^5=\ell_1^5+\cdots+\ell_8^5$, so that $r(G)=8$. Note $\alpha G=\alpha F=yz^3$. We compute again:

$$\begin{aligned}
\alpha^2G&=0, & \beta^2G&=12y^2z-20a^2(ay+bz)^3,\\
\alpha\beta G&=z^3, & \beta\gamma G&=3xz^2+4y^3-20ab(ay+bz)^3,\\
\alpha\gamma G&=3yz^2, & \gamma^2G&=6xyz-20b^2(ay+bz)^3.
\end{aligned}$$

If $a\ne 0$ then $\alpha\beta G$, $\alpha\gamma G$, $\beta^2G$ are linearly independent as $\beta^2G$ is the only one with a nonzero $y^5$ term. If $a=0$ then the same three derivatives are still linearly independent as they are distinct monomials. And $\beta\gamma G$, $\gamma^2G$ are linearly independent modulo the other derivatives because they involve different monomials with $x$. In conclusion, the nonzero derivatives of $G$ listed above are linearly independent, so $(G^\perp)_2$ is spanned by $\alpha^2$. It follows that $A^G$ has Hilbert function $1,3,5,5,3,1$, the same as $A^F$.

Now the same argument applies to $G$: $\alpha G=\alpha F=yz^3$, $\alpha^2G=0$, so $r(G)\ge\dim\mathrm{Diff}(yz^3)=8$, and if $r(G)=8$ then $r(G)\ge\dim\mathrm{Diff}(G)-\dim\mathrm{Diff}(\alpha G)=10$, hence $r(G)>8$. This contradicts the construction of $G$ which shows $r(G)=8$.

It follows that $r(F) > 9$, so $r(F) = 10$. □

*Remark* 20. The result of Theorem 1 is the best possible for degrees $d = 3, 5$: the result $r_{\max}(3, d) \geq 1 + ((d+1)/2)^2$ is equality for these degrees. For other degrees, and for $n > 3$, one may ask if this bound can be improved. Two potential routes for improvement suggest themselves. First, Carlini, et al, show a more general and potentially stronger version of Theorem 8, see [8, Corollary 3.4]. Second, one might try modifying the proof of Theorem 18 by taking $G$ of apolar length 2 less than the general apolar length, and showing that in appropriate cases $\mathbb{P}(D^{-1}(T_{k-1}G))$ is disjoint from not only a general translate of the Veronese but in fact from a general translate of the secant variety of the Veronese.

## References


1. J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), no. 2, 201–222.
2. Edoardo Ballico and Alessandro De Paris, *Generic power sum decompositions and bounds for the Waring rank*, `arXiv:1312.3494` [math.AG], Dec 2013.
3. Alessandra Bernardi and Kristian Ranestad, *On the cactus rank of cubic forms*, J. Symbolic Comput. **50** (2013), 291–297.
4. A. Białynicki-Birula and A. Schinzel, *Representations of multivariate polynomials by sums of univariate polynomials in linear forms*, Colloq. Math. **112** (2008), no. 2, 201–233.
5. Grigoriy Blekherman and Zach Teitler, *On maximum, typical, and generic ranks*, Math. Ann. (2014), DOI: 10.1007/s00208-014-1150-3.
6. Weronika Buczyńska and Jarosław Buczyński, *On differences between the border rank and the smoothable rank of a polynomial*, Glasgow Mathematical Journal **FirstView** (2014), 1–13.
7. Weronika Buczyńska, Jarosław Buczyński, and Zach Teitler, *Waring decompositions of monomials*, J. Algebra **378** (2013), 45–57.
8. Enrico Carlini, Maria Virginia Catalisano, Luca Chiantini, Anthony V. Geramita, and Youngho Woo, *Symmetric Tensors: Rank and Strassen's Conjecture*, `arXiv:1412.2975` [math.AC], Dec 2014.
9. Enrico Carlini, Maria Virginia Catalisano, Luca Chiantini, Anthony V. Geramita, and Youngho Woo, *e-computable forms and the Strassen conjecture*, `arXiv:1502.01107` [math.AC], Feb 2015.
10. Enrico Carlini, Maria Virginia Catalisano, and Anthony V. Geramita, *The solution to the Waring problem for monomials and the sum of coprime monomials*, J. Algebra **370** (2012), 5–14.
11. Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl. **30** (2008), no. 3, 1254–1279.
12. Pierre Comon and Bernard Mourrain, *Decomposition of quantics in sums of powers of linear forms*, Signal Processing **53** (1996), no. 2–3, 93–107.
13. Alessandro De Paris, *A proof that the maximal rank for plane quartics is seven*, `http://arxiv.org/abs/1309.6475`, Sep 2013.
14. ______, *Every ternary quintic is a sum of ten fifth powers*, `arXiv:1409.7643` [math.AG], Sep 2014.
15. Harm Derksen and Zach Teitler, *Lower bound for ranks of invariant forms*, `arXiv:1409.0061` [math.AG], Aug 2014.
16. Anthony V. Geramita, *Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, The Curves Seminar at Queen's, Vol. X (Kingston, ON, 1995), Queen's Papers in Pure and Appl. Math., vol. 102, Queen's Univ., Kingston, ON, 1996, pp. 2–114.
17. Erik Holmes, Paul Plummer, Jeremy Siegert, and Zach Teitler, *Maximum Waring ranks of monomials*, `arXiv:1309.7834` [math.AG], Apr 2014.
18. Anthony Iarrobino and Vassil Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman.

19. Joachim Jelisiejew, *An upper bound for the Waring rank of a form*, Arch. Math. (Basel) **102** (2014), no. 4, 329–336.
20. Johannes Kleppe, *Representing a homogenous polynomial as a sum of powers of linear forms*, Master's thesis, University of Oslo, 1999, `http://folk.uio.no/johannkl/kleppe-master.pdf`.
21. J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012.
22. J.M. Landsberg and Zach Teitler, *On the ranks and border ranks of symmetric tensors*, Found. Comp. Math. **10** (2010), no. 3, 339–366.
23. Kristian Ranestad and Frank-Olaf Schreyer, *On the rank of a symmetric form*, J. Algebra **346** (2011), 340–342.
24. Bruce Reznick, *On the length of binary forms*, Quadratic and Higher Degree Forms (New York) (K. Alladi, M. Bhargava, D. Savitt, and P. Tiep, eds.), Developments in Math., vol. 31, Springer, 2013, pp. 207–232.
25. B. Segre, *The Non-singular Cubic Surfaces*, Oxford University Press, Oxford, 1942.
26. Zach Teitler, *Geometric lower bounds for generalized ranks*, `arXiv:1406.5145` [math.AG], Jun 2014.

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