AN EXTENSION OF ABC-THEOREM

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Abstract. In this paper we generalize the abc-theorem for n-polynomials over $\mathbb{F}[x]$ in which $\mathbb{F}$ is an algebraically closed field of characteristic zero. This generalization is obtained by considering the Wronskian of functions over $\mathbb{F}[x]$. We also show that the Diophantine equation (The generalized Fermat-Catalan equation)

$$a_1^{m_1} + a_2^{m_2} + \cdots + a_{n-1}^{m_{n-1}} = a_n^{m_n},$$

where $a_1, a_2, \cdots, a_n \in \mathbb{F}[x]$ such that at most one of a_i's is constant, and $m_1, m_2, \cdots, m_n \in \mathbb{N}$, has no solution for which $a_i (i = 1, \cdots, n)$ are relatively prime by pairs provided that $n(n-2) \leq \min_{1 \leq i \leq n} \{m_i\}$.

Introduction

Although the arithmetic abc-conjecture is a great mystery, its algebraic counterpart is a rather easy theorem (abc-theorem). It looks like it was first noticed by W.W. Stothers [1]. Later on it was generalized and rediscovered independently by several people, including R.C. Mason [2] and J.H. Silverman [3].
Discovering the abc-theorem, opened a new way for investigating the Fermat's last theorem over the polynomials with coefficients in an algebraically closed field of characteristic zero. This theorem presented a very elementary proof of the Fermat's last theorem for polynomials. This led mathematician to give a variant of this theorem over the ring of integer numbers. Of course, this result has been stated as a conjecture and this conjecture has not been proved yet. Today this conjecture is known as the abc-conjecture. Let us state the original abc-theorem [1-4,8,9]. To do this, we need to introduce some notations. We denote the set of all polynomials of one variable x over $\mathbb{F}$ by $\mathbb{F}[x]$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero. We also consider the non-zero elements of $\mathbb{F}[x]$, as follows

$$f(x) = c \prod_{i=1}^{r} (x - \alpha_i)^{m_i},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_r$ are the distinct roots of f, $c \neq 0$ is a constant, and the positive integers m_i $(i = 1, 2, \cdots, r)$ are the multiplicities of the roots. The degree of the polynomial f is

$$\deg f = m_1 + m_2 + \cdots + m_r.$$

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The number of distinct roots of f will be denoted by $n_0(f)$. Thus, we have $n_0(f) = r$. If f, g are two nonzero polynomials, then in general

$$n_0(fg) \leq n_0(f) + n_0(g),$$

and the equality holds whenever f, g are relatively prime. Now, we state the abc-theorem.

The abc-Theorem (Stothers, Mason, Silverman). *Let $a, b, c \in \mathbb{F}[x]$ be non-constant relatively prime polynomials satisfying $a + b = c$. Then*

$$\max\{\deg a, \deg b, \deg c\} \leq n_0(abc) - 1.$$

The similar result for the ring of integers is well-known as the abc-conjecture. This conjecture has been stated by Oesterle and Masser [5,6] in 1986.

The abc-Conjecture (Oesterle, Masser). *Given $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that for all $a, b, c \in \mathbb{Z}$ whit $a + b = c$, we have the inequality*

$$\max\{|a|, |b|, |c|\} \leq C(\varepsilon)(N_0(abc))^{1+\varepsilon},$$

in which $N_0(abc)$ denotes the radical of abc. By radical function we mean

$$N_0(n) = \prod_{p|n} p \qquad (p \text{ is prime and } n \in \mathbb{N}).$$

Note that Stewart and Tijdeman gave some lower bounds for $C(\varepsilon)$ (cf [7]).

Generalizing ABC-Theorem

Now, we generalized the abc-theorem for n-functions. To do this, we need the following lemmas:

Lemma 1. *Suppose f is a nonzero polynomial in $\mathbb{F}[x]$. Then, we have*

$$\deg f - m.n_0(f) \leq \deg(f, f', \cdots, f^{(m)}), \tag{1}$$

where $(f, f', \cdots, f^{(m)})$ is the greatest common divisor of $f, f', \cdots, f^{(m)}$.

Needless to say that the derivative is considered as a purely algebraic operator over the elements of $\mathbb{F}[x]$. However, all known rules for derivatives in calculus text book can be easily proved by means of simple algebraic tools.

Proof of Lemma 1. Suppose $f(x) = c\prod_{i=1}^{r}(x - \alpha_i)^{m_i}$, in which $\alpha_1, \alpha_2, \cdots, \alpha_r$ are the distinct roots of f with multiplicities $m_1, m_2, \cdots, m_r$ respectively.

Case I. Suppose for any $i (1 \leq i \leq r)$ we have $m_i \leq m$. Then we get

$$\deg f = \sum_{i=1}^{r} m_i \leq mr = m.n_0(f) \leq m.n_0(f) + \deg(f, f', \cdots, f^{(m)}).$$

Case II. Now, we suppose that there exists an i such that $m_i > m$. Therefore, we have

$$(x - \alpha_i)^{m_i - m} \Big| f^{(j)} \qquad (j = 0, 1, \cdots, m),$$

and consequently,

$$(x - \alpha_i)^{m_i - m} \Big| (f, f', \cdots, f^{(m)}).$$

It is clear to see that,

$$\prod_{\substack{0<m_i-m\\1\leq i\leq r}} (x-\alpha_i)^{m_i-m} \Big| (f, f', \cdots, f^{(m)}).$$

Considering the degrees of the both sides of the above result, we obtain

$$\sum_{\substack{0<m_i-m\\1\leq i\leq r}} (m_i-m) \leq \deg(f, f', \cdots, f^{(m)}).$$

Since

$$\sum_{i=1}^{r}(m_i-m) \leq \sum_{\substack{0<m_i-m\\1\leq i\leq r}} (m_i-m),$$

we get

$$\sum_{i=1}^{r} (m_i-m) \leq \deg(f, f', \cdots, f^{(m)}),$$

or equivalently

$$\deg f - m.n_0(f) \leq \deg(f, f', \cdots, f^{(m)}),$$

and this completes our proof.

Remark 1. If $\mathrm{char}(\mathbb{F}) = 0$, then we conclude that

$$\deg f - m.n_0(f) \leq \deg(f, f^{(m)}) = \sum_{\substack{0<m_i-m\\1\leq i\leq r}} (m_i-m).$$

Definition 1. *Let $f_1, f_2, \cdots, f_n$ be functions over the ring $\mathbb{F}[x]$. The Wronskian of these functions is defined by,*

$$W[f_1, f_2, \cdots, f_n] = \det \left| f_j^{(i-1)} \right|_{1\leq i,j\leq n}.$$

Lemma 2. *If $\mathrm{char}(\mathbb{F}) = 0$ and $f_1, f_2, \cdots, f_n$ be linearly independent functions over $\mathbb{F}$ in $\mathbb{F}[x]$, then there exists an element x in $\mathbb{F}$, such that $W[f_1, f_2, \cdots, f_n](x) \neq 0$ (i.e. $W[f_1, f_2, \cdots, f_n](x)$ is a nonzero polynomial).*

Proof. Suppose for every $x \in \mathbb{F}$, we have

$$W[f_1, f_2, \cdots, f_n](x) = 0.$$

Therefore, there are constant numbers $c_i (i = 1, 2, \cdots, n)$ in $\mathbb{F}$, such that at least one of these c_i is nonzero and

$$c_1 \begin{pmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(n-1)}(x) \end{pmatrix} + \cdots + c_n \begin{pmatrix} f_n(x) \\ f_n'(x) \\ \vdots \\ f_n^{(n-1)}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

or

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0,$$

which is a contradiction with the linearly independence of $f_1, f_2, \cdots, f_n$.

Lemma 3. *Suppose* $\text{char}(\mathbb{F}) = 0$ *and* $f_1, f_2, \cdots, f_n$ *are nonzero functions in* $\mathbb{F}[x]$. *Then, for* $W[f_1, f_2, \cdots, f_n] \neq 0$*, we have*

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg(f_1 f_2 \cdots f_n) - \frac{n(n-1)}{2}. \tag{2}$$

Whenever $\deg f_1 = \cdots = \deg f_n$*, we get*

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg(f_1 f_2 \cdots f_n) - \frac{n(n-1)}{2} - 1. \tag{3}$$

Proof. We proceed it by mathematical induction on n. The initialization step $n = 1$, is clear. Suppose it holds for $n-1$ nonzero functions. By expanding the Wronskian determinant $W[f_1, f_2, \cdots, f_n]$ with respect to the first row, we obtain

$$W[f_1, f_2, \cdots, f_n] = \sum_{i=1}^{n} (-1)^{i+1} f_i . W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n']. \tag{4}$$

We have the following inequality for degrees

$$\deg W[f_1, f_2, \cdots, f_n] \leq \max_{1 \leq i \leq n} \{\deg f_i + \deg W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n']\},$$

and since $W[f_1, f_2, \cdots, f_n] \neq 0$, there exists an i such that the right-hand side has the greatest degree, namely

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg f_i + \deg W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n']. \tag{5}$$

Now, considering the induction hypothesis for the set of $(n-1)$-functions

$$f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n',$$

we get

$$\begin{aligned} \deg W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n'] &\leq \deg(f_1' \cdots f_{i-1}' f_{i+1}' \cdots f_n') - \frac{(n-1)(n-2)}{2} \\ &\leq \deg(f_1 \cdots f_{i-1} f_{i+1} \cdots f_n) - \frac{n(n-1)}{2}. \end{aligned} \tag{6}$$

Finally, by (5) and (6), we have

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg(f_1 f_2 \cdots f_n) - \frac{n(n-1)}{2}.$$

For proving (3), it is necessary to show that after expanding the determinant of $W[f_1, f_2, \cdots, f_n]$, the term with the highest degree is vanished. We prove this by induction on n, with $n \geq 2$. First we investigate the case $n = 2$. Since $\deg f_1 = \deg f_2$, we have $f_1(x) = a_k x^k + P(x)$ and $f_2(x) = b_k x^k + Q(x)$, where $P(x)$ and $Q(x)$ are two polynomials of degree at most $(k-1)$. So, we have

$$\begin{aligned} W[f_1, f_2] &= \begin{vmatrix} a_k x^k + P(x) & b_k x^k + Q(x) \\ k a_k x^{k-1} + P'(x) & k b_k x^{k-1} + Q'(x) \end{vmatrix} \\ &= a_k x^k Q'(x) + k b_k x^{k-1} P(x) - b_k x^k P'(x) - k a_k x^{k-1} Q(x). \end{aligned}$$

Now, assume its validity for any arbitrary $(n-1)$-functions. Then the proof is straight forward considering the relation (4). Now, we are ready to state our main result

Theorem 1. *Let* $f_n = f_1 + f_2 + \cdots + f_{n-1}$*, in which* f_i*'s are relatively prime* *by pairs in* $\mathbb{F}[x]$ *with* $\mathrm{char}(\mathbb{F}) = 0$ *and at most one of them is constant. Then, we have*

$$\max_{1\le i\le n} \deg f_i \le (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}, \tag{7}$$

and also

$$\min_{1\le i\le n} \deg f_i \le (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2} - 1. \tag{8}$$

Proof. For proving the first inequality, we distinguish between two cases. The proof of Case I, is analogous with [9, Theorem 1.2].

Case I. Let $f_1, f_2, \cdots, f_{n-1}$ be linearly dependent over $\mathbb{F}$. Now, the proof proceeds by induction on n. For $n = 3$, it is true; considering the results in [1-4]. Assume that the theorem is true for all cases $n', 3 \le n' < n$, and consider n polynomials. In equality $f_n = f_1 + f_2 + \cdots + f_{n-1}$, assume that $f_i (i = 1, 2, \cdots, n-1)$, are linearly dependent over $\mathbb{F}$. Note that, at most one of the $f_i (i = 1, 2, \cdots, n-1)$, is constant. Let $\{f_{i_1}, \cdots, f_{i_q}\}$, $q < n-1$, be a maximal linearly independent subset of the $f_i (i = 1, 2, \cdots, n-1)$. Since $n-1 \ge 2$, and f_j's are relatively prime by pairs, it follows that $q \ge 2$. So each $f_j, 1 \le j \le n-1$; j not one of the i_k, is a linear combination of the f_{i_k}, of the form

$$f_j = \lambda_1 f_{i_1} + \cdots + \lambda_q f_{i_q}, \tag{9}$$

where the $\lambda_k \in \mathbb{F}$, and at least two of these λ_k are not zero. Using our inductive hypothesis we apply the theorem to (9). This yields that if $\lambda_k \neq 0$, then

$$\deg f_{i_k} \le (q-1)n_0(f_j \prod_{k=1}^{q} f_{i_k}) - \frac{q(q-1)}{2},$$

and so that

$$\deg f_{i_k} \le (q-1)n_0(\prod_{i=1}^{n} f_i) - \frac{q(q-1)}{2}. \tag{10}$$

Now, since at most one of f_i is a constant, i.e. $n-1 \le n_0\left(\prod_{i=1}^{n} f_i\right)$, we yield that

$$(q-1)n_0(\prod_{i=1}^{n} f_i) - \frac{q(q-1)}{2} \le (n-2)n_0(\prod_{i=1}^{n} f_i) - \frac{(n-1)(n-2)}{2}. \tag{11}$$

Now, using (10) and (11), we have

$$\deg f_{i_k} \le (n-2)n_0(\prod_{i=1}^{n} f_i) - \frac{(n-1)(n-2)}{2}. \tag{12}$$

From (9) the same estimate as in (12) follows for $\deg f_j$. Thus the theorem is proved for such f_j and f_{i_k}. Inserting all the relations of the from (9) into the right side of equality $f_n = f_1 + f_2 + \cdots + f_{n-1}$, yields an equation of the form

$$f_r = \kappa_1 f_{i_1} + \cdots + \kappa_q f_{i_q}, \tag{13}$$

where the $\kappa_j \in \mathbb{F}$. Moreover, if one of these $\kappa_\nu = 0$, then the corresponding f_{i_ν} must be appeared in one of the equations (9) with a nonzero λ_ν. Hence, (12) is established for this f_{i_ν}. Finally, for those $\kappa_\nu \neq 0$, we treat (13) exactly as we did (9), (note that $q + 1 < n$), and obtain the estimate (12) for $\deg f_{i_\nu}$, and $\deg f_n$. This completes the induction in this case.

Case II. $f_1, f_2, \cdots, f_{n-1}$, are linearly independent over $\mathbb{F}$. By using Lemma 2, we have $W[f_1, f_2, \cdots, f_{n-1}] \neq 0$. Without lost of generality, we suppose that f_n has the greatest degree, and therefore it is necessary to prove that

$$\deg f_n \leq (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}.$$

Considering the equality $f_n = f_1 + f_2 + \cdots + f_{n-1}$, we have

$$W[f_1, \cdots, f_{n-2}, f_{n-1}] = W[f_1, \cdots, f_{n-2}, f_n].$$

It can be easily seen for any i $(i = 1, \cdots, n)$,

$$(f_i, f_i', \cdots, f_i^{(n-2)}) \Big| W[f_1, \cdots, f_{n-2}, f_{n-1}].$$

Since f_i's are relatively prime by pairs, we conclude that $(f_i, f_i' \cdots, f_i^{(n-2)})$'s are relatively prime. So, we get

$$\prod_{i=1}^{n} (f_i, f_i', \cdots, f_i^{(n-2)}) \Big| W[f_1, \cdots, f_{n-2}, f_{n-1}].$$

Now since $W[f_1, \cdots, f_{n-2}, f_{n-1}] \neq 0$, we conclude that

$$\sum_{i=1}^{n} \deg(f_i, f_i', \cdots, f_i^{(n-2)}) \leq \deg W[f_1, \cdots, f_{n-2}, f_{n-1}].$$

Using the relations (1) and (2), we obtain

$$\sum_{i=1}^{n} (\deg f_i - (n-2)n_0(f_i)) \leq \deg(f_1 f_2 \cdots f_{n-1}) - \frac{(n-1)(n-2)}{2}$$

or equivalently,

$$\deg f_n \leq (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}.$$

For proving (8), it is necessary to consider the case $\deg f_1 = \cdots = \deg f_n$. Now the proof is clear using the relation (3).

Remark 2. In the case where the number of constant polynomials are more than one, the inequality (7) is not valid in general case. For example if $f_1 = \cdots = f_5 = 1, f_6 = x$ and $f_7 = x + 5$, then it is not true. Indeed, finding similar inequality for the case that constant polynomials are more than one is an open question yet.

As an immediate result of the relation (7), we have:

Corollary 1. *With the assumption of the Theorem 1, we have*

$$\deg(f_1 f_2 \cdots f_n) \leq n(n-2)n_0(f_1 f_2 \cdots f_n) - \frac{n(n-1)(n-2)}{2}.$$

Corollary 2. *For $n \geq 3$, suppose $f_1, f_2, \cdots, f_n$ are non-constant and relatively prime by pairs. Then we obtain*

$$\frac{1}{n-2} < \frac{n_0(f_1)}{\deg f_1} + \frac{n_0(f_2)}{\deg f_2} + \cdots + \frac{n_0(f_n)}{\deg f_n}.$$

Proof. Without loss of generality, we suppose that $\deg f_1 \leq \cdots \leq \deg f_n$. Applying Theorem 1, yields

$$\deg f_n < (n-2)(n_0(f_1) + \cdots + n_0(f_n)).$$

Dividing the both sides of the above inequality by $(n-2)\deg f_n$, completes the proof.

Application to the generalized Fermat-Catalan Equation

Now, we deal with the generalized Fermat-Catalan equation [8].

Theorem 2. *Consider the generalized Fermat-Catalan equation as follows*

$$a_1^{m_1} + a_2^{m_2} + \cdots + a_{n-1}^{m_{n-1}} = a_n^{m_n}, \tag{14}$$

in which $a_1, a_2, \cdots, a_n$ *are elements of* $\mathbb{F}[x]$ *whit* $\mathrm{char}(\mathbb{F}) = 0$, *such that they are relatively prime by pairs and at most one of* a_i*'s is constant. Then the equation* (14) *with condition* $n(n-2) \leq m = \min_{1\leq i\leq n}\{m_i\}$ *has no solution in* $\mathbb{F}[x]$.

Proof. Suppose $f_1 = a_1^{m_1}, f_2 = a_2^{m_2}, \cdots, f_n = a_n^{m_n}$. These functions satisfy the conditions of Theorem 1. Thus we have

$$\deg(a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}) \leq n(n-2)n_0(a_1^{m_1}\cdots a_n^{m_n}) - \frac{n(n-1)(n-2)}{2}. \tag{15}$$

We also have,

$$m\deg(a_1a_2\cdots a_n) \leq \deg(a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}), \tag{16}$$

and

$$n_0(a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}) = n_0(a_1a_2\cdots a_n) \leq \deg(a_1a_2\cdots a_n). \tag{17}$$

Now considering the both relations (15)-(17), we get

$$m\deg(a_1a_2\cdots a_n) \leq n(n-2)\deg(a_1a_2\cdots a_n) - \frac{n(n-1)(n-2)}{2}, \tag{18}$$

or equivalently,

$$(m - n(n-2))\deg(a_1a_2\cdots a_n) \leq -\frac{n(n-1)(n-2)}{2}. \tag{19}$$

The last inequality result in $m - n(n-2) < 0$, which is in contradiction with our theorem's hypothesis. Therefore, we conclude that the Diophantine equation (14) has no solution in $\mathbb{F}[x]$.

Of course, there is in [10] a natural extension of the above result for several variables using the generalized Wronskian.

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