ON LINEAR SYSTEMS OF $\mathbb{P}^3$ WITH NINE BASE POINTS

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Abstract. We study special linear systems of surfaces of $\mathbb{P}^3$ interpolating nine points in general position having a quadric as fixed component. By performing degenerations in the blown-up space, we interpret the quadric obstruction in terms of linear obstructions for a quasi-homogeneous class. By degeneration we also prove a Nagata type result for $\mathbb{P}^2$ that implies a base locus lemma for the quadric. As an application we establish Laface-Ugaglia Conjecture for linear systems with multiplicities bounded by 8 and for homogeneous linear systems with multiplicity m and degree up to $2m+1$.

1. Introduction

The theory of linear systems is a classical object of study which is related to secant varieties, polynomial interpolation and to several interesting recently discovered applications. Even if linear systems have been studied for more than a century, basic questions, such as the dimensionality problem, are still open in general.

We denote by $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ interpolating s points in general position with multiplicities respectively $m_1, \ldots, m_s$. A linear system is said to be *non-special* if it has the (affine) expected dimension, which is $\mathrm{edim}(\mathcal{L}) = \max(\mathrm{vdim}(\mathcal{L}), 0)$, where the (affine) virtual dimension $\mathrm{vdim}(\mathcal{L})$ is defined as

$$\mathrm{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n}.$$

Special linear systems are those that have dimension strictly higher than the expected one and the *speciality* of the system is the difference $\dim(\mathcal{L}) - \mathrm{edim}(\mathcal{L})$.

In general it is challenging to compute the dimension of the linear systems. In order to classify the special linear systems, one has to understand first of all what are the *obstructions*, namely what are the varieties that, whenever contained with multiplicity in the base locus of $\mathcal{L}$, generate speciality. In [3, 4] these obstructions are named *special effect varieties*.

The well-known Alexander-Hirschowitz theorem ([1], see also [7, 30]), which concerns the case of linear systems with double points in $\mathbb{P}^n$, provides a list of special systems where the special effect varieties are linear cycles (when $d = 2$), a rational normal curve (when $d = 3$) or a quadric hypersurface (when $d = 4$).

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For higher multiplicities, the case of the plane has been deeply investigated by many authors and in this case the famous SGHG conjecture states that the obstructions are given by (-1)-curves [19, 23, 24, 32] (see also [14, 11, 12]).

In the higher dimensional case, in [5, 18] the authors extensively studied the linear special effect varieties in $\mathbb{P}^n$. In particular knowing the exact contribution to the speciality of any multiple linear cycle contained in the base locus allows to introduce the notion of *linear expected dimension*, (see [5, Definition 3.2]). We say that a system is *linearly non-special* when its dimension is equal to the linear expected dimension. This happens exactly when the only special effect varieties are linear.

We will devote the paper [6] to the investigation of linear systems in $\mathbb{P}^n$ for which rational normal curves through $n+3$ points are special effect curves.

It is a well-known fact that Cremona reduced linear systems of $\mathbb{P}^3$ do not contain rational normal curves in their base locus. Laface and Ugaglia conjectured [27] that for a Cremona reduced linear system the only special effect varieties are lines and quadric surfaces. The Laface-Ugaglia conjecture is known to be true if the number of points is less than or equal to 8 [15], and when the maximal multiplicity of the points is 5 [2].

In this paper we study linear systems in $\mathbb{P}^3$ with at least nine general multiple points for which the quadric hypersurface through nine of the base points, namely the fixed surface $Q := \mathcal{L}_{3,2}(1^9)$, is a special effect variety.

The first step is to prove a *base locus lemma* for quadric surfaces. Even a weak base locus lemma is not obvious to obtain. In fact such kind of results can be obtained as a consequence of *Nagata type* results, i.e. theorems which prove emptiness, for linear systems in $\mathbb{P}^2$ with ten points. In Section 4 we will establish a base locus lemma for the quadric surface through nine points (see Theorem 4.1) for a particular class of linear systems in $\mathbb{P}^3$. In order to prove this result, we study, via a suitable degeneration technique inspired by [10, 13], the emptiness of certain linear systems with ten points in $\mathbb{P}^2$.

The next step is to classify the special linear systems whose special effect varieties are quadrics. We focus in particular on the case of (Cremona reduced) linear system with nine points in $\mathbb{P}^3$, which is the first case where the speciality is not only due to linear obstructions.

Our goal is to understand precisely how much the quadric surface in the base locus contributes to the speciality of the system. Differently than in the linear case, here to give a formula which computes exactly the contribution to the speciality seems very difficult in general (see Remark 5.5 for more details).

Hence we focus first on some particular classes of linear systems, that are the homogeneous and the quasi-homogeneous ones.

The first case we study is given by the quasi-homogeneous linear systems $\mathcal{L}_{3,2m}(m^8, a)$, for $1 \leq a \leq m$. This class of systems behaves surprisingly well, indeed we are able to find an easy formula which relates the speciality with the multiplicity of the quadric in the base locus, see Theorem 3.1. The proof of this result is based on a degeneration argument, which allows to reduce the "mysterious" contribution of the quadric to the sum of two contributions given by linear special effect varieties in the degenerated systems.

We recall that in literature various degeneration arguments have been used to prove non-speciality results of linear systems in the plane [11, 12, 17] and in higher dimension [26, 30]. We also mention that the argument used in our proof was already divulged in [18, Section 7.2] in one particular example and is now generalized to a larger class.

In the case of degree $2m+1$, the relation between the speciality and the quadric becomes less clear even in the homogeneous case. Anyway in Theorem 4.10 we classify all the special homogeneous linear systems with nine points of multiplicity m and degree $2m+1$. In order to prove this result we will apply the emptiness results mentioned above and proved in Section 4.1.

In the last section, as an application of Theorems 3.1 and 4.10, we show that the Laface-Ugaglia conjecture holds for linear systems of any degree and nine points of multiplicities at most 8. In order to complete this proof (as also for the proof of Theorem 4.10) we need to perform some computations with the computer algebra system `Macaulay2` , [22].

We want to point out finally that the quadric hypersurfaces are *sporadic* special effect varieties. Indeed it is expected that they give contribution to the speciality of a linear system only in $\mathbb{P}^3$, for any degree, and in $\mathbb{P}^4$, for degree $d = 4, 6$. This two families of special linear systems are also between the few exceptions predicted by the Fröberg-Iarrobino conjecture for homogeneous linear systems, see [9, Conjecture 4.8]. We think that the understanding of the case of linear systems in $\mathbb{P}^3$ with nine points is the initial step in order to investigate these sporadic special systems.

This article is organized as follows. In Section 2 we give a brief description of the tools that we will use to prove our results.

In Section 3 we classify the case $\mathcal{L}_{3,2m}(m^8, a)$ and we give a geometric interpretation, via degenerations, of the quadric as special effect surface.

In Section 4 we completely classify the case $\mathcal{L}_{3,2m+1}(m^9)$; the main result is the base locus lemma Theorem 4.1.

In Section 5, we prove that Laface-Ugaglia Conjecture holds for linear systems with 9 base points of multiplicities $m_i \leq 8$.

2. Preliminaries

In this preliminary section we collect general results and techniques that will be used throughout the paper.

2.1. **Degenerations.** A natural approach to the dimensionality problem of linear systems is via degenerations. Degenerations allow to move the multiple base points of a linear system in special position and compute the dimension via a semicontinuity argument.

Ciliberto and Miranda in [11, 12] exploited a degeneration of the plane, originally proposed by Ran [31] to study higher multiplicity interpolation problems for planar linear systems with general multiple base points. This approach consists in degenerating the plane to a reducible surface, with two components intersecting along a line, and simultaneously degenerating the linear system to a limit linear system which is somewhat easier than the original one. In particular this degeneration argument allows to use induction either on the degree or on the number of imposed multiple points. This method was generalized by the third author to the higher dimensional cases of $\mathbb{P}^n$ [30] and of $(\mathbb{P}^1)^n$ with Laface [26].

Let $X \subseteq \mathbb{P}^N$ be a variety, let Δ be a complex disc with center at the origin and let $\mathcal{X} \to \Delta$ be a 1-dimensional embedded degeneration of X to the union of two varieties X^1, X^2, i.e. a 1-parameter family $\{X_t\}_{t\in\Delta}$ such that $X_t \cong X,\ t \neq 0$, and

$X_0 = X^1 \cup X^2$. Let $\mathcal{L}_t := \mathcal{L}$ be a line bundle on the general fibre. A limit $\mathcal{L}_0$ of $\mathcal{L}_t$ is a line bundle on X_0 obtained as fibred product of a line bundle $\mathcal{L}^1$ on X^1 and a line bundle $\mathcal{L}^2$ on X^2 over the intersection of the restricted line bundles ${\mathcal{L}^1}_{|Y}$ and ${\mathcal{L}^2}_{|Y}$. This provides a recursive formula for the dimension of $\mathcal{L}_0$ in terms of the dimensions of the involved linear systems on the two components:

$$\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}^2) + \dim(\hat{\mathcal{L}}^1) + \dim({\mathcal{L}^1}_{|Y} \cap {\mathcal{L}^2}_{|Y}),$$

where $\hat{\mathcal{L}}^i$ is the kernel of the restriction map $\mathcal{L}^i \to {\mathcal{L}^i}_{|Y}$, $i = 1, 2$. Upper semi-continuity implies the inequality $\dim(\mathcal{L}_t) \leq \dim(\mathcal{L}_0)$.

2.2. **Linear systems on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.** In order to study the base locus of linear systems on $\mathbb{P}^3$ through 9 general points, we want to understand their restrictions to the quadric surface $Q = \mathcal{L}_{3,2}(1^9) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The restriction of $\mathcal{L} = \mathcal{L}_{3,d}(m_1, \ldots, m_9)$ will be the linear series of curves of bidegree (d, d) on Q with 9 multiple points in general position, that we will denote as

$$\mathcal{L}_{\mathbb{P}^1 \times \mathbb{P}^1,(d,d)}(m_1, \ldots, m_9).$$

Not very much is known about such linear systems: Giuffrida, Maggioni, and Ragusa were among the first to study linear systems on a quadric surface in [20], for more recent results see e.g. [21]. As far as we know the only cases completely classified are the homogeneous double points [33] and triple points [25].

The following result allows to transform linear systems of given bidegree on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with multiple base points to linear systems on $\mathbb{P}^2$ with multiple base points, and vice-versa, by means of *cut-and-sew* of polygons.

The image of $\mathbb{P}^2$ blown-up at two points via the embedding given by the linear system $\mathcal{L}_{2,d_1+d_2-m}(d_1 - m, d_2 - m)$ based at two torus-invariant points (e.g. two coordinate points) is a toric projective surface whose defining polytope is combinatorially equivalent to the pentagon obtained by the triangle $(d_1 + d_2 - m)\Delta$ by cutting two triangles $(d_1 - m)\Delta$ and $(d_2 - m)\Delta$ from two corners, where Δ is the 2-simplex of $\mathbb{R}^2$, see Figure 1 on the left hand side.

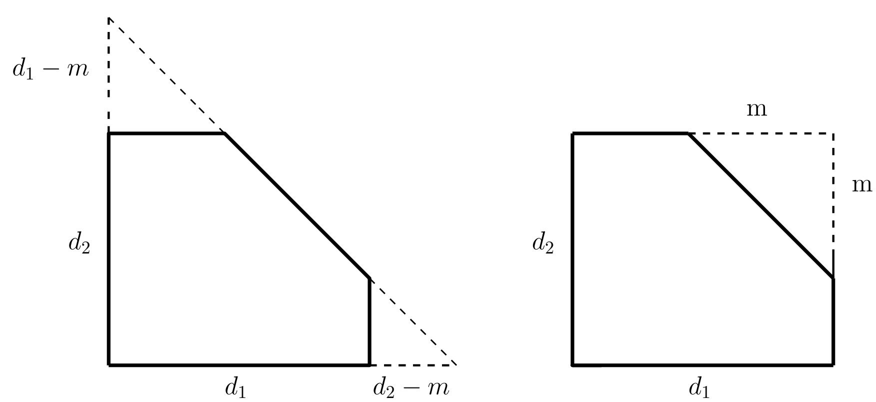

Figure 1. Two equivalent polytopes

Notice that the same polytope can be obtained from the rectangle $[0, d_1] \times [0, d_2] \subset \mathbb{R}^2$ by cutting the triangle $m\Delta$ from a corner. This interprets the above toric surface as the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ via the linear system of curves of bidegree

(d_1, d_2) with a point of multiplicity m, that is $\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)}(m)$, see Figure 1 on the right hand side.

In other terms, this is the birational map that factors in the blow-up of $\mathbb{P}^2$ at two points and the blow-down of the (-1)-line joining them.

This proves the following result.

Lemma 2.1. *If $m \le d_1, d_2$, then the following equality holds*

$$\dim(\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)}(m, m_1, \dots, m_s)) = \dim(\mathcal{L}_{2,d_1+d_2-m}(d_1-m, d_2-m, m_1, \dots, m_s)).$$

Remark 2.2. In [8, Theorem 1.1] the authors show how to convert linear systems on products of projective spaces $\mathbb{P}^{n_i}$ interpolating multiple points into linear systems in the projective space $\mathbb{P}^{\sum n_i}$ interpolating multiple points and multiple linear subspaces, and back. We point out that the case $m = 0$ in Lemma 2.1 falls into those equivalences, in the particular case of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

2.3. **Cremona transformations.** We recall the the standard Cremona transformation of $\mathbb{P}^n$ is the birational transformation defined by the following rational map:

$$\mathrm{Cr} : (x_0 : \cdots : x_n) \to (x_0^{-1} : \cdots : x_n^{-1}).$$

This map induces an action on the Picard group of the n-dimensional space blown-up at s points. Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_s)$ be a linear system based on s points in general position; we can assume, without loss of generality, that the first $n+1$ are the coordinate points. The Cremona action on $\mathcal{L}$ is described by the following rule (see for example [16, 27]). Set

$$c := m_1 + \cdots + m_{n+1} - (n-1)d,$$

then

$$\mathrm{Cr}(\mathcal{L}) = \mathcal{L}_{n,d-c}(m_1 - c, \dots, m_{n+1} - c, m_{n+2}, \dots, m_s)$$

and

$$\dim(\mathcal{L}) = \dim(\mathrm{Cr}(\mathcal{L})).$$

We will use this transformations in the cases $n = 2, 3$ to reduce the computation of the dimension of a linear system $\mathcal{L}$ to the computation of the dimension of its Cremona transfom $\mathrm{Cr}(\mathcal{L})$ that has lower degree and multiplicities whenever $c > 0$.

If $c \le 0$ we will say that the linear system $\mathcal{L}$ is *Cremona reduced*.

2.4. **Computing with `Macaulay2`.** In this paper we will need to perform some explicit computation in order to complete our classifications. In particular the proofs of Proposition 4.7, Lemma 4.8, Lemma 5.7 and Theorem 5.8 are computer aided.

We perform this computations by means of the computer algebra system `Macaulay2` . The procedure we use consists essentially in checking that several square matrices, randomly chosen, have maximal rank. We work over a field of characteristic 31991 and the proofs hold also in characteristic zero.

We used two scripts (one for linear systems in $\mathbb{P}^2$ and one in $\mathbb{P}^3$) available at this url `http://dipmat.univpm.it/~brambilla/NinePointsP3.html`, which allow to compute the dimension and the speciality of a linear system with given degree and multiplicities.

3. Quasi-homogeneous linear systems $\mathcal{L}_{3,2m}(m^8, a)$

In this section we describe a class of special linear systems in $\mathbb{P}^3$ with nine base points for which the quadric surface Q is the only special effect variety. We employ a double degeneration argument, similar to the one employed in [30] for linear systems with arbitrary general double points, that is based on the degeneration of the space described in Section 2.1. The linear system will degenerate into one that only has linear special effect varieties, and that is therefore understood by the results in [5, 18].

Fix integers a, m. Consider the *quasi-homogeneous* linear system in $\mathbb{P}^3$

$$\mathcal{L}(m,a) := \mathcal{L}_{3,2m}(m^8, a). \tag{3.1}$$

The main result of this section is the following.

Theorem 3.1. *If $1 \leq a \leq m$, the linear system (3.1) is special and*

$$\dim(\mathcal{L}(m,a)) = m - a + 1$$
$$h^1(\mathcal{L}(m,a)) = \binom{a+1}{3} + \binom{a}{2}.$$

Moreover the only special effect variety for $\mathcal{L}(m,a)$ is the quadric through nine points which is contained in the base locus with exact multiplicity a.

It was proved already in [28, Section 5] that for $a = m$ the linear system (3.1) has one element, that is the m-multiple of the quadric through the 9 points. This also implies that if $a > m$, the linear system (3.1) is empty. The case $a = 0$ was proved to be non-special in [15]. So the remaining cases to explore are $1 \leq a \leq m-1$; for the sake of completeness we include here the proof of the case $a = m$ as well.

Theorem 3.1, shows that the linear system (3.1) is special with dimension being a linear function of m and a. The only special effect variety is the quadric through the nine points which is contained with multiplicity a in the base locus and moreover, quite surprisingly, its contribution to the speciality, namely $h^1(\mathcal{L}(m,a))$ only depends in the multiplicity of containment of the quadric.

Remark 3.2. If we define $q(\mathcal{L}(m,a)) = \chi(\mathcal{L}(m,a)_{|Q})$ (see (5.1) in Section 5), then one can easily check the following:

$$q(\mathcal{L}(m,a)) = 1 - \binom{a+1}{2} < 0 \text{ iff } a \geq 2, \text{ and } q(\mathcal{L}) = 0 \text{ if } a = 1.$$

This in particular shows that Theorem 3.1 has the following immediate consequence:

Corollary 3.3. *Laface-Ugaglia Conjecture (see Conjecture 5.1 in Section 5) is true for any quasi-homogeneous linear system of the form (3.1).*

3.1. Degeneration of the blown-up $\mathbb{P}^3$ at 9 points.

In this section we give a detailed description of the degeneration techniques that we will employ to prove Theorem 3.1.

3.1.1. *First degeneration.* Consider the trivial family $\mathcal{V} = \mathbb{P}^3 \times \Delta \to \Delta$ with fibres $V_t \cong \mathbb{P}^3$, $t \in \Delta$. The blow-up of a point $p_0 \in V_0$ produces a flat morphism $\mathcal{X}' \to \Delta$ with general fibre $X'_t \cong \mathbb{P}^3$ and central fibre $X'_0 = \mathbb{F} \cup \mathbb{P}$, where $\mathbb{F} \cong \mathrm{Bl}_{p_0}\mathbb{P}^3$ is the pull-back of V_0 and $\mathbb{P} \cong \mathbb{P}^3$ is the exceptional divisor in the total space $\mathcal{X}'$. The two components $\mathbb{F}$ and $\mathbb{P}$ meet transversally along a surface $Y \cong \mathbb{P}^2$ that, as a divisor, belongs to the exceptional class of $\mathbb{F}$ and to the hyperplane class of $\mathbb{P}$. More precisely, if $E_0 := \mathbb{P}|_{\mathbb{F}}$ denotes the exceptional divisor of $p_0 \in V_0$, $H^{\mathbb{F}}$ the hyperplane class of $\mathbb{F}$ and $H^{\mathbb{P}}$ that of $\mathbb{P}$, with $H^{\mathbb{P}} \sim E_0$, we have $\mathrm{Pic}(\mathbb{F}) = \langle H^{\mathbb{F}}, E_0\rangle$ and $\mathrm{Pic}(\mathbb{P}) = \langle H^{\mathbb{P}}\rangle$.

We choose 7 general points on $\mathbb{F}$ and 2 points on $\mathbb{P}$ and we consider these as limits of 9 general points in the general fibre X'_t. Precisely, for $t \in \Delta$ let $\{p_1(t) \dots, p_9(t)\}$ be a general collection of points and assume that $p_1(0), \dots, p_7(0) \in \mathbb{F}$ and $p_8(0), p_9(0) \in \mathbb{P}$. Consider $\widetilde{\mathcal{X}}'$ the blow-up of $\mathcal{X}'$ along the horizontal curves $\{p_i(t)\}_{t\in\Delta}$, with $\widetilde{X}'_t$, $t \in \Delta$, fibres and $\mathcal{E}_i$, $i = 1, \dots, 9$, exceptional divisors. Write also $E_i := \mathcal{E}_i|_{\widetilde{X}'_t}$, for $t \in \Delta$, $i = 1, \dots, 9$. The general fibre is $\widetilde{X}'_t \cong \mathrm{Bl}_{p_1,\dots,p_9}(\mathbb{P}^3)$ so that $\mathrm{Pic}(\widetilde{X}_t) = \langle H, E_1, \dots, \dots, E_9\rangle$. The central fibre is described by $\mathrm{Pic}(\mathbb{F}) = \langle H^{\mathbb{F}}, E_0, E_1, \dots, E_7\rangle$ and $\mathrm{Pic}(\mathbb{P}) = \langle H^{\mathbb{P}}, E_8, E_9\rangle$, where by abuse of notation $\mathbb{F}$ and $\mathbb{P}$ are also the pull-backs in $\widetilde{\mathcal{X}}'$ of the components of X'_0.

3.1.2. *Second degeneration.* We further specialize the points by sending a point from each component of X'_0 to the intersection. Precisely, consider the trivial family $\mathcal{X}'' := X'_0 \times \Delta'$ and, on each fibre over $s \in \Delta'$, take a collection of general points $\{p_1(s), \dots, p_7(s)\} \subset \mathbb{F}$ and $\{p_8(s), p_9(s)\} \subset \mathbb{P}$ such that, on the central fibre, $p_1(0), p_9(0) \in \mathbb{F} \cap \mathbb{P}$.

Consider $\widetilde{\mathcal{X}}''$ the blow-up of $\mathcal{X}''$ along the horizontal curves $\{p_i(s)\}_{s\in\Delta'}$, with $\mathcal{E}_i$ exceptional divisors, $i = 1, \dots, 9$. The components of the fibres are described by the same Picard groups as the components of $\widetilde{X}'_0$ (see Secton 3.1.1). We use the symbols $\mathbb{F}_0$ and $\mathbb{P}_0$ to denote the pull-backs of the components of the central fibre over Δ', X''_0, and the symbol Y_0 for their intersection. Notice that $Y_0 \cong \mathrm{Bl}_{p_1,p_9}(\mathbb{P}^2)$ is a plane blown-up at two points.

The combination of the two above subsequent degenerations produces a degeneration of $\mathrm{Bl}_{p_1,\dots,p_9}(\mathbb{P}^3)$, the blow-up of $\mathbb{P}^3$ at 9 general points, to the union of blown-up spaces $\widetilde{X}''_0 = \mathbb{F}_0 \cup \mathbb{P}_0$ intersecting along a blown-up plane Y_0.

3.1.3. *Intersection table on the central fibre.* Notice that, as a divisor on $\mathbb{F}_0$ (or on $\mathbb{P}_0$), the surface Y_0 is represented by the class $E_0 - E_9$ (resp. $H^{\mathbb{P}} - E_1$).

One can compute the restrictions of any divisor on $\mathbb{F}_0$ or on $\mathbb{P}_0$ to Y_0, by means of the following intersection table for the generators of the Picard groups:

- $H^{\mathbb{P}}|_{Y_0} =: h$,
- $E_1|_{Y_0} =: e_1$,
- $E_2|_{Y_0} = 0$

and

- $H^{\mathbb{F}}|_{Y_0} = 0$,
- $E_0|_{Y_0} = -h$,
- $E_i|_{Y_0} = 0$, $i = 3, \dots, 8$,
- $E_9|_{Y_0} =: e_9$.

In this notation we have $\mathrm{Pic}(Y_0) = \langle h, e_1, e_9\rangle$.

3.2. **The limit linear system.** Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be a linear system of degree-d surfaces in $\mathbb{P}^3$ with nine assigned multiple points in general position. Let D be the corresponding divisor in the blown-up space $\mathrm{Bl}_{p_1,\dots,p_9}(\mathbb{P}^3)$.

In the notation of Section 3.1.1, consider on $\mathcal{X}'$ the twisted line bundle $\mathcal{O}_{\mathcal{X}'}(d) \otimes \mathcal{O}_{\mathcal{X}'}(-\delta\mathbb{P})$. It restricts to $\mathcal{O}_{\mathbb{P}^3}(d)$ on X'_t and, for $t = 0$, to $\mathcal{O}_{\mathbb{F}}(dH_{\mathbb{F}} - \delta E_0)$ on $\mathbb{F}$ and to $\mathcal{O}_{\mathbb{P}}(\delta H^{\mathbb{P}})$ on $\mathbb{P}$. By following the first degeneration we can consider the linear system $\mathcal{L}_t = \mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ on $X'_t \cong \mathbb{P}^3$ and its limit $\mathcal{L}'_0$ on X'_0. We denote by D'_0 the corresponding divisor class in the blown-up central fibre $\widetilde{X}'_0$

By following the second degeneration and blowing-up the nine points on each fibre (see Section 3.1.2), we obtain the limit divisor D''_0 in the blown-up central fibre $\tilde{X}''_0$ that is given by divisors $D^{\mathbb{F}_0}$ and $D^{\mathbb{P}_0}$ on the two components. We consider the restriction maps to Y_0, $D^{\mathbb{F}_0} \to R^{\mathbb{F}_0} := D^{\mathbb{F}_0}|_{Y_0}$ and $D^{\mathbb{P}_0} \to R^{\mathbb{P}_0} := D^{\mathbb{P}_0}|_{Y_0}$ and denote by $\hat{D}^{\mathbb{F}_0}$ and $\hat{D}^{\mathbb{P}_0}$ the kernels respectively. Let $R_0 := R^{\mathbb{F}_0} \cap R^{\mathbb{P}_0}$ denote the intersection of the restricted divisors.

Lemma 3.4. *In the notation of above, for $i \geq 0$ we have*

$$h^i(\widetilde{X}''_0, D''_0) = h^i(\mathbb{P}_0, \hat{D}^{\mathbb{P}_0}) + h^i(\mathbb{F}_0, \hat{D}^{\mathbb{F}_0}) + h^i(Y_0, R_0).$$

Proof. Notice first of all that the assertion holds if we replace h^i by χ, the Euler characteristic. The equality holds for $i = 0$ by construction. Indeed the divisor D''_0 on $\widetilde{X}''_0$, or its associated line bundle, is obtained as fibred product of $D^{\mathbb{F}_0}$ and $D^{\mathbb{P}_0}$ over R_0, see Section 2.1. Finally, since all cohomology groups with $i \geq 2$ vanish, the assertion holds for $i = 1$. □

Lemma 3.5. *In the notation of above, we have*

$$h^i(Bl_{p_1,\ldots,p_s}(\mathbb{P}^n), D) \leq h^i(\widetilde{X}'_0, D'_0) \leq h^i(\widetilde{X}''_0, D''_0),\ i = 0, 1.$$

Proof. The inequalities hold for $i = 0$ by the property of upper semi-continuity of the two degenerations. As $\chi(\mathrm{Bl}_{p_1,\ldots,p_9}(\mathbb{P}^3), D) = \chi(\widetilde{X}'_0, D'_0) = \chi(\widetilde{X}''_0, D''_0)$ and all higher cohomology groups vanish, equalities hold for $i = 1$ as well. □

Remark 3.6. The above construction as well as Lemma 3.4 and Lemma 3.5 is potentially applicable in more general contexts for linear systems in any $\mathbb{P}^n$ and with arbitrary number of points and multiplicities by choosing different specializations and twists, as it was done for instance by the last author in [30]. Nevertheless, it is not easy to find a good degeneration in general.

3.3. **Proof of Theorem 3.1.** In order to prove the theorem, we need the following result.

Proposition 3.7. *The following linear systems are non-special with dimension equal to the virtual dimension: $\mathcal{L}_{3,2m}(m+1, m^6, m-1)$, $\mathcal{L}_{3,2m-1}(m^4, (m-1)^4)$.*

This can be easily deduced from [15] where the authors deal with Cremona reduced linear systems with eight base multiple points. However we include the proof here for the sake of completeness.

Proof. One can easily check that $\mathrm{vdim}(\mathcal{L}_{3,2m}(m+1, m^6, m-1)) = 0$. The statement is obviously true for $m = 1$. By performing two subsequent Cremona transformations of $\mathbb{P}^3$ (see Section 2.3) we reduce from m to $m - 1$. Hence we conclude by induction on m.

Similarly, one proves that $\dim(\mathcal{L}_{3,2m-1}(m^4, (m-1)^4)) = \mathrm{vdim}(\mathcal{L}_{3,2m-1}(m^4, (m-1)^4)) = 0$ by induction on m. □

Proof of Theorem 3.1. Let $Q = \mathcal{L}_{3,2}(1^9)$ be the quadric surface through nine points. The obvious inclusion of linear systems $\mathcal{L} - aQ \subseteq \mathcal{L}$ implies the inequality $\dim(\mathcal{L} - aQ) \leq \dim(\mathcal{L})$. But $\mathcal{L} - aQ = \mathcal{L}_{3,2(m-a)}((m-a)^8)$ and, by Proposition 3.7,

$\dim(\mathcal{L}_{3,2(m-a)}((m-a)^8)) = \mathrm{vdim}(\mathcal{L}_{3,2(m-a)}((m-a)^8)) = m-a+1 \geq 1$. Hence $m-a+1 \leq \dim(\mathcal{L}(m,a))$.

We prove the inverse inequality by degeneration. Let D denote the divisor in $\mathrm{Bl}_{p_1,\ldots,p_9}(\mathbb{P}^3)$ corresponding to $\mathcal{L}(m,a)$:

$$D = 2mH - m\sum_{i=1}^{8} E_i - aE_9.$$

In the notation of Section 3.2, now with $d = 2m$, choose $\delta = m$. In the space $\widetilde{\mathcal{X}}''$ of the second degeneration, we have the following divisors on the components of the central fibre $\widetilde{X}_0''$:

$$D^{\mathbb{F}_0} = 2mH^{\mathbb{F}} - mE_0 - m\sum_{i=1}^{7} E_i,$$
$$D^{\mathbb{P}_0} = mH^{\mathbb{P}} - mE_8 - aE_9.$$

We consider the restriction maps to Y_0, and we obtain the following kernel divisors

$$\hat{D}^{\mathbb{F}_0} = 2mH^{\mathbb{F}} - (m+1)E_0 - (m-1)E_1 - m\sum_{i=2}^{7} E_i,$$
$$\hat{D}^{\mathbb{P}_0} = (m-1)H^{\mathbb{P}} - mE_8 - (a-1)E_9.$$

Firstly, on the component $\mathbb{F}_0$ of the central fibre we have the following. By Proposition 3.7, we obtain that both $D^{\mathbb{F}_0}$ and $\hat{D}^{\mathbb{F}_0}$ are non-special, so that the first cohomology groups vanish; moreover the second is non-effective, namely $h^0 = 0$.

Secondly, on the exceptional component of $\widetilde{X}_0''$ we have the following. The divisor $D^{\mathbb{P}_0}$ is (only) linearly obstructed and has $h^1(D^{\mathbb{P}_0}) = \binom{a+1}{3}$ caused by a line of multiplicity a, see [5]. Moreover the kernel $\hat{D}^{\mathbb{P}_0}$ is non-effective and has $h^1(\hat{D}^{\mathbb{P}_0}) = \binom{a+1}{3}$, see [18].

The above implies that both $D^{\mathbb{F}_0}$ and $D^{\mathbb{P}_0}$ cut the complete linear series on the intersection Y_0 of the components. Using Sections 3.1.3 and 3.2 and the notation there introduced, we have

$$R^{\mathbb{P}_0} = mh - ae_9, \quad R^{\mathbb{F}_0} = mh - me_1.$$

Since $R^{\mathbb{F}_0}$ and $R^{\mathbb{P}_0}$ meet transversally on Y_0, their intersection is given by

$$R_0 = mh - me_1 - ae_9.$$

One computes $h^0(R_0) = m-a+1$ and $h^1(R_0) = \binom{a}{2}$, the speciality being given by a line of multiplicity a, see [5]. Finally, by Lemma 3.4 we obtain $h^0(D_0'') = m-a+1$ and $h^0(D_0'') = \binom{a+1}{3} + \binom{a}{2}$. Now we conclude the proof of the first part of the theorem by upper semi-continuity, see Lemma 3.5.

To prove the last sentence of the theorem, we simply notice that the linear system $\mathcal{L}(m,a)$ splits as follows:

$$\mathcal{L}(m,a) = aQ + \mathcal{L}_{3,2(m-a)}((m-a)^8).$$

The second addend in the right hand side is the moving part of $\mathcal{L}$ and is non-special by Proposition 3.7. This concludes the proof. □

Remark 3.8. In the proof of Theorem 3.1, we argued that the speciality of $\mathcal{L}(m,a)$ is given by aQ and equals the speciality of the limit D''_0 that, using Lemma 3.4, is given by a line of multiplicity a in the base locus of $\hat{D}^{\mathbb{P}_0}$ and a line of multiplicity a in the base locus of R_0.

A geometric interpretation is the following. Let us denote by $\mathcal{L}^{\mathbb{P}_0,\mathbf{m}}$ the *matching linear system* defined by the matching conditions imposed by $R^{\mathbb{F}_0}$ to $R^{\mathbb{P}_0}$, so that we have the following exact sequence of sheaves

$$0 \to |\hat{D}^{\mathbb{P}_0}| \to \mathcal{L}^{\mathbb{P}_0,\mathbf{m}} \to |R_0| \to 0.$$

The emptiness of $|\hat{D}^{\mathbb{P}_0}|$ implies that the limit linear system $|D''_0|$ is the matching linear system; it is of the form $\mathcal{L}_{3,m}(m,m,a)$ where the second and third points, p_8, p_9, are general in $\mathbb{P}_0$, while the first point, p_1, is on the intersection and is the one giving the matching.

In particular, if we follow the quadric $Q = \mathcal{L}(1,1)$ in the degeneration process, by simply setting $m = a = 1$ in the above, we see that its limit is given by a matching linear system on $\mathbb{P}_0$ of the form $\mathcal{L}_{3,1}(1,1,1)$, based at the points p_8, p_9 and p_1 of the central fibre as above. This linear system has only one element that is the plane spanned by the three points. This plane is the special effect variety for the limit of $\mathcal{L}(m,a)$; it is contained with multiplicity a in the base locus and it contributes by $\binom{a+1}{3} + \binom{a}{2}$ to the speciality.

A *weak base locus lemma* for the quadric Q in the case of $\mathcal{L}(m,a)$, is just an easy application of Lemma 2.1. Indeed to prove that Q is contained in the base locus of $\mathcal{L}(m,a)$ with multiplicity at least a, it is enough to show that for every m and a, the restriction $\mathcal{L}(m,a)|_Q$ is empty and this is equivalent to prove that $\mathcal{L}_{2,3m}(m^9,a)$ is empty, which is a well-known fact.

In the next section we will see that in general to obtain such a result is extremely difficult, mostly because of the very little knowledge of linear systems on $\mathbb{P}^1 \times \mathbb{P}^1$.

4. Homogeneous linear systems $\mathcal{L}_{3,2m+1}(m^9)$

In this section we consider linear systems with nine points of multiplicity m and degree $2m+1$. For this class of linear system it is more difficult to understand the relation between the speciality and the presence of the quadric as special effect variety. Even to compute the multiplicity of containment of the quadric in the base locus is not an obvious task. The main result of this section is in fact a vanishing result for linear systems in $\mathbb{P}^2$ which allows to deduce a base locus lemma for the quadric.

Given a linear system $\mathcal{L} = \mathcal{L}_{3,d}(m_1,\ldots,m_s)$ with $s \geq 9$, let Q be the unique quadric surface through the first nine points. Consider the restriction exact sequence

$$0 \to \mathcal{L} - Q \to \mathcal{L} \to \mathcal{L}_{|Q} \to 0. \tag{4.1}$$

The linear system $\mathcal{L}_{|Q}$ is contained in the linear system of the curves of bidegree (d,d) in $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ with nine multiple points and denoted by $\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(d,d)}(m_1,\ldots,m_9)$ as in Section 2.2.

By Lemma 2.1 we know that the system $\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1(d,d)}(m_1,\ldots,m_9)$ has the same dimension as the system $\mathcal{L}_{2,2d-m_1}(d-m_1,d-m_1,m_2,\ldots,m_s)$ in $\mathbb{P}^2$ and in particular the first system is empty if and only if the second one is.

The main part of this chapter is devoted to prove, via degeneration techniques, emptiness results for linear systems in $\mathbb{P}^2$ with ten multiple points. As a straightforward consequence of Theorem 4.2 below, we obtain the following (weak) base locus lemma for the quadric. Let α be any positive integer.

Theorem 4.1 (Quadric base locus lemma)**.** *Let $\mathcal{L} = \mathcal{L}_{3,2m+\alpha}(m^9, m_{10} \ldots, m_s)$ be a non-empty linear system. If $m > 9\alpha$, then the quadric Q through the first nine points is contained in the base locus of $\mathcal{L}$.*

We remark that a major difference between the quadric through nine points in $\mathbb{P}^3$ and the linear cycles in $\mathbb{P}^n$ is in the geometry of their normal bundles. For the last ones the normal bundles are toric bundles so we understand their cohomology groups [5, 18] while for the first one the cohomological information is highly non-trivial.

4.1. **Emptiness of linear systems with ten points in $\mathbb{P}^2$.** The goal of this section is to find a good bound for m to have emptiness of certain linear systems in $\mathbb{P}^2$. More precisely we will prove the following result, which implies Theorem 4.1.

Theorem 4.2. *The linear system $\mathcal{L} = \mathcal{L}_{2,3m+2\alpha}((m+\alpha)^2, m^8)$ is empty for any $m > 9\alpha$.*

We will perform an appropriate degeneration, both of the plane (more precisely, the blown-up at the ten general points) and of the line bundle. The full analysis requires several different degenerations, depending on α and m. Even though this technique was applied before in [10, 13] for homogeneous linear systems with ten points, we will present in detail our approach.

4.1.1. *The first degeneration.* We first consider the trivial family over a disc Δ and blow-up a point in the central fibre. We get a family $\mathcal{X} \to \Delta$ over Δ, where the general fibre X_t for $t \neq 0$ is a $\mathbb{P}^2$, and the central fibre X_0 is the union of two surfaces $\mathbb{P} \cup \mathbb{F}$, where $\mathbb{P} \cong \mathbb{P}^2$ is a projective plane, $\mathbb{F} \cong \mathbb{F}_1$ is a plane blown-up at a point, and $\mathbb{P}$ and $\mathbb{F}$ meet transversally along a smooth rational curve E which is the exceptional divisor on $\mathbb{F}$ and a line on $\mathbb{P}$ (see Figure 2).

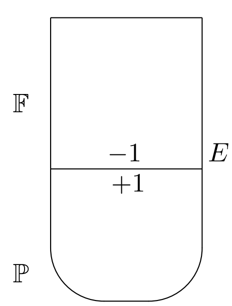

FIGURE 2. The degeneration of the plane

We now choose four general points on $\mathbb{P}$ and six general points on $\mathbb{F}$. Consider these ten points as limits of ten general points in the general fibre X_t and blow these points up in the family $\mathcal{X}$.

We denote by $\mathcal{X}' \to \Delta$ this new family. The general fibre X'_t for $t \neq 0$ is a plane blown-up at ten general points. The central fibre X'_0, shown in Figure 3, is the union $\mathbb{P}_0 \cup \mathbb{F}_0$ where:

- $\mathbb{P}_0$ is a plane blown-up at four general points;
- $\mathbb{F}_0$ is a plane blown-up at seven general points;
- $\mathbb{P}_0$ and $\mathbb{F}_0$ meet transversally along a smooth rational curve E which is a (-1)-curve on $\mathbb{F}_0$, whereas $E^2 = 1$ on $\mathbb{P}_0$ (i.e. it is a line).

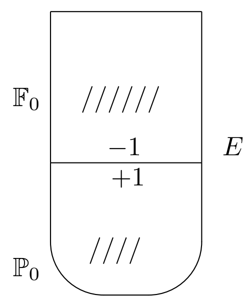

FIGURE 3. The degeneration of the blown-up plane

4.1.2. *The second degeneration.* This degeneration was first introduced in [13], we provide the construction of the degeneration together with the limit bundles computation for the sake of completeness. The interested reader should also consult [10]. We denote by C the (-1)-curve on $\mathbb{F}$ passing through six points that meets the double curve E in two points p_1 and p_2, at the form $\mathcal{L}_3(2, 1^6)$. We consider the family obtained in Section 4.1.1, $\mathcal{X}' \to \Delta$, and we blow-up twice the cubic C on $\mathbb{F}$ and then contract the first exceptional divisor created. In this way, we will obtain a new family $\mathcal{X}'' \to \Delta$ whose general fibre is still a plane blown-up at ten points and whose special fibre over the origin becomes the union of four surfaces. We abuse notations and denote by $\mathbb{F}$ and $\mathbb{P}$ the surfaces of the central fibre in the second degeneration and by $\mathbb{S}$ and $\mathbb{T}$ the exceptional divisors created by the double blow-up of C.

Blow-up C, obtaining the ruled surface $\mathbb{T}$, which is isomorphic to $\mathbb{F}_1$; $\mathbb{T}$ meets $\mathbb{F}$ along C. Notice that the double curve C also represents the negative section of $\mathbb{T}$. The blow-up will create on the surface $\mathbb{P}$ two exceptional divisors G_1 and G_2. These G_i are also fibres of the ruling of $\mathbb{T}$.

Now blow-up C again, creating the ruled surface $\mathbb{S}$. This time $\mathbb{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$; $\mathbb{S}$ meets $\mathbb{F}$ along C, and it meets $\mathbb{T}$ along the negative section. The blow-up affects the surface $\mathbb{P}$, creating two more exceptional divisors F_1 and F_2 which are (-1)-curves on $\mathbb{P}$. By abusing notation we denote by G_1, G_2 their proper transforms that are now (-2)-curves. The surface $\mathbb{S}$ now occurs with multiplicity two in the central fibre of the degeneration, since it was obtained by blowing-up a double curve.

We may now blow $\mathbb{S}$ down the other way. This contracts C on the surface $\mathbb{F}$, and contracts the negative section of $\mathbb{T}$, so that $\mathbb{T}$ becomes a $\mathbb{P}^2$ (by abusing notation, we still denote by $\mathbb{T}$ its image after the contraction of $\mathbb{S}$). The surface $\mathbb{P}$ has the two curves F_1 and F_2 identified. In [10, 13] this operation is called a 2-*throw* of C on $\mathbb{P}$.

4.1.3. *Degenerating the line bundles.* We will now describe limits of line bundles on $\mathbb{P}^2$ via the double degeneration. The limit bundles are bundles in the central fibre, X_0'', of the family $\mathcal{X}'' \to \Delta$ that agree on the intersection of the double curves.

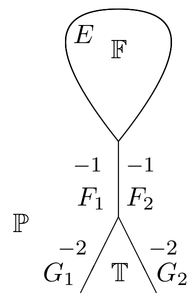

FIGURE 4. The second degeneration

Remark 4.3. The bundle on $\mathbb{P}$ can be interpreted in the geometry of $\mathbb{P}$ where two new compound multiple points have been created, namely two pairs of infinitely near points. We will use the notation of [13] $[m_1, m_2]$, to indicate a multiple point m_1 and an infinitely near multiple point m_2. More precisely, we adopt the notation $[m_1, m_2]$ to denote $m_2F_1 + m_1(F_1 + G_1)$.

Remark 4.4. In this section, as in Section 3.2, we will describe limits of divisors, and not limits of linear systems. This difference is emphasized in Remark 4.3. In particular, Proposition 4.5 should be understood as describing all possible limit divisors D in the linear system $\mathcal{L}_{2,d}((m+\alpha)^2, m^8)$ on the central fibre X''_0. However, in order to simplify the language and also to be consistent with notation previously used in [10, 13], in this section we abuse notations and we use the linear system terminology. We must also emphasize now that this degeneration is different than the one we exploited in Section 3.2. More precisely, only the first degeneration of the blown-up projective space $\mathbb{P}^3$, described in Sections 3.2 and 4.1.1 coincide. In Section 3.2 this degeneration was denoted by $\widetilde{\mathcal{X}}'$ while in Section 4.1.1 it was denoted by $\mathcal{X}'$. However, in Section 3.2 the second degeneration was obtained by specializing points on the intersection of the two components while in Section 4.1.2 the second degeneration is obtained from flopping a negative curve. In order to highlight this major difference we choose different notations. More precisely even if both represent degenerations of blown-up projective projective spaces $-\mathbb{P}^3$ in Section 3.2 and $\mathbb{P}^2$ here$-$ we will denote them by $\widetilde{\mathcal{X}}''$ and $\mathcal{X}''$ in Sections 3.2 and Sections 4.1.2 respectively.

We will now determine all possible limit bundles of the linear system on the general fibre

$$\mathcal{L}_{2,d}((m+\alpha)^2, m^8).$$

- The line bundle on $\mathbb{P}$ must be of the form $\mathcal{L}_\delta(m^4, [a,b], [a,b])$, where δ, a, b represent the twisting parameters. This line bundle meets the 4 times blown-up line $\delta - 2a - 2b$ times. The system on $\mathbb{F}$ is of the form $\mathcal{L}_{\mathbb{F}} = \mathcal{L}_{2,t}(\delta - 2a - 2b, y^4, y'^2)$, for some t, y, y'. The assumption that $\mathcal{L}_{\mathbb{F}}$ doesn't meet the cubic C implies that the degree of $\mathcal{L}_{\mathbb{F}}$ has to be even, write $t = 2e$. Indeed $0 = \mathcal{L}_{\mathbb{F}} \cdot C = 3t - 2\delta + 4a + 4b - 4y - 2y'$.

  Moreover, because $y = m - a - b$ and $y' = m - a - b + \alpha$, then one can check that

$$\delta = 3e - 3m + 5a + 5b - \alpha.$$

- Consider the intersection of $\mathbb{S}$ and $\mathbb{F}$ that is a fibre on $\mathbb{S}$ and the cubic on $\mathbb{F}$. Note that $\mathcal{L}_{\mathbb{S}}$ is a horizontal bundle so it must have bidegree $(b, 0)$. Moreover $\mathcal{L}_{\mathbb{T}}$ meets a fibre $a-b$ times and does not meet the negative section B. Hence
$$\mathcal{L}_{\mathbb{T}} = \mathcal{L}_{2,a-b}.$$
- The last parameter to be determined is e. We compute it by observing that the limit bundle should have degree d, that is the degree on the bundle on the general fibre. By pulling-back a line in the plane we get a line on $\mathbb{P}$, a fibre on $\mathbb{F}$, a fibre on $\mathbb{T}$ and a fibre on $\mathbb{S}$. Therefore the intersection number with all the bundles from above will have to add up to d. We obtain
$$e = \frac{d-3(a+b)}{2}.$$

Solving this system of linear equations we obtain
$$\mathcal{L}_{\mathbb{F}} = \mathcal{L}_{2,d-3a-3b}\left(\frac{3d}{2} - 3m - \frac{3(a+b)}{2} - \alpha, (m-a-b)^4, (m-a-b+\alpha)^2\right).$$

The normal bundle of the surface $\mathbb{S}$ has bidegree $(-1,-1)$ so we can contract the other ruling. This will contract the cubic C on $\mathbb{F}$ to a point by a series of Cremona transformations to $\mathcal{L}_{\mathbb{F}}$, see Section 2.3. After contracting the surface $\mathbb{S}$ the bundle on $\mathbb{F}$ becomes
$$\mathcal{L}_{\mathbb{F}} = \mathcal{L}_{2,3m-\frac{d}{2}-\frac{3(a+b)}{2}+\alpha}\left(0, \left(2m - \frac{d}{2} - \frac{a+b}{2} + \alpha\right)^4, \left(2m - \frac{d}{2} - \frac{a+b}{2}\right)^2\right).$$
The zero multiplicity of $\mathcal{L}_{\mathbb{F}}$ represents the image of the cubic after the Cremona transformations. Since we are contracting the surface $\mathbb{S}$ we will simply ignore this multiplicity. We recall that contracting the cubic on $\mathbb{F}$ will also affect the surface on $\mathbb{P}$ by identifying the last two (-1)-curves created on $\mathbb{P}$, namely F_1 and F_2.

For the future analysis we will work with the normalization of $\mathbb{P}$, so we will consider F_1 and F_2 disjoint as before. We obtain the following result.

Proposition 4.5. *All limits of the bundle $\mathcal{L}_d((m+\alpha)^2, m^8)$ are of the following form for some choice of the parameters a and b:*

- $\mathcal{L}_{\mathbb{P}} = \mathcal{L}_{2,\frac{3d}{2}-3m+\frac{a+b}{2}-\alpha}\left(m^4, [a,b], [a,b]\right)$,
- $\mathcal{L}_{\mathbb{F}} = \mathcal{L}_{2,3m-\frac{d}{2}-\frac{3(a+b)}{2}+\alpha}\left((2m-\frac{d}{2}-\frac{a+b}{2}+\alpha)^4, (2m-\frac{d}{2}-\frac{a+b}{2})^2\right)$,
- $\mathcal{L}_{\mathbb{T}} = \mathcal{L}_{2,a-b}$.

We study the effectivity of $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ for $d = 3m+2\alpha$ and $m > 9\alpha$. Notice that by substituiting $d = 3m+2\alpha$ we obtain the following bundles on $\mathbb{P}$ and $\mathbb{F}$
$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}_{2,\frac{3m}{2}+\frac{a+b}{2}+2\alpha}\left(m^4, [a,b], [a,b]\right),$$
$$\mathcal{L}_{\mathbb{F}} = \mathcal{L}_{2,\frac{3m}{2}-\frac{3(a+b)}{2}}\left(\left(\frac{m}{2} - \frac{a+b}{2}\right)^4, \left(\frac{m}{2} - \frac{a+b}{2} - \alpha\right)^2\right).$$

Remark 4.6. The following two statements are obvious

- The linear system on $\mathbb{T}$, $\mathcal{L}_{\mathbb{T}} = \mathcal{L}_{2,a-b}$, is nonempty if and only if $a \geq b$.
- The linear system on $\mathbb{F}$, $\mathcal{L}_{\mathbb{F}}$ is non-empty if any only if $a+b \leq m$.

We will now analyse the linear system on $\mathbb{P}$. We denote by Q_i the four quartics $\mathcal{L}_4(2^3, 1, [1, 1]^2)$ on $\mathbb{P}$ and we see that these (-1)-curves split off the system if $m \geq 8$. Indeed

$$\mathcal{L}_{\mathbb{P}} Q_i = \mathcal{L}_{2, \frac{3m}{2}+\frac{a+b}{2}+2\alpha}(m^4, [a, b], [a, b])\ \mathcal{L}_{2,4}(2^3, 1, [1, 1]^2) = 8\alpha - m.$$

We further apply a series of four Cremona transformations based rspectively at the points $1-2-3, 4-5-8, 4-6-7, 1-2-3$ to contract the four quartics. Note that this series of these Cremona transformations contracts the four quartics to a point at the same time.

$$\mathrm{Cr}(\mathcal{L}_{\mathbb{P}}) = \mathcal{L}_{2, \frac{a+b}{2}-\frac{5m}{2}+18\alpha}((8\alpha-m)^4, [a-m+4\alpha, b-m+4\alpha], [a-m+4\alpha, b-m+4\alpha])$$

For $m \geq 8$ the exceptional divisors corresponding to the first four points are (-1)-curves that split off the system. These exceptional divisors represent the four quadrics; we will remove them and forget the zero multiplicities created. The residual system is

$$\mathcal{L}'_{\mathbb{P}} = \mathcal{L}_{2, \frac{a+b}{2}-\frac{5m}{2}+18\alpha}([a-m+4\alpha, b-m+4\alpha], [a-m+4\alpha, b-m+4\alpha])$$

It is obvious that $\mathcal{L}_{\mathbb{P}}$ is empty if and only if $\mathcal{L}'_{\mathbb{P}}$ is empty.

We are now ready to prove the main result of this section.

Proof of Theorem 4.2. We want to prove that $\mathcal{L} = \mathcal{L}_{2,3m+2\alpha}((m+\alpha)^2, m^8)$ is empty for $m > 9\alpha$.

We assume by contradiction that there are some values of the parameters a and b for which both linear systems $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ are non-empty in the central fibre of the degeneration. If $\mathcal{L}_{\mathbb{P}}$ is non-empty then the degree of $\mathcal{L}'_{\mathbb{P}}$ is positive. In particular

$$a + b \geq 5m - 36\alpha.$$

On the other hand, since $\mathcal{L}_{\mathbb{F}}$ is non-empty, by Remark 4.6, we must have

$$a + b \leq m.$$

These two inequalities lead to a contradiction, hence the linear system $\mathcal{L}_{2,3m+2\alpha}((m+\alpha)^2, m^8)$ is empty. □

In particular, Theorem 4.2 gives the following consequence.

Proposition 4.7. *If $m \geq 8$, then the linear system $\mathcal{L}_{2,3m+2}((m+1)^2, m^8)$ is empty.*

Proof. First, we check cases $m = 8, 9$ by computer. For $m \geq 10$ we apply Theorem 4.1 with $\alpha = 1$. □

4.2. **Classification of homogeneous linear systems $\mathcal{L}_{3,2m+1}(m^9)$.** We are now in position to prove the complete classification of homogeneous linear systems in $\mathbb{P}^3$ of degree $2m + 1$ and with nine points.

An easy consequence of Proposition 4.7 is the following.

Lemma 4.8. *The linear system $\mathcal{L}_{2,3m+2}((m+1)^2, m^8)$ satisfies:*

$$\mathrm{h}^0(\mathcal{L}_{2,3m+2}((m+1)^2, m^8)) = \chi(\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(2m+1,2m+1)}(m^9))$$

for $m \leq 8$ and it is empty for $m \geq 8$.

Proof. We check by computer the statement for $m \leq 7$. For $m \geq 8$ we use Proposition 4.7. □

By Corollary 2.1, the previous lemma have the following straightforward consequence.

Corollary 4.9. *The linear system $\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(2m+1,2m+1)}(m^9)$ is non-special for every $m\geq 1$ and it is empty for $m\geq 8$.*

Theorem 4.10. *A linear system $\mathcal{L}=\mathcal{L}_{3,2m+1}(m^9)$ is special if and only if $m\geq 9$. In particular we have:*

- *$\dim(\mathcal{L}_{3,2m+1}(m^9))=\operatorname{vdim}(\mathcal{L}_{3,2m+1}(m^9))$ for $m\leq 8$;*
- *$\dim(\mathcal{L}_{3,2m+1}(m^9))=60$ for $m\geq 7$;*
- *the quadric Q through the nine base points is in the base locus of $\mathcal{L}$ with multiplicity $m-7$, for any $m\geq 8$.*

Proof. The restriction exact sequence (4.1) gives in this case:

$$0\to\mathcal{L}_{3,2(m-1)+1}((m-1)^9)\to\mathcal{L}_{3,2m+1}(m^9)\to\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(2m+1,2m+1)}(m^9)\to 0.$$

By using induction on $m\geq 1$ and Corollary 4.9, we deduce that the linear system is non-special if and only if $m\leq 8$ (notice that in case $m=8$ we have $\chi(\mathcal{L}_{\mathbb{P}^1\times\mathbb{P}^1,(17,17)}(8^9))=0$). In order to prove that the quadric is contained in the base locus of $\mathcal{L}$ with multiplicity $m-7$, it is enough to use Corollary 4.9 and to notice that $\dim(\mathcal{L}_{3,15}(7^9))\neq\dim(\mathcal{L}_{3,13}(6^9))$. □

A straightforward consequence of the previous theorem is the following:

Corollary 4.11. *Laface-Ugaglia Conjecture (see Conjecture 5.1 in Section 5) is true for any homogeneous linear system with nine points of multiplicity m and degree $d\leq 2m+1$.*

5. Proof of Laface-Ugaglia Conjecture for linear systems with 9 points and multiplicities bounded by 8

Let $\mathcal{L}=\mathcal{L}_{3,d}(m_1,\ldots,m_9)$ be the linear system of degree d hypersurfaces of $\mathbb{P}^3$ with s general multiple points of multiplicities $m_1,\ldots,m_9$. In this section we will assume that $d\geq m_1\geq m_2\geq\ldots\geq m_9$. Let $Q=\mathcal{L}_{3,2}(1^9)$ be the unique quadric surface through the nine base points. We adopt the following notation

$$q(\mathcal{L})=\chi(\mathcal{L}_{|Q})=(d+1)^2-\sum_{i=1}^{9}\binom{m_i+1}{2}. \tag{5.1}$$

Laface and Ugaglia formulated their conjecture in [27, Conjecture 4.1] and [29, Conjecture 6.3]. Following the definition of linear speciality introduced in [5], we can reformulate this conjecture in the following way.

Conjecture 5.1 (Laface-Ugaglia Conjecture)**.** *Given a Cremona reduced linear system $\mathcal{L}$ in $\mathbb{P}^3$, we have*

(1) if $q(\mathcal{L})\leq 0$, then $\mathrm{h}^0(\mathcal{L})=\mathrm{h}^0(\mathcal{L}-Q)$;
(2) if $q(\mathcal{L})>0$, then $\mathcal{L}$ is linearly non-special.

Remark 5.2. Since $\mathcal{L}$ is Cremona reduced, i.e. $m_1+m_2+m_3+m_4\leq 2d$, it does not contain any plane in the base locus. Hence Conjecture 5.1 says that if $q(\mathcal{L})>0$, then $\mathcal{L}$ is special if and only if $m_1+m_2-d\geq 2$ and in this case:

$$\mathrm{h}^0(\mathcal{L})=\operatorname{ldim}(\mathcal{L})=\chi(\mathcal{L})+\sum_{i,j}\binom{m_i+m_j-d+1}{3},$$

where ldim denotes the affine linear dimension, see [18, Definition 1.2].

Remark 5.3. If $q(\mathcal{L}) \leq 0$ and $\mathrm{h}^0(\mathcal{L}) = \mathrm{h}^0(\mathcal{L}-Q)$, from the exact sequence (4.1) we obtain that

$$\mathrm{h}^1(\mathcal{L}) = \mathrm{h}^1(\mathcal{L}-Q) - q(\mathcal{L}).$$

This means that the quadric Q is a special effect surface for the linear system $\mathcal{L}$.

Remark 5.4. We point out that a quadric surface in the base locus can give speciality even if it is simple. Consider e.g. the linear system $\mathcal{L} = \mathcal{L}_{3,8}(4^7, 3^2)$ which have $\mathrm{h}^0(\mathcal{L}) = 6$, $\mathrm{h}^1 = 1 = -q(\mathcal{L})$. This system contains in its base locus the quadric Q through the nine points, but does not contain $2Q$.

This behaviour is different from the case of linear special effect varieties, for which any linear cycle of dimension l contributes to the speciality only if its multiplicity in the base locus is at least $l+1$.

Remark 5.5. Notice that when a linear system $\mathcal{L}$ has a quadric surface as special effect variety, then the computation of $\mathrm{h}^1(\mathcal{L})$ is quite difficult in general. In fact the quasi-homogeneous systems classified in Section 3 form a very special family for which we understand completely the situation, but this does not happen in the general case.

Let $\mathcal{L}$ be a linear system with $q(\mathcal{L}) \leq 0$. Assume that

- $q(\mathcal{L}-kQ) \leq 0$ for any $0 \leq k \leq \overline{k}$ and $q(\mathcal{L}-(\overline{k}+1)Q) > 0$,
- $\mathcal{L}-kQ$ restricts to non-special linear systems on the quadric Q, for any $0 \leq k \leq \overline{k}$.

Then, by using Remark 5.3, we get the following formula:

$$\mathrm{h}^1(\mathcal{L}) = -\sum_{k=0}^{\overline{k}} q(\mathcal{L}-kQ).$$

By applying this formula to the special case of quasi-homogeneous systems $\mathcal{L}_{3,2m}(m^8, a)$ and using Remark 3.2, we recover exactly the formula $\mathrm{h}^1(\mathcal{L}) = \binom{a+1}{3} + \binom{a}{2}$ of Theorem 3.1. In this case $\overline{k} = a$.

The problem in general is to determine the value of $\overline{k}$. Let us see an example: if $\mathcal{L} = \mathcal{L}_{3,13}(8, 6^8)$, then $q(\mathcal{L}) = -8$, $q(\mathcal{L}-q) = -4$, $q(\mathcal{L}-2Q) = -1$, while $q(\mathcal{L}-3Q) = 1 > 0$, hence we have $\mathrm{h}^1(\mathcal{L}) = 8+4+1 = 13$ and in this case $\overline{k} = 2$.

Remark 5.6. Given two vectors $v = (m_1, \ldots, m_s)$ and $v' = (m'_1, \ldots, m'_s)$ in $\mathbb{N}^s$, we write $v' \leq v$ if and only if $m'_i \leq m_i$ for any $1 \leq i \leq s$.

It is well-known that if a linear system $\mathcal{L}_{n,d}(v)$ is non-special and non-empty, then also any linear system $\mathcal{L}_{n,d}(v')$ is non-special and non-empty for any vector $v' \leq v$.

Now we give the proof of Laface-Ugaglia Conjecture for any linear system with nine points of multiplicities bounded by 8. We start with a lemma whose proof is essentially computational.

Lemma 5.7. *If a linear system $\mathcal{L} = \mathcal{L}_{3,d}(m_1, \ldots, m_9)$ is such that $m = \max(m_i) \leq 8$ and $d < 2m$, then it satisfies Conjecture 5.1.*

Proof. First of all it is clear that if $d < m$ the system is empty, so we assume $d \geq m$. Assume that $\mathcal{L}$ is Cremona reduced, that is

$$m_1 + m_2 + m_3 + m_4 \leq 2d. \tag{5.2}$$

Now if $d = m$, then from (5.2) we have that $m_1 = m$ and $m_2 < m$. Therefore by applying [5, Theorem 5.3], we have that if $\sum_{i=1}^{9} m_i \leq 3d + 2$ then $\mathcal{L}$ is linearly non-special. Hence we can also assume

$$\sum_{i=1}^{9} m_i > 3d + 2. \tag{5.3}$$

For any $m \leq 8$ there are only the following systems satisfying conditions (5.2) and (5.3): $\mathcal{L}_{3,6}(6, 2^8)$ and $\mathcal{L}_{3,7}(7, 3, 2^7)$. It is easy to check that these two system are linearly non-special, hence they satisfy the conjecture.

Assume now that $d \geq m + 1$. We know, by [2], that Laface-Ugaglia conjecture is true for any linear system with multiplicities bounded by 5. So we can assume $6 \leq m \leq 8$.

Moreover, by applying again [5, Theorem 5.3], we have that if $\sum_{i=1}^{9} m_i \leq 3d+3$ then $\mathcal{L}$ is linearly non-special. Hence we can also assume

$$\sum_{i=1}^{9} m_i > 3d + 3. \tag{5.4}$$

Now we list all the possible linear systems which satisfy conditions (5.2) and (5.4), for any $5 \leq m \leq 8$ and any $m + 1 \leq d \leq 2m - 1$.

Then we prove that all the cases in the list satisfy the conjecture using the following procedure. For any degree we start to check the cases $\mathcal{L} = \mathcal{L}_{3,d}(m_1, \ldots, m_8) = \mathcal{L}_{3,d}(v)$ for the higher vectors v. We compute $\mathrm{h}^0(\mathcal{L})$ by means of the computer system `Macaulay2` as explained in Section 2.4.

If $\mathcal{L}$ is non-special and non-empty, then we know, by Remark 5.6, that also any linear system $\mathcal{L}_{3,d}(v')$ is non-special and non-empty for any vector $v' \leq v \in \mathbb{Z}^9$, hence we greatly decrease the number of cases to be checked.

If $\mathcal{L}$ is linearly non-special, then we apply [5, Lemma 5.5 and Remark 5.6] and we obtain again that any system $\mathcal{L}_{3,d}(v')$, for $v' \leq v \in \mathbb{Z}^9$, is linearly non-special. Hence we further reduce the number of cases to be checked and we obtain at the end the lists contained in Tables 1, 2, 3. Notice that in the tables the special and linearly non-special systems are marked with a $*$.

By applying this procedure we complete the proof of the lemma. □

We give now the main result of this section:

Theorem 5.8. *Conjecture 5.1 is true for any linear system $\mathcal{L}_{3,d}(m_1, \ldots, m_9)$ such that $m = \max(m_i) \leq 8$.*

Proof. If the degree $d \leq 2m - 1$ the result follows from Lemma 5.7.

If $d = 2m$, by Theorem 3.1, we know that the quasi homogeneous linear systems $\mathcal{L}_{3,2m}(m^8, a)$ are special if and only if $2 \leq a \leq m$ and they clearly satisfy Conjecture 5.1. Arguing as in Lemma 5.7, in order to complete the proof we need to check all the linear systems satisfying (5.2) and (5.4) for any $6 \leq m \leq 8$.

The list of these cases (reduced by Remark 5.6) is contained in Table 4 and we checked all of them by computer.

Now if $d \geq 2m+1$, we know by Theorem 4.10 that the linear system $\mathcal{L}_{3,2m+1}(m^9)$ is non-special and non-empty. Then we deduce that any homogeneous linear system $\mathcal{L}_{3,d}(m^9)$ for $d \geq 2m + 1$ is also non-special and non-empty.

Finally we deduce that any (non-homogeneous) linear system $\mathcal{L}_{3,d}(m_1, \ldots, m_9)$ with $m_i \leq m$ is non-special and non-empty, by Remark 5.6. This completes the proof. □

5.1. **Future directions.** We conclude this paper by pointing out possible future directions (both theoretical and computational) in establishing the Laface-Ugaglia Conjecture for nine points. On the one hand, one can introduce further degenerations of $\mathbb{P}^2$ in order to obtain a better bound in the base locus lemma, Theorem 4.1. On the other hand, the combination of the results of Section 4 and pf similar computer-based computations as the one performed in this section could improve the bound on the multiplicities of Theorem 5.8.

5.2. **Tables.**

Table 1. The case $m = 6$

degree	$(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9)$	q	h[0]	h[1]
11	(6, 5, 5, 5, 5, 5, 5, 5, 5)	3	28	0
11	(6, 6, 5, 5, 5, 5, 5, 5, 4)	2	22	0
11	(6, 6, 5, 5, 5, 5, 5, 5, 5)	-3	10	3
11	(6, 6, 6, 4, 4, 4, 4, 4, 4)	21	76	0
10	(6, 5, 5, 4, 4, 4, 4, 4, 4)	10	40	0
10	*(6, 6, 4, 4, 4, 4, 4, 4, 4)	9	35	1
10	*(6, 6, 5, 3, 3, 3, 3, 3, 3)	28	80	1
9	(6, 4, 4, 4, 4, 4, 4, 4, 3)	3	14	0
9	(6, 4, 4, 4, 4, 4, 4, 4, 4)	-1	5	1
9	*(6, 5, 4, 3, 3, 3, 3, 3, 3)	18	50	1
9	*(6, 6, 3, 3, 3, 3, 3, 3, 3)	16	42	4
8	*(6, 4, 3, 3, 3, 3, 3, 3, 3)	8	20	1

TABLE 2. The case $m = 7$

degree	$(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9)$	q	h^0	h^1
13	(7, 6, 6, 6, 6, 6, 6, 6, 6)	0	28	0
13	(7, 7, 6, 6, 6, 6, 6, 5, 5)	5	42	0
13	(7, 7, 6, 6, 6, 6, 6, 6, 4)	4	36	0
13	(7, 7, 6, 6, 6, 6, 6, 6, 5)	-1	22	1
13	(7, 7, 6, 6, 6, 6, 6, 6, 6)	-7	10	10
13	(7, 7, 7, 5, 5, 5, 5, 5, 5)	22	98	0
12	(7, 6, 6, 5, 5, 5, 5, 5, 5)	9	49	0
12	*(7, 7, 5, 5, 5, 5, 5, 5, 5)	8	43	1
12	*(7, 7, 6, 4, 4, 4, 4, 4, 4)	32	112	1
11	(7, 5, 5, 5, 5, 5, 5, 5, 4)	1	15	0
11	(7, 5, 5, 5, 5, 5, 5, 5, 5)	-4	5	5
11	*(7, 6, 5, 4, 4, 4, 4, 4, 4)	20	70	1
11	*(7, 6, 6, 3, 3, 3, 3, 3, 3)	38	110	2
11	*(7, 7, 4, 4, 4, 4, 4, 4, 4)	18	60	4
11	*(7, 7, 5, 3, 3, 3, 3, 3, 3)	37	105	4
10	*(7, 5, 4, 4, 4, 4, 4, 4, 4)	8	28	1
10	*(7, 5, 5, 3, 3, 3, 3, 3, 3)	27	74	2
10	* (7, 6, 4, 3, 3, 3, 3, 3, 3)	26	70	4
10	*(7, 7, 3, 3, 3, 3, 3, 3, 3)	23	58	10
9	*(7, 4, 4, 3, 3, 3, 3, 3, 3)	16	38	2
9	*(7, 5, 3, 3, 3, 3, 3, 3, 3)	15	35	4
8	*(7, 3, 3, 3, 3, 3, 3, 3, 3)	5	9	8

Table 3. The case $m = 8$

degree	$(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9)$	q	h^0	h^1
15	(8, 7, 7, 7, 7, 7, 7, 7, 6)	3	52	0
15	(8, 7, 7, 7, 7, 7, 7, 7, 7)	-4	28	4
15	(8, 8, 7, 7, 7, 7, 7, 6, 6)	2	44	0
15	(8, 8, 7, 7, 7, 7, 7, 7, 5)	1	37	0
15	(8, 8, 7, 7, 7, 7, 7, 7, 6)	-5	22	6
15	(8, 8, 7, 7, 7, 7, 7, 7, 7)	-12	10	22
15	(8, 8, 8, 6, 6, 6, 6, 6, 6)	22	120	0
14	(8, 7, 7, 6, 6, 6, 6, 6, 6)	7	56	0
14	*(8, 8, 6, 6, 6, 6, 6, 6, 6)	6	49	1
14	*(8, 8, 7, 5, 5, 5, 5, 5, 5)	35	147	1
14	*(8, 8, 8, 4, 4, 4, 4, 4, 4)	57	203	3
13	(8, 6, 6, 6, 6, 6, 6, 5, 5)	4	34	0
13	(8, 6, 6, 6, 6, 6, 6, 6, 4)	3	28	0
13	(8, 6, 6, 6, 6, 6, 6, 6, 5)	-2	15	2
13	(8, 6, 6, 6, 6, 6, 6, 6, 6)	-8	5	13
13	*(8, 7, 6, 5, 5, 5, 5, 5, 5)	21	91	1
13	*(8, 7, 7, 4, 4, 4, 4, 4, 4)	44	154	2
13	*(8, 8, 5, 5, 5, 5, 5, 5, 5)	19	79	4
13	*(8, 8, 6, 4, 4, 4, 4, 4, 4)	43	148	4
12	*(8, 6, 5, 5, 5, 5, 5, 5, 5)	7	35	1
12	*(8, 6, 6, 4, 4, 4, 4, 4, 4)	31	105	2
12	*(8, 7, 5, 4, 4, 4, 4, 4, 4)	30	100	4
12	*(8, 8, 4, 4, 4, 4, 4, 4, 4)	27	85	10
11	*(8, 5, 5, 4, 4, 4, 4, 4, 4)	18	56	2
11	*(8, 6, 4, 4, 4, 4, 4, 4, 4)	17	52	4
11	*(8, 6, 5, 3, 3, 3, 3, 3, 3)	36	98	5
11	*(8, 7, 4, 3, 3, 3, 3, 3, 3)	34	90	10
11	*(8, 8, 3, 3, 3, 3, 3, 3, 3)	30	74	20
10	*(8, 4, 4, 4, 4, 4, 4, 4, 4)	5	14	8
10	*(8, 5, 4, 3, 3, 3, 3, 3, 3)	24	56	5
10	*(8, 6, 3, 3, 3, 3, 3, 3, 3)	22	50	10
9	*(8, 4, 3, 3, 3, 3, 3, 3, 3)	12	21	11

Table 4. The case $d = 2m$

$(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9)$	q	h^0	h^1
(6, 6, 6, 6, 6, 5, 5, 5, 5)	4	35	0
(6, 6, 6, 6, 6, 6, 5, 5, 4)	3	29	0
(6, 6, 6, 6, 6, 6, 5, 5, 5)	-2	16	2
(6, 6, 6, 6, 6, 6, 6, 4, 4)	2	23	0
(6, 6, 6, 6, 6, 6, 6, 5, 3)	1	18	0
(6, 6, 6, 6, 6, 6, 6, 5, 4)	-3	11	3
(6, 6, 6, 6, 6, 6, 6, 5, 5)	-8	6	13
(7, 7, 7, 7, 7, 6, 6, 6, 6)	1	36	0
(7, 7, 7, 7, 7, 7, 6, 6, 5)	0	29	0
(7, 7, 7, 7, 7, 7, 6, 6, 6)	-6	16	8
(7, 7, 7, 7, 7, 7, 7, 5, 4)	4	37	0
(7, 7, 7, 7, 7, 7, 7, 5, 5)	-1	23	1
(7, 7, 7, 7, 7, 7, 7, 6, 3)	2	26	0
(7, 7, 7, 7, 7, 7, 7, 6, 4)	-2	18	2
(7, 7, 7, 7, 7, 7, 7, 6, 5)	-7	1	10
(7, 7, 7, 7, 7, 7, 7, 6, 6)	-13	6	26
(8, 8, 8, 8, 7, 7, 7, 7, 7)	5	69	0
(8, 8, 8, 8, 8, 7, 7, 7, 6)	4	61	0
(8, 8, 8, 8, 8, 7, 7, 7, 7)	-3	36	3
(8, 8, 8, 8, 8, 8, 7, 6, 6)	3	53	0
(8, 8, 8, 8, 8, 8, 7, 7, 5)	2	46	0
(8, 8, 8, 8, 8, 8, 7, 7, 6)	-4	29	4
(8, 8, 8, 8, 8, 8, 7, 7, 7)	-11	16	19
(8, 8, 8, 8, 8, 8, 8, 6, 5)	1	38	0
(8, 8, 8, 8, 8, 8, 8, 6, 6)	-5	23	6
(8, 8, 8, 8, 8, 8, 8, 7, 3)	3	35	0
(8, 8, 8, 8, 8, 8, 8, 7, 4)	-1	26	1
(8, 8, 8, 8, 8, 8, 8, 7, 5)	-6	18	8
(8, 8, 8, 8, 8, 8, 8, 7, 6)	-12	11	22
(8, 8, 8, 8, 8, 8, 8, 7, 7)	-19	6	45

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