ON MAXIMUM, TYPICAL, AND GENERIC RANKS

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Abstract. We show that for several notions of rank including tensor rank, Waring rank, and generalized rank with respect to a projective variety, the maximum value of rank is at most twice the generic rank. We show that over the real numbers, the maximum value of the real rank is at most twice the smallest typical rank, which is equal to the (complex) generic rank.

1. Introduction

In many areas of applied mathematics, machine learning, and engineering, the notion of shortest decomposition of a vector into simple vectors is of prime importance. See for example [CM96, BCMT10, BBCM11, Lan12] and [DSS09, Chapter 4]. The length of the shortest decomposition is usually called the **rank** of the vector.

In this article we consider the rank of a vector with respect to a variety over an arbitrary field $\mathbb{F}$, but we will highlight the real and complex situations later on. Let $X \subset \mathbb{FP}^n$ be a projective variety and let $\hat{X} \subset \mathbb{F}^{n+1}$ be the affine cone over X. The variety X is called **nondegenerate** if X (or equivalently $\hat{X}$) is not contained in a hyperplane. In this case, for any vector $v \in \mathbb{F}^{n+1}$, $v \neq 0$, we can define the **rank of v with respect to X** (**X-rank** of v for short) as follows:

$$\operatorname{rank}_X(v) = \min r \quad \text{such that} \quad v = \sum_{i=1}^{r} x_i \quad \text{where} \quad x_i \in \hat{X},$$

i.e., the X-rank of v is the length of the shortest decomposition of v into elements of $\hat{X}$. We will assume that the variety X is irreducible, as is the case in the applications of interest.

For example, tensor rank (real or complex) is rank with respect to the Segre variety, symmetric tensor rank (also called Waring rank) is rank with respect to the Veronese variety, and anti-symmetric tensor rank is rank with respect to the Grassmannian variety. See section 3 for more discussion of examples.

A rank r is called **generic** if the vectors of X-rank r contain a Zariski open subset of $\mathbb{F}^{n+1}$. It is well-known that over any algebraically closed field there is a unique generic X-rank for a nondegenerate variety X. Over $\mathbb{C}$, we can equivalently define rank r to be generic if the set of vectors of rank r contains an open subset of $\mathbb{C}^{n+1}$ with respect to the standard product topology. A significant effort has gone into the calculation of the generic rank for various varieties X, and the generic rank is fairly well understood for various tensor ranks over $\mathbb{C}$, see section 3.

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Much less is known about the maximal X-rank. One of our main results shows that the generic rank and the maximal rank cannot be too far apart:

Theorem 1. *Let $X \subset \mathbb{FP}^n$ be an irreducible nondegenerate variety over an algebraically closed field $\mathbb{F}$. Let $r_{\max}$ be the maximum value of rank with respect to X and let r_{gen} be the generic value of rank with respect to X. Then*

$$r_{\max} \leq 2r_{\text{gen}}.$$

While the above bound is elementary, we will show in Section 3 that it strongly improves many existing bounds on the maximal tensor rank. Also, due to its generality it applies to any notion of tensor rank: regular, symmetric and antisymmetric. We show in Theorem 6 that the above bound can be slightly improved to $r_{\max} \leq 2r_{\text{gen}} - 2$ in the special case where the Zariski closure of the vectors of rank $r_{\text{gen}} - 1$ forms a hypersurface in $\mathbb{F}^{n+1}$.

Over the real numbers a rank r is called **typical** if the set of vectors of rank r contains an open subset of $\mathbb{R}^{n+1}$ with respect to the Euclidean topology. Unlike over the complex numbers, there may exist more than one typical rank. However, the lowest typical rank is equal to the generic rank over the complexification of X. While this is probably well-known to the experts, we did not find it explicitly stated in the literature in this generality:

Theorem 2. *Let $X \subset \mathbb{RP}^n$ be an irreducible nondegenerate real projective variety. Let r_0 be the minimal typical rank with respect to X and let r_{gen} be the generic rank with respect to the complexification $X_{\mathbb{C}} = X \otimes \mathbb{C}$. Then*

$$r_0 = r_{\text{gen}}.$$

We are able to show a very similar and elementary bound for maximal X-rank with respect to real varieties as well:

Theorem 3. *Let $X \subset \mathbb{RP}^n$ be an irreducible nondegenerate real projective variety. Let r_0 be the minimal typical rank with respect to X, and let $r_{\max}$ be the maximum value of rank with respect to X. Then*

$$r_{\max} \leq 2r_0.$$

Again, despite being elementary the above theorem provides the best known bounds on maximal real tensor rank in many cases. In fact, much less is known about the maximal real rank, so the bound of Theorem 3 is even stronger.

2. Proofs of Main Theorems

We begin by working over an algebraically closed field, then we work over $\mathbb{R}$.

2.1. **Rank over algebraically closed fields.** Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety over an algebraically closed field $\mathbb{F}$. Since X spans $\mathbb{P}^n$, choosing a basis from points of X shows that every point has rank at most $n+1$. The best general bound on maximal rank with respect to X is to our knowledge:

$$r_{\max}(X) \leq n + 1 - \dim(X), \tag{1}$$

see [Ger96], remarks following Theorem 7.6, and [LT10, Prop. 5.1]. We now prove Theorem 1, which usually provides a much better bound on the maximal rank:

Proof of Theorem 1. Every point is a sum of two general points, each having rank r_{gen}, and therefore the maximal rank is at most $2r_{\text{gen}}$.

Explicitly, let U be a Zariski dense open subset of points of rank exactly r_{gen}. Let $q \in \mathbb{P}^n$ be any point and let p be any point in U. The line L through q and p intersects U at another point p' (in fact, at infinitely many more points). Since p and p' span L, q is a linear combination of p and p'. Since p and p' each have rank r_{gen}, q has rank at most $2r_{\text{gen}}$. □

We now describe a slight improvement to this theorem in certain cases. First we recall the definition of the secant variety.

Definition 4. The r-th **secant variety** $\sigma_r(X)$ is the Zariski closure of the set of points of rank at most r.

Since points of X span $\mathbb{P}^n$ the $(n+1)$-st secant variety $\sigma_{n+1}(X)$ certainly fills $\mathbb{P}^n$. Note that the generic rank with respect to X is the least r such that $\sigma_r(X) = \mathbb{P}^n$.

The following map will be useful. Let $\hat{X}$ be the affine cone over X and let $\Sigma_{r,X} : \hat{X}^r \to \mathbb{A}^{n+1}$ be the map

$$\Sigma_{r,X}(x_1, \ldots, x_r) = x_1 + \cdots + x_r.$$

The image of $\Sigma_{r,X}$ is precisely the affine cone over the set of points of rank r or less. The secant variety $\sigma_r(X)$ is the Zariski closure of the projectivization of $\Sigma_{r,X}(\hat{X}^r)$. (See for example [SS06] where a slight variant of this map is used to describe the defining ideal of $\sigma_r(X)$.)

The set of points of rank r_{gen} contains a dense Zariski open subset of $\sigma_{r_{\text{gen}}}(X) = \mathbb{P}^n$. For $r < r_{\text{gen}}$ the set of points of rank $\le r$ contains a dense subset of $\sigma_r(X)$; this dense subset can be taken to be open, and to consist of points of rank equal to r:

Lemma 5. *Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety over an algebraically closed field. Let $r \le r_{\text{gen}}$. The set of points of rank equal to r contains a dense Zariski open subset of $\sigma_r(X)$.*

Proof. The projectivization of the image of $\Sigma_{r,X}$ is dense in $\sigma_r(X)$ and constructible by Chevalley's theorem. Every dense constructible set contains a dense open subset. The set of points of rank less than r is a closed subset as it is the image of $\Sigma_{r-1,X}$ and is a proper subset as otherwise, $r_{\text{gen}} \le r_{\max} \le r - 1$. Removing it leaves a dense open subset of the points of rank equal to r. □

Now we can give a slight improvement to Theorem 1 if the secant variety $\sigma_{r_{\text{gen}}-1}(X)$ is a hypersurface.

Theorem 6. *Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety over an algebraically closed field. Suppose X is not a hypersurface, but for some r, $\sigma_r(X)$ is a hypersurface; necessarily $r = r_{\text{gen}} - 1$. Then $r_{\max} \le 2r = 2r_{\text{gen}} - 2$.*

Proof. In particular $r \ge 2$. Then $\deg \sigma_r(X) \ge r + 1 \ge 3$ [CJ01]. Let $U \subset \sigma_r(X)$ be a dense Zariski open subset of points with rank r. Let $q \in \mathbb{P}^n$ be any point. For $p \in \sigma_r(X)$, the line L through p and q intersects the hypersurface $\sigma_r(X)$ in at least one more point p', distinct from p and q. If $p \in U$ is general then so is p'. So q can be written as a linear combination of two points $p, p' \in U$, each with rank r, showing that q has rank at most $2r$. □

See Example 9 for a case in which this bound is sharp.

Finally we mention a simple generalization of Theorem 6 and (1):

Proposition 7. *Suppose $\sigma_k(X)$ has codimension c. Let s be the maximum rank of points on $\sigma_k(X)$. Then $r_{\max} \leq \max\{s,(c+1)k\}$.*

Proof. Let $q \in \mathbb{P}^N$. If $q \in \sigma_k(X)$ then $r_X(q) \leq s$. Otherwise, a general c-plane through q is spanned by its intersection with $\sigma_k(X)$, which gives q as a linear combination of $c+1$ general points on $\sigma_k(X)$ which each have rank k. □

The upper bound (1) is given by $k = s = 1$. Theorem 6 is the case $c = 1$ (plus the observation that $s \leq 2k$). We believe that the best bounds resulting from this proposition are just these previously observed extreme cases, and intermediate values of k and c probably do not give interesting new bounds. Perhaps if some secant variety of X is highly degenerate, this bound might be interesting.

2.2. **Real varieties.** Now we consider rank with respect to a real variety. As before, let $X \subset \mathbb{RP}^n$ be an irreducible nondegenerate variety.

Let $X_{\mathbb{C}} = X \otimes \mathbb{C}$ be the complexification of X, i.e. the variety in $\mathbb{CP}^n$ defined by the same equations that define $X \subset \mathbb{RP}^n$. A priori, the real rank of $v \in \mathbb{R}^{n+1}$ with respect to X may be strictly greater than the (complex) rank of the same point $v \in \mathbb{C}^{n+1}$ with respect to the complexification $X_{\mathbb{C}}$. Moreover, the maximum real rank with respect to X may be strictly greater than the maximum (complex) rank with respect to $X_{\mathbb{C}}$. See for example [Rez13b].

An integer r is called a **typical rank** if it is the rank of every point in some nonempty open subset of $\mathbb{R}^{n+1}$ in the Euclidean topology. In contrast to the closed field case, there may be more than one typical rank. See for example [Ble12].

We now show Theorem 2. It is proved for triple tensor products $\mathbb{C}^\ell \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ in [Fri12, Thm. 7.1]. (See also results and references in [Fri12, §7] regarding the maximum typical rank.)

Proof of Theorem 2. Certainly $r_0 \geq r_{\text{gen}}$: $\sigma_{r_{\text{gen}}-1}(X_{\mathbb{C}})$ is contained in a hypersurface and it is defined over $\mathbb{R}$. Therefore $\sigma_{r_{\text{gen}}-1}(X_{\mathbb{C}})$ is contained in a hypersurface defined over $\mathbb{R}$, and hence so is $\sigma_{r_{\text{gen}}-1}(X)$. Thus $r_{\text{gen}}-1$ is not a typical rank, nor is any rank less than $r_{\text{gen}}-1$.

On the other hand, $r = r_{\text{gen}}$ is a typical rank. Since $\hat{X}$ and $\hat{X}^r$ are semialgebraic sets and the map $\Sigma_{r,X}$ is a linear projection, the image $\mathcal{S} = \Sigma_{r,X}(\hat{X}^r)$ is a semialgebraic set by the Tarski-Seidenberg theorem [BCR98]. We can write $\mathcal{S}$ as a finite union $B_1 \cup \cdots \cup B_t$ of semialgebraic sets, where each B_i is nonempty and defined by a finite set of real polynomial equations $f(x) = 0$ and inequalities $f(x) > 0$ [BCR98]. If the definition of B_i includes an equation then B_i is contained in a hypersurface. Since $\mathcal{S}$ is Zariski dense, there must be at least one B_i whose definition consists solely of inequalities. Removing points of rank less than r_{gen} if necessary, this B_i is an open set in the Euclidean topology consisting of points of rank r_{gen}. This shows r_{gen} is a typical rank. □

Combining the above result with Theorem 1 shows that the maximum complex rank with respect to X is at most twice the lowest typical rank. But the maximum real rank may be greater than the maximum complex rank. So in Theorem 3 we show, analogously to Theorem 1, that the maximum real rank with respect to X is also bounded by twice the least typical rank.

Proof of Theorem 3. Let $B \subset \mathbb{R}^{n+1}$ be a small open ball in which every point has rank r_0. Then $B - B$ is an open neighborhood of the origin in which every point p is a sum (difference) of two points of rank r_0, so $r(p) \le 2r_0$. But every nonzero point has a scalar multiple in $B - B$ and rank is invariant under scalar multiplication. □

3. Applications to Tensor Rank

We now apply our bound on the maximal rank with respect to X to various tensor ranks over $\mathbb{C}$ and $\mathbb{R}$. We also discuss the relation between our bound and previously known bounds on the maximal tensor rank. Note that there seems to be relatively little known about upper bounds for real tensor rank. With rare exceptions previously known upper bounds on maximal rank are over $\mathbb{C}$, while our Theorems also give the same upper bounds over $\mathbb{R}$.

3.1. **Symmetric Tensor Rank.** Symmetric tensors correspond to homogeneous polynomials (forms). The symmetric tensor rank of a homogeneous form F of degree d in n variables (equivalently n-variate symmetric tensor of order d) is the least number r of terms needed to write F as a linear combination of dth powers of linear forms,

$$F = c_1\ell_1^d + \cdots + c_r\ell_r^d.$$

This corresponds precisely to the rank of a form F with respect to the d-th Veronese variety $\nu_d(\mathbb{P}^{n-1})$. This is also known as the **Waring rank** of F. For example, since $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$, the Waring rank of xy is 2 (as long as the characteristic of the field is not 2). See [CM96, IK99, Lan12] for introductions to Waring rank. We limit our discussion to the fields $\mathbb{R}$ and $\mathbb{C}$.

We denote the maximal Waring rank $r_{\max}(n,d)$. Classically, $r_{\max}(n,2) = n$ and $r_{\max}(2,d) = d$ are well-known, but to our knowledge, only two other values of maximal rank over $\mathbb{C}$ are known: $r_{\max}(3,3) = 5$ [Yer32, CM96, LT10] and $r_{\max}(3,4) = 7$ [Kle99, BGI11, Par13].

The vector space of forms of degree d in n variables has dimension $\binom{n+d-1}{n-1}$, so trivially $r_{\max}(n,d) \le \binom{n+d-1}{n-1}$ (by taking a basis consisting of powers of linear forms). Several improvements are known: $r_{\max}(n,d) \le \binom{n+d-1}{n-1} - n + 1$ [Ger96, LT10]; better, $r_{\max}(n,d) \le \binom{n+d-2}{n-1}$ [BBS08]; and up till now the best known upper bound for complex Waring rank is [Jel13]:

$$r_{\max}(n,d) \le \binom{n+d-2}{n-1} - \binom{n+d-6}{n-3}. \tag{2}$$

We denote the generic complex Waring rank—the Waring rank of a general form in n variables of degree d, that is, one with general coefficients—by $r_{\text{gen}}(n,d)$. Its value is given by the Alexander–Hirschowitz theorem [AH95]: $r_{\text{gen}}(n,d) = \lceil \frac{1}{n}\binom{n+d-1}{n-1} \rceil$, except if $(n,d) = (n,2)$, $(3,4)$, $(4,4)$, $(5,4)$, $(5,3)$. In the exceptional cases, $r_{\text{gen}}(n,2) = n$ and for the rest, $r_{\text{gen}}(n,d) = \lceil \frac{1}{n}\binom{n+d-1}{n-1} \rceil + 1$.

We immediately obtain the following Corollary:

Corollary 8. *The maximal real Waring rank of a real form of degree $d \ge 3$ in n variables is at most:*

$$r_{\max}(n,d) \le 2\left\lceil \frac{1}{n}\binom{n+d-1}{n-1} \right\rceil,$$

except $r_{\max}(3,4) \le 12$, $r_{\max}(4,4) \le 20$, $r_{\max}(5,4) \le 30$, $r_{\max}(5,3) \le 16$. The same upper bound holds for the complex Waring rank.

Asymptotically, Jelisiejew's upper bound is $nd/(n+d-1)$ times the generic rank. Our bound is asymptotically better than Jelisiejew's upper bound, but worse for some small cases. In the following table, $r^J_{\max}$ denotes Jelisiejew's upper bound (2) and $r^*_{\max}$ denotes our upper bound, $r^*_{\max} = 2r_{\text{gen}}$. The exact maximum is listed in the two cases where it is known.

n	d	r_{gen}	$r^J_{\max}$	$r^*_{\max}$	$r_{\max}$
3	3	4	5	8	5
3	4	6	9	12	7
3	5	7	14	14	
3	6	10	20	20	
3	7	12	27	24	
3	8	15	35	30	

n	d	r_{gen}	$r^J_{\max}$	$r^*_{\max}$
4	3	5	9	10
4	4	10	18	20
4	5	14	32	28
4	6	21	52	42
4	7	30	79	60
4	8	42	114	84

Example 9. The upper bound of Theorem 1 is almost sharp for bivariate forms, $n = 2$. In that case the maximum rank is $r_{\max} = d$ and the generic rank is $r_{\text{gen}} = \lceil \frac{d+1}{2} \rceil$, so $r_{\max} = 2r_{\text{gen}} - 2$ if d is even, $r_{\max} = 2r_{\text{gen}} - 1$ if d is odd. It is known that in the case d is even, the $(r_{\text{gen}} - 1)$st secant variety is a hypersurface, defined by the vanishing of the determinant of the middle—that is, $(d/2)$th—catalecticant [Syl51]. In this case, Theorem 6 is sharp.

The hypersurface condition happens for Veronese varieties $X = \nu_d(\mathbb{P}^{n-1})$ if and only if

$$(r_{\text{gen}} - 1)n - 1 = \binom{n+d-1}{n-1} - 2.$$

This happens if and only if $\binom{n+d-1}{n-1} \equiv 1 \pmod{n}$. For example, if $n = 2$ and d is even then $\binom{n+d-1}{n-1} = d + 1$ is odd; other instances include $\binom{14}{2} \equiv 1 \pmod{3}$, $\binom{14}{4} \equiv 1 \pmod{5}$.

3.2. **Tensor Rank.** Tensor rank of a tensor of order d in n variables corresponds precisely to the rank with respect to the Segre variety $\sigma(\mathbb{P}^n \times \cdots \times \mathbb{P}^n)$, with d factors. The generic tensor rank has been well-studied and it is known for several families of tensors.

Example 10 (Tensors of format $2 \times \cdots \times 2$)**.** The generic rank of tensors in $(\mathbb{C}^2)^{\otimes n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ is $\lceil \frac{2^n}{n+1} \rceil$ [CGG11]. Therefore the maximum rank (real or complex) is at most $2\lceil \frac{2^n}{n+1} \rceil$. For $n > 7$ this is better than the bound 2^{n-2} given in [SSM13, Cor. 6.2] (see also [Sta12] for the bound $3 \cdot 2^{n-3}$).

Example 11 (Tensor rank in triple products)**.** It is known that the maximum rank of tensors in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is at most $\binom{n+1}{2}$ if n is odd, $\frac{n(n+2)}{2} - 1$ if n is even, see [AS79, AL80, SMS10].

It is also known that the generic rank of tensors in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is 5 when $n = 3$, or $\lceil \frac{n^3-1}{3n-2} \rceil$ when $n > 3$ [Lan12, Thm. 3.1.4.3]. Therefore by Theorem 3 the maximum rank of tensors in $\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ is at most 10 when $n = 3$, or $2\lceil \frac{n^3-1}{3n-2} \rceil \approx \frac{2}{3}n^2$ when $n > 3$.

The same upper bound holds for the complex tensor rank, but this is not as good as the previously known bounds. (See also [Lan12, Cor. 3.1.2.1], [BL13, Cor. 3.5] and [Fri12, Cor. 6.7].)

3.3. **Waring problem with higher degree terms.** A variant of Waring rank is the number of terms needed to write a homogeneous polynomial of degree kd as a linear combination of kth powers of d-forms. See [FOS12, Rez13a, CO13]. In [FOS12] it is shown that a generic complex kd-form in $n+1$ variables is a sum of at most k^n kth powers of d-forms, and no fewer when d is sufficiently large. Therefore every complex kd-form is a sum of at most $2k^n$ kth powers of d-forms, and every real kd-form is a real linear combination (possibly including negative coefficients) of at most $2k^n$ kth powers of real d-forms.

3.4. **Antisymmetric Tensor Rank.** The rank of an alternating tensor $T \in \bigwedge^k \mathbb{C}^n$ is the least number of terms needed to write T as a linear combination of simple wedges. It is given by the rank with respect to a Grassmannian in its Plücker embedding. In [AOP12] it is shown that the generic rank of an alternating tensor in $\bigwedge^3 \mathbb{C}^n$ is asymptotically $\frac{n^2}{18}$. Therefore the maximum rank of such a tensor is asymptotically less than or equal to $\frac{n^2}{9}$.

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