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# On the typical rank of real polynomials (or symmetric tensors) with a fixed border rank

**Edoardo Ballico**



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**Abstract** Let $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$, $b(m+1) < \binom{m+d}{m}$, denote the set of all degree $d$ real homogeneous polynomials in $m+1$ variables (i.e. real symmetric tensors of format $(m+1)\times\cdots\times(m+1)$, $d$ times) which have border rank $b$ over $\mathbb{C}$. It has a partition into manifolds of real dimension $\le b(m+1)-1$ in which the real rank is constant. A typical rank of $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$ is a rank associated to an open part of dimension $b(m+1)-1$. Here we classify all typical ranks when $b \le 7$ and $d, m$ are not too small. For a larger sets of $(m,d,b)$ we prove that $b$ and $b+d-2$ are the two first typical ranks. In the case $m = 1$ (real bivariate polynomials) we prove that $d$ (the maximal possible a priori value of the real rank) is a typical rank for every $b$.

**Keywords** symmetric tensor rank · Veronese variety · real rank · typical rank · secant variety · border rank · bivariate polynomial

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## 1 Introduction

Fix an integer $m > 0$ and call $\mathbb{K}$ either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For every integer $d \ge 0$ let $\mathbb{K}[x_0,\dots,x_m]_d$ denote the $\mathbb{K}$-vector space of all degree $d$ polynomials in the variables $x_0,\dots,x_m$ and with coefficients in $\mathbb{K}$. Now assume $d > 0$ and fix $f \in \mathbb{K}[x_0,\dots,x_m]_d\setminus\{0\}$. The rank or the symmetric tensor rank $r_{\mathbb{K}}(f)$ of $f$ with respect to $\mathbb{K}$ is the minimal integer $s > 0$ such that

$$f = c_1\ell_1^d + \cdots + c_s\ell_s^d \qquad (1)$$

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E. Ballico
Dept. of Mathematics, University of Trento, 38123 Trento (Italy)
Tel.: 39+0461281646
Fax: 39+0461281624
E-mail: ballico@science.unitn.it

for some $\ell_i \in \mathbb{K}[x_0,\dots,x_m]_1$ and some $c_i \in \mathbb{K}$ ([23], §5.4). If either $\mathbb{K}=\mathbb{C}$ or $d$ is odd and $\mathbb{K}=\mathbb{R}$, then we may take $c_i = 1$ for all $i$ without any loss of generality. If $d$ is even and $\mathbb{K}=\mathbb{R}$, then we may take $c_i \in \{-1,1\}$ without any loss of generality. We may see any symmetric tensor of format $(m+1)\times\cdots\times(m+1)$ ($d$ products) as a polynomial $f \in \mathbb{K}[x_0,\dots,x_m]_d$. Any tensor has a tensor rank, but it is not known if for a symmetric tensor its rank as a tensor and its rank as a polynomial are the same. Comon's conjecture asks if symmetric rank is equal to rank for all symmetric tensors. In this paper we will always consider the symmetric rank of a symmetric tensor $T$, i.e. we will see $T$ as as a polynomial $f$ and take $r_{\mathbb{K}}(f)$ as the integer associated to $T$; we call it the symmetric tensor rank of $T$ or $f$. These notions appears in several different topics in engineering ([7], [14], [20], [21], [19], [22], [25], [26], [28], [29], [30]). The book [23] contains a huge bibliography which contains both the applied side and the theoretical side of this topic. In many applications the data are known only approximatively. In this case over $\mathbb{C}$ there is a non-empty and dense open subset $\mathcal{U}$ of $\mathbb{C}[x_0,\dots,x_m]_d \cong \mathbb{C}^{r+1}$, $r := \binom{m+d}{m}-1$, such that all $f\in\mathcal{U}$ have the same rank $r_{\mathbb{C}}(f)$ and this rank is called the generic rank. If $m=1$ and $d\ge 2$, then $\lfloor (d+2)/2\rfloor$ is the generic rank ([8], [18], [23], [24]). In the case $m\ge 2$ the generic rank is also known by a theorem of Alexander and Hirschowitz: for each $d\ge 3$ the generic rank is $\lceil \binom{m+d}{m}/(m+1)\rceil$ except in 4 well-studied exceptional cases ([1], [2], [10], [16]). The situation in the case $\mathbb{K}=\mathbb{R}$ is more complicated, because $\mathbb{R}[x_0,\dots,x_m]_d \cong \mathbb{R}^{r+1}$ has several non-empty open subsets for the euclidean topology in which the real rank is constant, but these constants are not the same for different open sets. The ranks with respect to $\mathbb{R}$ arising in these open subsets are called the typical ranks ([18], [15]). Only the bivariate cases was recently solved ([9]). However, quite often we get polynomials (or symmetric tensors) with some constraints and one may study the generic rank (case $\mathbb{K}=\mathbb{C}$) or the typical ranks (case $\mathbb{K}=\mathbb{R}$) for polynomials with those constraints. An algebraic constraint which is quite studied over $\mathbb{C}$ is the border rank ([23], Chapter 5), which we now define. We always assume $d\ge 2$. Let $\nu_d : \mathbb{P}^m(\mathbb{K}) \to \mathbb{P}^r(\mathbb{K})$, $r := \binom{m+d}{m}-1$, be the order $d$ Veronese embedding, i.e. the embedding induced by the vector space $\mathbb{K}[x_0,\dots,x_m]_d$. Set $X_{m,d}(\mathbb{K}) := \nu_d(\mathbb{P}^m(\mathbb{K}))$. Let $Y \subset \mathbb{P}^n(\mathbb{K})$ be any set spanning $\mathbb{P}^n(\mathbb{K})$. For any $P\in\mathbb{P}^n(\mathbb{K})$ the $Y$-rank $r_Y(P)$ or $r_{Y,\mathbb{K}}(P)$ of $P$ with respect to $\mathbb{K}$ is the minimal cardinality of a set $S\subset Y$ such that $P\in\langle S\rangle$, where $\langle\ \rangle$ denote the linear span. Any $f\in\mathbb{K}[x_0,\dots,x_m]_d\setminus\{0\}$ induces $P\in\mathbb{P}^r(\mathbb{K})$ and $r_{X_{m,d}(\mathbb{K})}(P) = r_{\mathbb{K}}(f)$. Now assume that $Y\subset\mathbb{P}^n$ is a geometrically integral variety defined over $\mathbb{K}$. For each integer $b>0$ let $\sigma_b(Y(\mathbb{C}))$ denote the closure of the union of all linear spaces $\langle S\rangle$ with $S\subset Y(\mathbb{C})$ and $\sharp(S)=b$ (or $\le b$). For any $P\in\mathbb{P}^r(\mathbb{C})$ the border rank $br(P)$ of $P$ is the minimal integer $b\ge 1$ such that $P\in\sigma_b(X_{m,d}(\mathbb{C}))$. The set $\sigma_b(X_{m,d}(\mathbb{C}))$ is an integral variety defined over $\mathbb{R}$ and we may look at its real points $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$. The integer $\alpha(m,d,b) := \dim(\sigma_b(X_{m,d}(\mathbb{C})))$ is known (in the case $m\ge 2$ by the quoted theorem of Alexander-Hirschowitz, while it was classically known that $\alpha(1,d,b)=\min\{d,2b-1\}$ for all $d,b$ ([8], [24], [23], Chapter 5). The set $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$ has a partition into topological manifolds of dimensions $\le \alpha(m,d,b)$ with finitely many connected open subsets of dimension $\alpha(m,d,b)$ with the additional condition that on each of these connected pieces the symmetric rank is constant. We call any such symmetric rank a typical $b$-rank or a typical rank for the border rank $b$. Notice that we use $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$, not something constructed only using $X_{m,d}(\mathbb{R})$. In many cases we have the equations of $\sigma_b(X_{m,d}(\mathbb{C}))$ and hence we may check if $P \in (\sigma_b(X_{m,d}(\mathbb{C})) \setminus \sigma_{b-1}(X_{m,d}(\mathbb{C})))$ ([23], Chapter 7, [27], and references therein).

In section 3 we study the bivariate case and prove that $d$ is a typical rank for every border rank, i.e. we prove the following result.

**Theorem 1** *Fix integers $b, a, d$ such that $2 \le b \le (d+2)/2$, $d \ge$ and $1 \le a \le b/2$. Fix $Q_1, \dots, Q_a \in \mathbb{P}^1(\mathbb{C})$ and $P_1, \dots, P_{b-2a} \in \mathbb{P}^1(\mathbb{R})$ such that $P_i \ne P_j$ for all $i \ne j$ and $\sharp(\{Q_1, \dots, Q_a, \sigma(Q_1), \dots, \sigma(Q_a)\}) = 2a$. Set*

$$A := \{Q_1, \dots, Q_a, \sigma(Q_1), \dots, \sigma(Q_a), P_1, \dots, P_{b-2a}\}$$

*and $M := \langle \nu_d(A) \rangle(\mathbb{R})$. $M$ is a $(b-1)$-dimensional real vector space and there is a non-empty open subset of $M$ in the euclidean topology such that $r_{\mathbb{R}}(P) = d$ for all $P \in U$.*

We recall that in the bivariate case $d$ is the maximum of all real ranks. Hence knowing that $d$ is a typical rank for each border rank $\ne 1$ is the worst news we could get on this subject. We prove a finer result which shows that the degree $d$ bivariate polynomials with real symmetric rank $d$ are ubiquitous (Theorem 9 and Remark 4).

In section 2 we study the multivariate case. We first consider the border ranks $\le 7$. In each case we give all the typical ranks for border rank $b \le 7$ if, say, $m \ge \max\{2, b-1\}$ and $d$ is large (see Theorems 2,...,7). We also describe all the real ranks when $b = 2$ (see Theorem 2). For every $m \ge 2$, $b \ge 2$ and for all $d \ge 2b-1$ we prove that $b$ and $b+d-2$ are the two smallest typical ranks for the border rank $b$ (Theorem 8).

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## 2 The multivariate case

Let $\sigma : \mathbb{C} \to \mathbb{C}$ be the complex conjugate and write $\sigma$ also for the involution induced by conjugation on each projective space $\mathbb{P}^y(\mathbb{C})$. Hence $\mathbb{P}^y(\mathbb{R}) = \{P \in \mathbb{P}^y(\mathbb{C}) : \sigma(P) = P\}$.

*Remark 1* Fix integers $m \ge 1$, $b > 0$, $d \ge 3$ such that $\sigma_b(X_{m,d}(\mathbb{C})) \subsetneq \mathbb{P}^r(\mathbb{C})$ (e.g. assume $b(m+1) < \binom{m+d}{m}$). Obviously $b$ is a typical rank of $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$.

*Remark 2* Fix positive integers $m, d, b$ such that $b \ge 2$ and $d \ge 2b-1$. We have $\dim_{\mathbb{C}}(\sigma_b(X_{m,d}(\mathbb{C}))) = b(m+1)-1$. For any $P \in (\sigma_b(X_{m,d}(\mathbb{C})) \setminus \sigma_{b-1}(X_{m,d}(\mathbb{C})))$ there is a unique zero-dimensional scheme $Z \subset \mathbb{P}^m(\mathbb{C})$ such that $\deg(Z) = b$ and $P \in \langle \nu_d(Z) \rangle$ ([8], Proposition 11, [12], Lemma 2.1.6 and proof of Theorem

1.5.1, [4], Remark 1 and Lemma 1). Now assume $P \in \sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$. Since $\sigma(P) = P$, the uniqueness of $Z$ implies $\sigma(Z) = Z$, i.e. the scheme $Z$ is defined over $\mathbb{R}$. Hence the finite set $J := Z_{red}$ is defined over $\mathbb{R}$. Set $e := \sharp(J)$. Since $J$ is defined over $\mathbb{R}$, there are an integer $a$ such that $0 \le 2a \le e$, distinct points $Q_1, \dots, Q_a \in \mathbb{P}^m(\mathbb{C}) \setminus \mathbb{P}^m(\mathbb{R})$ and $e - 2a$ distinct points $P_j \in \mathbb{P}^m(\mathbb{R})$, $1 \le j \le e - 2a$, such that $J = \{P_1, \dots, P_{e-2a}, Q_1, \sigma(Q_1), \dots, Q_a, \sigma(Q_a)\}$; the only restriction is that $Q_i \ne \sigma(Q_j)$ for all $i, j$. Now we vary $J \subset \mathbb{P}^m(\mathbb{C})$ fixing $e$ and $a$, i.e. we vary $P_1, \dots, P_{e-2a} \in \mathbb{P}^m(\mathbb{R})$ and $Q_1, \dots, Q_a \in \mathbb{P}^m(\mathbb{C}) \setminus \mathbb{P}^m(\mathbb{R})$ with the restrictions $P_i \ne P_j$ for all $i \ne j$, $Q_i \ne Q_j$ for all $i \ne j$ and $Q_h \ne \sigma(Q_k)$ for all $h, k$. Notice that $\dim(\langle \nu_d(J) \rangle) = e - 1$ for any such $J$, because $d \ge e - 1$. If $d \ge 2e - 1$ and $J \ne J'$, then we also get $\langle \nu_d(J) \rangle \cap \langle \nu_d(J') \rangle = \langle \nu_d(J \cap J') \rangle$. Hence in this way we get an $(me+e-1)$-dimensional real manifold of $\sigma_e(X_{m,d}(\mathbb{C}))(\mathbb{R})$. We will say that $(e - 2a, a)$ is the type of $J$ or of this $(me + e - 1)$-dimensional real manifold or of any $P \in \langle \nu_d(J) \rangle(\mathbb{R}) \setminus (\cup_{J' \subsetneq J} \langle \nu(J') \rangle)$.

**Lemma 1** *Fix positive integers $m, t$. Let $S \subset \mathbb{P}^m(\mathbb{C})$ be a finite subset such that $\langle S \rangle = \mathbb{P}^m(\mathbb{C})$ and $h^1(\mathcal{I}_S(t)) > 0$. Then $\sharp(S) \ge t + m + 1$.*

*Proof* The lemma is true if $m = 1$. Hence we may assume $m \ge 2$ and use induction on $m$. The case $t = 0$ is true for arbitrary $m$, because $S$ spans $\mathbb{P}^m(\mathbb{C})$. Since $S$ spans $\mathbb{P}^m(\mathbb{C})$, there is a hyperplane $H \subset \mathbb{P}^m(\mathbb{C})$ spanned by $m$ points of $S$. First assume $h^1(H, \mathcal{I}_{H \cap S}(t)) > 0$. Since $S \cap H$ spans $H$, the inductive assumption gives $\sharp(H \cap S) \ge t + m$. Since $S$ spans $\mathbb{P}^m(\mathbb{C})$, we have $\sharp(S) > \sharp(S \cap H) \ge m + t$. Now assume $h^1(H, \mathcal{I}_{H \cap S}(t)) = 0$. The Castelnuovo's exact sequence

$$0 \to \mathcal{I}_{S \setminus S \cap H}(t-1) \to \mathcal{I}_S(t) \to \mathcal{I}_{S \cap H, H}(t) \to 0$$

gives $h^1(\mathcal{I}_{S \setminus S \cap H}(t-1)) > 0$. Hence $\sharp(S \setminus S \cap H) \ge t + 1$ (obvious if $t = 1$, [8], Lemma 34, if $t - 1 > 0$). Since $\sharp(S \cap H) \ge m$, we get $\sharp(S) \ge t + m + 1$.

We recall the following weak form of [3], Theorem 1.

**Lemma 2** *Fix positive integers $m, t$. Let $S \subset \mathbb{P}^m(\mathbb{C})$, $m \ge 4$, be a finite subset such that $h^1(\mathcal{I}_S(t)) > 0$. Assume $\sharp(S) \le 4t + m - 5$. Then either there is a line $T \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(S \cap T) \ge t + 2$ or there is a plane $E \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(E \cap S) \ge 2t + 2$ or there is a 3-dimensional linear subspace $F \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(F \cap S) \ge 3t + 2$.*

**Lemma 3** *Fix integers $k \ge 1$, $e \ge 0$, $t \ge 1$, $m$ such that $m \ge 2k - 1 + e$. Fix $k$ lines $L_1, \dots, L_k \subset \mathbb{P}^m(\mathbb{C})$, a set $E \subset L_1 \cup \dots \cup L_k$ such that $\sharp(E \cap L_i) \le t+1$ for all $i$ and a set $F \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(F) = e$ and $\dim(\langle L_1 \cup \dots \cup L_k \cup F \rangle) = 2k - 1 + e$. Then $h^1(\mathcal{I}_{E \cup F}(t)) = 0$.*

*Proof* Set $N := \langle L_1 \cup \dots \cup L_k \rangle$ and $\Lambda := \langle N \cup F \rangle$. Since $\dim(\Lambda) = 2k - 1 + e$, we have $\dim(N) = 2k - 1$ and $F \cap N = \emptyset$. Since $E \cup F \subset \Lambda$ and $h^1(\Lambda, \mathcal{I}_{E \cup F, \Lambda}(t)) = h^1(\mathcal{I}_{E \cup F}(t))$, it is sufficient to prove the lemma in the case $m = 2k - 1 + e$. Since the case $k = 1$ and $e = 0$ is obvious, we may assume $k + e \ge 2$ and use induction on the integer $k + e$.

(a) Assume $e>0$ and fix $P \in F$. Set $F' := F \setminus \{P\}$. Notice that $H := \langle L_1 \cup \cdots \cup L_k \cup F' \rangle$ is a hyperplane of $\Lambda$ and that $\{P\} = (E\cup F)\setminus (E\cup F)\cap H$. Hence we have an exact sequence on $\Lambda$:

$$0 \to \mathcal{I}_{\{P\}}(t-1) \to \mathcal{I}_{E\cup F}(t) \to \mathcal{I}_{E\cup F',H}(t) \to 0 \quad (2)$$

Since $t-1 \ge 0$, we have $h^1(\mathcal{I}_{\{P\}}(t-1)) = 0$. The inductive assumption gives $h^1(H, \mathcal{I}_{E\cup F',H}(t)) = 0$. Hence (2) gives $h^1(\mathcal{I}_{E\cup F}(t)) = 0$.

(b) Assume $e=0$ and hence $k \ge 2$ and $F = \emptyset$. Set $N' := \langle L_1 \cup \cdots \cup L_{k-1} \rangle$. Since $\dim(N) = 2k-1$, we have $\dim(N') = 2k-3$. Hence there is a hyperplane $M$ containing $N'$ and a point of $E \cap L_k$ (or just containing $N'$ if $E\cap L_k = \emptyset$). Since $\sharp(E \setminus E\cap M) \le t$, we have $h^1(\mathcal{I}_{E\setminus E\cap M}(t-1)) = 0$. Look at the following exact sequence on $N$:

$$0 \to \mathcal{I}_{E\setminus E\cap M}(t-1) \to \mathcal{I}_E(t) \to \mathcal{I}_{E\cap M, M}(t) \to 0 \quad (3)$$

Since $\sharp(E \setminus E\cap M) \le t$, we have $h^1(\mathcal{I}_{E\setminus E\cap M}(t-1)) = 0$ (e.g., by [8], Lemma 34). Hence (3) gives $h^1(\mathcal{I}_E(t)) = 0$.

**Lemma 4** *Fix integers $a \in \{1,2,3\}$ and $e$ with $0 \le e \le 7-2a$. Assume $m \ge 2a+e-1$ and $d \ge 4a+2e$. Fix $Q_i \in \mathbb{P}^m(\mathbb{C})$, $1 \le i \le a$, such that $\sharp(\{Q_1,\dots,Q_a,\sigma(Q_1),\dots,\sigma(Q_a)\}) = 2a$ and $\dim(\langle\{Q_1,\dots,Q_a,\sigma(Q_1),\dots,\sigma(Q_a)\}\rangle = 2a-1$. Set $D_i := \langle\{Q_i,\sigma(Q_i)\rangle$. If $e=0$, then set $A := \{Q_1,\dots,Q_a,\sigma(Q_1),\dots,\sigma(Q_a)\}$. If $e>0$, then take $P_1,\cdots P_e \in \mathbb{P}^m(\mathbb{R})$ such that $\sharp(A) = 2a+e$, where*

$$A := \{P_1,\dots,P_e,Q_1,\dots,Q_a,\sigma(Q_1),\dots,\sigma(Q_a)\}.$$

*Fix $P \in \langle \nu_d(A)\rangle(\mathbb{R})$ such that $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$. Then $A$ is the only scheme evincing $r_{\mathbb{C}}(P)$, $r_{\mathbb{R}}(P) = 2a+e$ and every $B \subset \mathbb{P}^m(\mathbb{R})$ evincing $r_{\mathbb{R}}(P)$ contains $d$ points on each $D_i$ and (if $e>0$), $P_1,\dots,P_e$.*

*Proof* Notice that $\dim(\langle\nu_d(A)\rangle) = \sharp(A)-1$. Since $A$ is a finite set, it has finitely many proper subsets. Hence $P$ exists and the set of all such points $P$ is an open and dense subset of the real vector space $\langle\nu_d(A)\rangle(\mathbb{R})$. The set $A$ is unique ([12], Theorem 1.5.1, or Remark 2).

(a) Fix $A' \subsetneq A$ such that $A' \ne \emptyset$. Since $P \in \langle\nu_d(A)\rangle$ and $P \notin \langle\nu_d(A_1)\rangle$ for any $A_1 \subsetneq A$, there is a unique $P' \in \langle\nu_d(A')\rangle$ such that $P \in \langle\{P'\}\cup\nu_d(A\setminus A')\rangle$. Since $\sigma(A) = A$ and $\sigma(P) = P'$, if $A'$ is defined over $\mathbb{R}$ the uniqueness of $P'$ implies $\sigma(P') = P'$.

(b) Take $B \subset \mathbb{P}^m(\mathbb{R})$ evincing $r_{\mathbb{R}}(P)$. Hence $P \notin \langle\nu_d(B')\rangle$ for any $B' \subsetneq B$. Since $B \ne A$, [4], Lemma 1, gives $h^1(\mathcal{I}_{A\cup B}(d)) > 0$. Set $W_0 := A\cup B$. Set $D_i := \langle\{Q_i,\sigma(Q_i)\}\rangle$. Each $D_i$ is a line defined over $\mathbb{R}$. We have $\langle\cup_{i=1}^a D_i\rangle = \langle\{Q_1,\sigma(Q_1),\dots,Q_a,\sigma(Q_a)\}\rangle$ and hence

$$\dim(\langle\cup_{i=1}^a D_i\rangle) = \dim(\langle\{Q_1,\sigma(Q_1),\dots,Q_a,\sigma(Q_a)\rangle) = 2a-1.$$

By step (a) there is a unique $O_i \in \langle\nu_d(D_i)\rangle(\mathbb{R})$ such that $P \in \langle\{O_1,\dots,O_a\}\cup \nu_d(\{P_1,\dots,P_e\})\rangle$. Since each $O_i$ has rank $\le d$ with respect to the rational normal curve $\nu_d(D_i)$ ([18], Proposition 2.1), we get $\sharp(B) \le e+ad$. Hence

$\sharp(A \cup B) \le 2e + ad + 2a$. If $a \le 2$, then we get $\sharp(A \cup B) \le 2d + 10$. If $a = 3$, then $\sharp(A \cup B) \le 3d + 7$. If $(a, e) = (3, 1)$ (resp. $(a, e) = (3, 0)$) we have $d \ge 14$ (resp. $d \ge 12$) and $m \ge 6$ (resp. $m \ge 5$). Hence if $a = 3$ we have $2m - 4 + 3d > \sharp(A \cup B)$, unless $a = 3$, $e = 0$ and $m = 5$.

(c) Let $H_1 \subset \mathbb{P}^m(\mathbb{C})$ be a hyperplane such that $b_1 := \sharp(W_0 \cap H_1)$ is maximal. Set $W_1 := W_0 \setminus W_0 \cap H_1$. For each integer $i \ge 2$ define recursively the hyperplane $H_i \subset \mathbb{P}^m(\mathbb{C})$, the integer $b_i$ and the set $W_i$ in the following way. Let $H_i \subset \mathbb{P}^m(\mathbb{C})$ be a hyperplane such that $b_i := \sharp(W_{i-1} \cap H_i)$ is maximal. Set $W_i := W_{i-1} \setminus W_{i-1} \cap H_i$. For each integer $i \ge 1$ we have an exact sequence

$$0 \to \mathcal{I}_{W_i}(d - i) \to \mathcal{I}_{W_{i-1}}(d + 1 - i) \to \mathcal{I}_{W_{i-1} \cap H_i, H_i}(d + 1 - i) \to 0 \qquad (4)$$

Since $h^1(\mathcal{I}_{W_0}(d)) > 0$, (4) implies the existence of an integer $i > 0$ such that $h^1(H_i, \mathcal{I}_{W_{i-1} \cap H_i, H_i}(d+1-i)) > 0$. We call $g$ the minimal such an integer. The sequence $\{b_i\}_{i \ge 0}$ is non-decreasing. Any $m$ points of $\mathbb{P}^m(\mathbb{C})$ are contained in a hyperplane. Hence if $b_i \le m - 1$, then $b_{i+1} = 0$. Since $\sharp(W_0) \le 2a + ad + e$ and $m \ge 2a + e - 1$, we get $b_i = 0$ for all $i > (d + 2)/2$. Hence $g \le (d + 2)/2$. Since $h^1(H_g, \mathcal{I}_{W_{g-1} \cap H_g}(d + 1 - g)) > 0$, we have $b_g \ge d + 3 - g$ ([8], Lemma 34). Since $b_i \ge b_g$ for all $i \le g$, we get $g(d + 3 - g) \le 2a + ad + e$. Until step (d) we assume $g \ge 2$. Assume for the moment $b_g \ge 2d + 4 - 2g$. Since $b_i \ge b_g$ for all $i \le g$, we get $g(2d + 4 - 2g) \le 2a + ad + e \le ad + 7$. Set $\psi(t) := 2t(d + 2 - t)$. The real function $\psi(t)$ is increasing if $t \le (d + 2)/2$ and decreasing if $t > (d + 2)/2$. Since $2 \le g \le (d + 2)/2$, $\psi(2) = 4d > ad + 7$ and $\psi(d+2)/2 = (d+2)^2 > ad+7$, we get a contradiction. Hence $b_g \le 2(d+1-g)+1$. By [8], Lemma 34, there is a line $T \subset H_i$ such that $\sharp(T \cap W_{g-1}) \ge d + 3 - g$. Since $b_g > 0$, $W_{g-2}$ spans $\mathbb{P}^m$. Hence $b_{g-1} \ge (m - 2) + d + 3 - g$. Since $b_i \ge b_{g-1}$ for all $i \le g - 1$, we get $2a + ad + e \le (g - 1)(m - 2) + g(d + 3 - g)$. Set $\phi(t) := t(d+1+m-t)-m+2$. Since $\phi(t)$ is increasing if $t \le (d+1+m)/2$, $2 \le g \le (d + 2)/2$, and $\phi(3) = 2m - 4 + 3d > \sharp(A \cup B)$ (unless $m = 5$, $a = 3$ and $e = 0$), we get $g = 2$, unless $a = 3$, $e = 0$ and $m = 5$; in this case we only get $g \le 3$.

(c1) Assume for the moment $m = 5$, $a = 3$, $e = 0$ and $g = 3$. We have $b_3 \ge d$ and $b_1 \ge b_2 \ge d + 3$. Hence $b_4 = 0$, $b_3 = d$, $b_2 = b_1 = d + 3$ and $W_2 \subset T$ and $\sharp(W_2) = d$. Let $H_1' \subset \mathbb{P}^5(\mathbb{C})$ be a hyperplane containing $T$ and with $b_1' := \sharp(H_1' \cap W_0)$ maximal among the hyperplanes containing $T$. Set $W_1' := W_0 \setminus W_0 \cap H_1'$. If $h^1(H_1', \mathcal{I}_{W_0 \cap H_1', H_1'}(d)) > 0$, then go to step (d). Now assume $h^1(H_1', \mathcal{I}_{W_0 \cap H_1', H_1'}(d)) = 0$. From the exact sequence (4) with $i = 1$ and $H_1'$ instead of $H_1$ we get $h^1(\mathcal{I}_{W_1'}(d - 1)) > 0$. Since $W_0$ spans $\mathbb{P}^5$, we have $b_1' \ge d + 3$. Since $b_1 \ge b_1'$, we get $b_1' := b_1$. Hence any $J \subset W_0 \setminus W_0 \cap T$ with $\sharp(J) \le 4$ is linearly independent. Since $\sharp(W_1') = 2d + 3$ and $m = 5$, we easily get $h^1(\mathcal{I}_{W_1'}(d - 1)) = 0$ (Lemma 3 is stronger), a contradiction.

(c2) Now assume $g = 2$. Set $U_0 := W_0$. Let $M_1$ be a hyperplane containing $T$ and with $c_1 := \sharp(M_1 \cap W_0)$ maximal among the hyperplanes containing $T$. Set $U_1 := U_0 \setminus U_0 \cap M_1$. Define recursively the hyperplane $M_i \subset \mathbb{P}^m(\mathbb{C})$, the integer $c_i$ and the set $U_i$ in the following way. Let $M_i \subset \mathbb{P}^m(\mathbb{C})$ be a hyperplane such that $c_i := \sharp(U_{i-1} \cap M_i)$ is maximal. Set $U_i := U_{i-1} \setminus U_{i-1} \cap M_i$. For each integer $i \geq 1$ we have an exact sequence like (4) with $U_{i-1}$ and $U_i$ instead of $W_{i-1}$ and $W_i$ and with $M_i$ instead of $H_i$. Hence there is a minimal integer $f \geq 1$ such that $h^1(M_f, \mathcal{I}_{U_{f-1}\cap M_f, M_f}(d+1-f)) > 0$. Assume for the moment $f \geq 2$. Notice that $c_1 \geq d+m-1$, that $c_{i+1} \leq c_i$ for all $i \geq 2$ and that $c_m \geq m$ for all $m < f$. As in the case with $g$ we get a contradiction, unless $f = 2$. First assume $c_2 \geq 2d$. Hence $c_1 + c_2 \geq 4d > 3d + 7 \geq \sharp(A \cup B)$, a contradiction. Now assume $c_2 \leq 2d - 1$. Since $h^1(M_2, \mathcal{I}_{U_1\cap M_2, M_2}(d-1)) > 0$, there is a line $T' \subset M_2$ such that $\sharp(T' \cap U_0) \geq d + 1$. Hence $\sharp((T' \cup T) \cap W_0) \geq 2d + 2$. Since $b_2 > 0$, $W_0$ spans $\mathbb{P}^m$. Hence there is a hyperplane containing $T' \cup T$ and $m - 4$ further points of $W_0 \setminus W_0 \cap (T \cup T')$. Hence $b_1 \geq 2d + m - 2$. Since $b_2 \geq d + 1$, we get $ad + 7 \leq 3d + m - 1$. Hence $a = 3$ and $5 \leq m \leq 8$. Take a hyperplane $H''$ containing $T' \cup T$ and at least $m - 4$ further points and assume $h^1(H'', \mathcal{I}_{W_0\cap H'', H''}(d)) = 0$. An exact sequence (4) with $H''$ instead of $H_1$ gives $h^1(\mathcal{I}_{W_0\setminus W_0\cap H''}(d-1)) > 0$. The proof of the inequality $c_2 \leq 2d - 1$ gives $\sharp(W_0 \setminus W_0 \cap H'') \leq 2d - 1$. Hence there is a line $T''$ such that $\sharp(T'' \cap (W_0 \setminus W_0 \cap H'')) \geq d + 1$. If $m \geq 6$, then there is a hyperplane containing $T \cup T' \cup T''$ and hence $b_1 \geq 3d+3$; hence $b_1 + b_2 \geq 4d+4 > 3d+7$, a contradiction. Now assume $m = 5$, $a = 3$ and $e = 0$ and $\dim(\langle T \cup T' \cup T' \rangle) = 5$. See step (d5) for this case.

(d) In this step we cover the case $g = 1$, the case $g \geq 2$ and $f = 1$ the case $h^1(H_1', \mathcal{I}_{W_0\cap H_1', H_1'}(d)) > 0$, the case $h^1(H_1'', \mathcal{I}_{W_0\cap H_1'', H_1''}(d)) > 0$ and conclude the proof of the case $g = 2$, $m = 5$, $a = 3$ and $e = 0$. In these cases (except the last one) we have a hyperplane $H \subset \mathbb{P}^m(\mathbb{C})$ defined over $\mathbb{R}$ and with $h^1(H, \mathcal{I}_{H\cap W_0}(d)) > 0$.

(d1) Assume for the moment $W_0 \subset H$. In this case we may use induction on $m$, because the inclusion $A \subset H$ implies $m > 2a+ad+e-1$. Hence from now on we assume that $W_0 \setminus W_0 \cap H \neq \emptyset$. We also reduce (decreasing if necessary $m$) to the case in which $W_0$ spans $\mathbb{P}^m(\mathbb{C})$. Since $W_0$ spans $\mathbb{P}^m(\mathbb{C})$, we may take $H$ with the additional condition that $W_0 \cap H$ spans $H$.

(d2) Assume for the moment $h^1(\mathcal{I}_{W_0\setminus W_0\cap H}(d-1)) = 0$. By [6], Lemma 5, we have $A \setminus A \cap H = B \setminus B \cap H$. Hence $\sharp(B \setminus B \cap H) \leq e$ and $H$ contains $\langle\{Q_1, \sigma(Q_1), \dots, Q_a, \sigma(Q_a)\}\rangle$. Since $A \subset H$ and $W_0 \nsubseteq H$, we have $e > 0$. Set $e' := \sharp(W_0 \setminus W_0 \cap H)$ and write $F_1 := W_0 \setminus W_0 \cap H$, $A_1 := A \setminus F_1$ and $B_1 := B \setminus F_1$. Take either $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Since $d \geq 2a + e - 1$, we have $\langle\nu_d(A_1)\rangle(\mathbb{K}) \cap \langle\nu_d(F_1)\rangle(\mathbb{K}) = \emptyset$. Since $d \geq ad + e - 1$, we have $\langle\nu_d(B_1)\rangle(\mathbb{K}) \cap \langle\nu_d(F_1)\rangle(\mathbb{K}) = \emptyset$. Since $d \geq \sharp(A \cup B) - 1$, we have $\langle\nu_d(A)\rangle(\mathbb{K}) \cap \langle\nu_d(B)\rangle(\mathbb{K}) = \langle\nu_d(A \cap B)\rangle(\mathbb{K})$. Hence taking $A_1$ and $B_1$ instead of $A$ and $B$ we reduce to the case $(a', e') = (a, e - \sharp(F_1))$ and in this case we get that $W_0 \setminus F_1$ is contained in a hyperplane (a case inductively solved). Hence from now on we assume $h^1(\mathcal{I}_{W_0\setminus W_0\cap H}(d-1)) > 0$. By [8], Lemma 34, we have $\sharp(W_0) - \sharp(W_0 \cap H) \geq d+1$. Lemma 1 gives $\sharp(W_0 \cap H) \geq d + m - 2$.

(d3) Assume for the moment $\sharp(W_0) - \sharp(W_0 \cap H) \geq 2d$. Hence $2a+e+ad \leq 3d + m - 2$. Hence $a = 3$ and $5 \leq m \leq 6$. See Step (d7).

(d4) Now assume $\sharp(W_0 \setminus W_0 \cap H) \leq 2d - 1$. By [8], Lemma 34, there is a line $T_1 \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(T_1 \cap (W_0 \setminus W_0 \cap H)) \geq d + 1$. Let $R_1 \subset \mathbb{P}^m(\mathbb{C})$ be a hyperplane such that $R_1 \supset T_1$ and $m_1 := \sharp(W_0 \cap R_1)$ is maximal among the hyperplanes containing $T_1$. Since $\sharp(T_1 \cap (W_0 \setminus W_0 \cap H)) \ge d+1$, we have $m_1 \ge d+m-1$. Assume for the moment $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R_1}(d-1)) = 0$. By [6], Lemma 5, we have $A \setminus A \cap H = B \setminus B \cap H$. Hence $\sharp(B \setminus B \cap R_1) \le e$ and $R_1$ contains $\langle \{Q_1, \sigma(Q_1), \dots, Q_a, \sigma(Q_a)\} \rangle$. Since $A \subset R_1$ and $W_0 \nsubseteq R_1$, we have $e > 0$. As in step (d2) we reduce to a case with a smaller $e$, say $e'$, for which we run all the proof from step (b) on. In case (d1) we see that we get a smaller $m$ for $e' < e$. Hence after finitely many steps we run only with cases with $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R_1}(d-1)) > 0$.

Now assume $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R_1}(d-1)) > 0$. See step (d7) for the case $\sharp(W_0 \setminus W_0 \cap R_1)) \ge 2d$. Here we assume $\sharp(W_0 \setminus W_0 \cap R_1)) \le 2d-1$. Hence there is a line $T_2$ such that $\sharp(T_2 \cap (W_0 \setminus W_0 \cap R_1)) \ge d+1$ ([8], Lemma 34). Let $R_1 \subset \mathbb{P}^m(\mathbb{C})$ be a hyperplane such that $R_1 \supset T_1 \cup T_2$ and $W_0 \cap R_2$ spans $R_2$. Notice that $\sharp(R_2 \cap W) \ge m+2d-2$. First assume $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R_2}(d-1)) = 0$. We conclude as at the beginning of step (d2). Now assume $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R_2}(d-1)) > 0$. Since $4d-m-1 > \sharp(W_0)$, we have $\sharp(W_0 \setminus W_0 \cap R_2) \le 2d-1$. Hence $\sharp(W_0 \setminus W_0 \cap R_2)) \ge d+1$ (it implies $a=3$) and there is a line $T_3$ such that $\sharp(T_3 \cap (W_0 \setminus W_0 \cap R_2)) \ge d+1$. First assume that either $m \ge 6$ or $\dim(\langle T_1 \cup T_2 \cup T_3 \rangle) \le 4$. In these cases there is a hyperplane $R_3$ containing $T_1 \cup T_2 \cup T_3$ and spanned by some of the points of $W_0$. Since $T_1 \cup T_2 \subset R_2$ and $\sharp(T_3 \cap (W_0 \setminus W_0 \cap R_2)) \ge d+1$, we have $\sharp(W_0 \cap R_3) \ge 3d+3+(m-5)$. Hence $\sharp(W_0) \ge 3d+m-2$. Hence either $m=5$, $a=3$ and $e=0$ or $m=6$, $a=3$ and $e=1$.

(d5) Now assume $m=5$, $a=3$, $e=0$ and $\dim(\langle T_1 \cup T_2 \cup T_3 \rangle) = 5$. Hence $T_i \ne T_j$ for all $i \ne j$. Since $\sharp(A \cap T_j) \le 2$, we have $\sharp(B \cap T_j) \ge 2$. Hence each line $T_j$ is defined over $\mathbb{R}$. Hence either $T_j \cap A = \emptyset$ or $T_j = \langle \{Q_h, \sigma(Q_h)\} \rangle$ for some $h \in \{1,2,3\}$. All triples of complex lines in $\mathbb{P}^5(\mathbb{C})$ whose union spans $\mathbb{P}^5(\mathbb{C})$ are projectively equivalent. Hence we immediately see that the sheaf $\mathcal{I}_{T_1 \cup T_2 \cup T_3}(2)$ is spanned by its global sections. Since $W_0$ is finite, we get the existence of a complex quadric hypersurface $J$ such that $T_1 \cup T_2 \cup T_3 \subset J$ and $J \cap W_0 = (T_1 \cup T_2 \cup T_3) \cap W_0$. Since $\deg(T_i \cap W_0) \ge d+1$ for all $i$ and $T_i \cap T_j = \emptyset$ for all $i \ne j$, we have $\sharp(W_0 \setminus W_0 \cap J) \le 3$. Hence $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap J}(d-2)) = 0$. As in [6], Lemma 5, we get $A \setminus A \cap J = B \setminus B \cap J$. Since $A \cap B = \emptyset$, we get $A \cup B \subset J$. Since $J \cap W_0 = (T_1 \cup T_2 \cup T_3) \cap W_0$. Hence $A \cup B \subset T_1 \cup T_2 \cup T_3$. Since $A \subset T_1 \cup T_2 \cup T_3$ we get that (up to renaming the points $Q_1, Q_2, Q_3$) $T_i = \langle \{Q_i, \sigma(Q_i)\} \rangle$. To conclude the lemma in this case we only need to prove that $\sharp(B \cap T_i) = d$ for all $i$. Up to now we only know that $\sharp(B \cap T_i) \ge d-1$. Since $\sharp(B) \le 3d$, it is sufficient to prove that $\sharp(B \cap T_i) \ge d$ for all $i$. Assume that this is not the case and that, up to a permutation of the indices $1,2,3$, there is $h \in \{1,2,3\}$ such that $\sharp(T_i \cap B) = d-1$ if and only if $i \ge h$. Since $A \subset T_1 \cup T_2 \cup T_3$, there are $O_i \in \langle \nu_d(T_i)(\mathbb{C}) \rangle$ such that $P \in \langle \{O_1, O_2, O_3\} \rangle$. Since $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, we get $P \notin \langle E \rangle$ for any $E \subsetneq A_i$ and that the points $O_i$ are unique. The uniqueness of $O_i$ implies $O_i \in \langle \nu_d(T_i)(\mathbb{R}) \rangle$. Fix $i < h$ (if any). Take the union of $B \cap T_j$ for all $j \ne i$ and a set computing the real rank of $O_i$ with respect to the rational normal curve $\nu_d(T_i)$. Since $\sharp(B) = r_{\mathbb{R}}(P)$, we get $\sharp(B \cap T_i) = d$ for all $i < h$ (if any). There is a hyperplane $M \subset \mathbb{P}^m(\mathbb{C})$ such that $M$ contains $T_i$ for all $i < h$ and exactly one point of $T_j \cap B$ for all

$j \geq 4$. Hence $W_0 \setminus W_0 \cap M$ is the union of $d$ points of $T_j$ for all $j \geq h$. Lemma 3 implies $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap M}(d-1)) = 0$. Hence $A \setminus A \cap M = B \setminus B \cap M$ ([6], Lemma 5). Hence $B \subset M$, a contradiction.

(d6) Now assume $m = 6$, $a = 3$ and $e = 1$. This case is done as in step (d5), because $d-2 \geq 11$ and $\sharp(W_0 \setminus W_0 \cap (T_1 \cup T_2 \cup T_3)) \leq 4$. We stated Lemma 3 in the case $e > 0$ to allow its quotation here.

(d7) In this step we conclude the proof of steps (d3) and (d4). We assume the existence of a hyperplane $H \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(W_0) - \sharp(W_0 \cap H) \geq 2d$, $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap H}(d-1)) > 0$ and $W_0 \cap H$ spans $H$. Hence $a = 3$ and it is sufficient to check the cases $m = 5$, $e = 0$ and $m = 6$, $e = 1$. By Lemma 2 there are $i \in \{1, 2, 3\}$ and an $i$-dimensional linear subspace $J_i \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(T_1 \cap (W_0 \setminus W_0 \cap H)) \geq i(d-1)+2$. If $i = 1$, then we continue as in step (d4), just using the line $T_1$. In all other cases take a hyperplane $R$ containing $J_i$ and with maximal $\alpha := \sharp(R \cap W_0)$. We have $\alpha \geq i(d-1)+2+m-i$. In step (d2) we proved that a contradiction arises unless $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R}(d-1)) > 0$. Hence $\sharp(W_0) - \alpha \geq d+1$. If $i = 3$, then we get $\sharp(W_0) \geq d + 1 + 3d - 1 + m - 4$, a contradiction. Now assume $i = 2$. Since $2d + \alpha > \sharp(W_0)$, we have $\sharp(W_0 \setminus W_0) \leq 2d - 1$. Since $h^1(\mathcal{I}_{W_0 \setminus W_0 \cap R}) > 0$, there is a line $T_4$ such that $\sharp(T \cap (W_0 \setminus W_0 \cap R) \geq d + 1$. Since $m \geq 5$ and $R \supset J_2$, there is a hyperplane containing $J_2$ and $T_4$. The maximality property of $\alpha$ gives $\alpha \geq 3d + 1$. Hence $\sharp(W_0) \leq (3d + 1) + (d + 1)$, a contradiction.

**Theorem 2** *Fix integers $m \geq 1$ and $d \geq 3$ and any $P \in \sigma_2(X_{m,d}(\mathbb{C}))(\mathbb{R}) \setminus X_{m,d}(\mathbb{R})$. Then either $r_{\mathbb{R}}(P) = 2$ or $r_{\mathbb{R}}(P) = d$ and both 2 and $d$ are typical real rank for $\sigma_2(X_{m,d})$. We have $r_{\mathbb{R}}(P) = d$ if and only if either $P$ is a point of the tangential variety $\tau(X_{m,d}(\mathbb{C}))(\mathbb{R})$ or $r_{\mathbb{C}}(P) = 2$ and the only $A \subset \mathbb{P}^m(\mathbb{C})$ evincing $r_{\mathbb{C}}(P)$ is of the form $\{Q, \sigma(Q)\}$ for some $Q \in \mathbb{P}^m(\mathbb{C}) \setminus \mathbb{P}^m(\mathbb{R})$. 2 and $d$ are the typical ranks of $\sigma_2(X_{m,d}(\mathbb{C}))(\mathbb{R})$.*

*Proof* By Remark 2 there is a unique scheme $Z \subset \mathbb{P}^m(\mathbb{C})$ such that $\deg(Z) = 2$, $P \in \langle \nu_d(Z) \rangle$ and $Z$ is defined over $\mathbb{R}$. First assume that $Z$ is not reduced and set $\{O\} := Z_{red}$. Since $\sigma(Z) = Z$, we have $O \in \mathbb{P}^m(\mathbb{R})$ and $D := \langle Z \rangle \subseteq \mathbb{P}^m(\mathbb{C})$ is a line defined over $\mathbb{R}$. In this case we have $r_{\mathbb{C}}(P) = d$ ([8], Theorem 32). Hence $r_{\mathbb{R}}(P) \geq d$. Since $P \in \langle \nu_d(D) \rangle$, $r_{\mathbb{R}}(P)$ is at most the real rank of $P$ with respect to the rational normal curve $\nu_d(D)$, we have $r_{\mathbb{R}}(P) \leq d$. Hence $r_{\mathbb{R}}(P) = d$. Now assume that $Z$ is reduced. If $Z = \{P_1, P_2\} \subset \mathbb{P}^m(\mathbb{R})$ with $P_1 \neq P_2$, then $r_{\mathbb{R}}(P) = 2$. Now assume $Z = \{Q, \sigma(Q)\}$ for some $Q \in \mathbb{P}^m(\mathbb{C}) \setminus \mathbb{P}^m(\mathbb{R})$. Set $T := \langle \{Q, \sigma(Q)\}$. Since the line $T$ is defined over $\mathbb{R}$ and $P \in \langle \nu_d(T) \rangle$, the bivariate case gives $r_{\mathbb{R}}(P) \leq d$ ([18], Proposition 2.1).

*Claim :* Let $B \subset \mathbb{P}^m(\mathbb{R})$ be any set evincing $r_{\mathbb{R}}(P)$. Then $\sharp(B) = d$ and $B \subset T(\mathbb{R})$.

*Proof of the Claim:* Since $r_{\mathbb{R}}(P) \leq d$, we have $\sharp(B) \leq d$. Set $A := \{Q, \sigma(Q)\}$. Since $A \neq B$, $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$, $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$, we have $h^1(\mathcal{I}_{A \cup B}(d)) > 0$ ([4], Lemma 1). Since $\sharp(A \cup B) \leq d+2$, [8], Lemma 34, gives $\sharp(A \cup B) = d+2$ and the existence of a line $T' \subset \mathbb{P}^m(\mathbb{C})$ such that $A \cup B \subset T'$. Since $\sharp(A \cup B) = d + 2$, we have $\sharp(B) = d$. Since $A$ spans $T$, we have $T' = T$.

The last part of Remark 2 gives that 2 and $d$ are the typical ranks of $\sigma_2(X_{m,d}(\mathbb{C}))(\mathbb{R})$.

**Theorem 3** *Fix integers $m \ge 2$ and $d \ge 6$. Then 3 and $d+1$ are the typical ranks of $\sigma_3(X_{m,d}(\mathbb{C}))(\mathbb{R})$.*

**Theorem 4** *Fix integers $m \ge 3$ and $d \ge 8$. For all $m \ge 3$ the typical ranks of $\sigma_4(X_{m,d}(\mathbb{C}))(\mathbb{R})$ are 4, $d+2$ and $2d$.*

**Theorem 5** *Fix integers $m \ge 4$ and $d \ge 10$. The typical ranks of $\sigma_5(X_{m,d}(\mathbb{C}))(\mathbb{R})$ are 5, $d+3$ and $2d+1$.*

**Theorem 6** *Fix integers $m \ge 5$ and $d \ge 12$. The typical ranks of $\sigma_6(X_{m,d}(\mathbb{C}))(\mathbb{R})$ are 6, $d+4$, $2d+2$ and $3d$.*

**Theorem 7** *Fix integer $m \ge 6$ and $d \ge 14$. The typical ranks of $\sigma_7(X_{m,d}(\mathbb{C}))(\mathbb{R})$ are 7, $d+5$, $2d+3$ and $3d+1$.*

*Proofs of Theorems 3, 4, 5, 6 and 7.* Fix the border rank $b \in \{3,4,5,6,7\}$. Obviously $b$ is a typical rank for the border rank. Notice that in the statement of the theorem concerning the border rank $b$ we assumed $m \ge b-1$. For any integer $a$ such that $1 \le a \le b/2$ we have $d \ge 4a + 2(b-2a)$. Hence the assumptions of Lemma 4 are satisfied for the data $m, d, a, e := b-2a$. Lemma 4 says that all typical ranks for the border rank $b$ are obtained taking an integer $a \in \{1, \dots, \lfloor b/2 \rfloor\}$, setting $e := b-2a$ and then describing a subset of $A \in \mathbb{P}^m(\mathbb{C})^b$ with $\sigma(A) = A$ and associated to the typical rank $ad+e$. □

**Theorem 8** *Fix integers $m \ge 2$, $b \ge 2$ and $d \ge 2b-1$. Then $b$ and $b+d-2$ are the first two typical ranks of $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$.*

*Proof* Obviously $b$ is the minimal typical rank. Take an integer $g$ such that $b < g \le b+d-2$ and take a sufficiently general $P$ in an $(mb+b-1)$-dimensional open subset of $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R}) \setminus \sigma_{b-1}(X_{m,d}(\mathbb{C}))(\mathbb{R})$ with real symmetric rank $g$. Since $d \ge 2b-1$, there is a unique zero-dimensional scheme $A \subset \mathbb{P}^m(\mathbb{C})$ such that $\deg(A) = b$ and $P \in \langle \nu_d(A) \rangle$ (Remark 2). Moreover $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. For a general $P$ we may assume that $A$ is reduced and that its Hilbert function is the Hilbert function of a general set of $b$ points of $\mathbb{P}^m(\mathbb{C})$, i.e. $h^1(\mathcal{I}_A(t)) = \max\{0, b - \binom{m+t}{m}\}$ for all $t \in \mathbb{N}$. In particular no 3 of the points of $A$ are collinear. The uniqueness of $A$ implies $\sigma(A) = A$. Take $B \subset \mathbb{P}^m(\mathbb{R})$ evincing $r_{\mathbb{R}}(P)$. We have $h^1(\mathcal{I}_{A\cup B}(d)) > 0$ ([4], Lemma 1). Since $\sharp(A \cup B) \le 2b+d-2 \le 2d+1$, there is a line $T \subset \mathbb{P}^m(\mathbb{C})$ such that $\sharp(A \cup B) \ge d+2$. Since $\sharp(A \cap T) \le 2$, we have $\sharp(B \cap T) \ge d$. Since $\sharp(B \cap T) \ge 2$, the line $T$ is defined over $\mathbb{R}$. Since $A \cup B$ is a finite set, there is a hyperplane $H \subset \mathbb{P}^m(\mathbb{C})$ such that $H \supseteq T$ and $H \cap (A \cup B \setminus (A \cup B) \cap T) = \emptyset$. Since $\sharp(A \cup B) - \sharp(H \cap (A \cup B)) \le d-1 \le d$, we have $h^1(\mathcal{I}_{A\cup B \setminus H \cap (A\cup B)}(d-1)) = 0$. Hence $A \setminus A \cap H = B \setminus B \cap H$ ([6], Lemma 5), i.e. $A \setminus A \cap T = B \setminus B \cap T$. Since $\sharp(A \cap T) \le 2$, $\sharp((A \cup B) \cap T) \ge d+2$ and $g \le b+d-2$, we get $g = b+d-2$, $\sharp(A \cap T) = 2$, $\sharp(B \cap T) = d$ and $A \cap B \cap T = \emptyset$. Hence $g = d+b-2$ and at least

$b-2$ of the points of $A$ are real. Since $g > b$, not all points of $b$ are real. We get that $A = \{Q, \sigma(Q)\} \cup (B \setminus B \cap T)$ for some $Q \in \mathbb{P}^m(\mathbb{C}) \setminus \mathbb{P}^m(\mathbb{R})$. Conversely take any $P' \in \langle \nu_d(A) \rangle(\mathbb{R}) \setminus (\cup_{A' \subsetneq A} \langle \nu_d(A') \rangle)$, with $A = \{Q, \sigma(Q)\} \cup F$ for some $F \subset \mathbb{P}^m(\mathbb{R})$ with $\sharp(F) = b-2$. Since $d \geq 2b-1$, we have $r_{\mathbb{C}}(P') = b$ and $P'$ has border rank $b$. Since $d \geq 2b-1$, $A$ is the only set with cardinality $\leq b$ such that $P' \in \langle \nu_d(A) \rangle(\mathbb{C})$. Since $Q \notin \mathbb{P}^m(\mathbb{R})$, we have $r_{\mathbb{R}}(P') > b$. Varying $Q$ and $F$ we cover a non-empty open subset of $\sigma_b(X_{m,d}(\mathbb{C}))(\mathbb{R})$. The first part of the proof gives $r_{\mathbb{R}}(P') \geq d+b-2$. Set $D := \langle \{Q, \sigma(Q)\} \rangle$. $D$ is a line defined over $\mathbb{R}$. Since $A$ evinces $r_{\mathbb{C}}(P')$, the set $\langle \nu_d(\{Q, \sigma(Q)\} \rangle \cap \langle \nu_d(F) \rangle$ is a unique point, $P_1$. Since $\sigma(A) = A$ and $\sigma(P) = P$, the uniqueness of $P_1$ implies $\sigma(P_1) = P_1$. Since $P_1$ has real rank $\leq d$ with respect to the rational normal curve $\nu_d(D)$ ([18], Proposition 2.1), we get $r_{\mathbb{R}}(P') \leq r_{\mathbb{R}}(P_1) + \sharp(F) \leq d+b-2$. Hence $r_{\mathbb{R}}(P') = d+b-2$. Hence $d+b-2$ is a typical rank.

*Remark 3* We may go further, up to the range $2d+b-4$ for all $m \geq 2$ if $d \gg \max\{m, b\}$, but the case $m = 2$ is quite different and the case $m = 3$ requires a different proof from the case $m \geq 4$ (roughly speaking, after doing the case $m = 3$ one may do the case $m > 3$ by induction on $m$ as in steps (d1) and (d2) of the proof of Lemma 4).

## 3 The bivariate case

In this section we prove Theorem 1 by proving a stronger result, which shows that bivariate polynomials with real rank $d$ are ubiquitous and " typical " even in very small pieces of $\sigma_b(X_{1,d}(\mathbb{C}))(\mathbb{R})$.

Fix an integer $b$ such that $2 \leq b \leq (d+2)/2$. We need to prove that $d$ is a typical rank of $\sigma_b(X_{1,d}(\mathbb{C}))(\mathbb{R})$. We prove the following stronger result.

**Theorem 9** *Fix integers $d, b, a$ such that $b \geq 2$, $d \geq 2b-1$ and $2 \leq 2a \leq b$. Fix $Q_1, \ldots, Q_a \in \mathbb{P}^1(\mathbb{C})$ and $P_i \in \mathbb{P}^1(\mathbb{R})$, $1 \leq i \leq b-2a$. Set*

$$A := \langle \{P_1, \ldots, P_{b-2a}, Q_1, \sigma(Q_1), \ldots, Q_a, \sigma(Q_a)\} \rangle.$$

*Assume $\sharp(A) = b$ and set $M := \langle \nu_d(A) \rangle(\mathbb{R})$. Then the real projective space $M$ has dimension $b-1$ and there is a non-empty open subset $U \subset M$ for the euclidean topology such that $r_{\mathbb{R}}(P) = d$ for all $P \in U$.*

*Proof* Since $d \geq b-1$, the real projective space $M$ has dimension $b-1$. Choose real homogeneous coordinates on $\mathbb{P}^1(\mathbb{C})$ so that $Q_j = \alpha_j \in \mathbb{C} \setminus \mathbb{R}$ and $P_h = \beta_h \in \mathbb{R}$. The space $M$ parametrizes (up to a non-zero real multiplicative constant) all degree $d$ homogeneous polynomial $f$ of the form

$$\sum_{j=1}^{a} (c_j (z-\alpha_j)^d + \overline{c_j} (z-\overline{\alpha_j})^d) + \sum_{h=1}^{b-2a} d_h (z-\beta_h)^d \quad c_j \in \mathbb{C},\ d_h \in \mathbb{R} \qquad (5)$$

Set $D := \langle \nu_d(\{Q_1, \sigma(Q_1)\} \rangle \subset \mathbb{P}^d(\mathbb{C})$. $D$ is a line defined over $\mathbb{R}$ which intersects the rational normal curve $X_{1,d}(\mathbb{C})$ at two distinct complex conjugate points.

Fix any $O \in D(\mathbb{R})$. Up to a real change of coordinates we may assume $\alpha_1 = i$ and hence $\overline{\alpha}_1 = -i$. Hence the polynomial $f_0$ associated to $O$ is of the form

$$f_O = c(z-i)^d + \overline{c}(z-i)^d,\ c \neq 0$$

We claim that $r_{\mathbb{R}}(O) = d$. Indeed, $r_{\mathbb{R}}(O) \le d$ by [18], Proposition 2.1. Since $O \notin \{Q_1, \sigma(Q_1)\}$, we have $r_{\mathbb{C}}(O) = 2$. As in the Proof of the Claim in the proof of Theorem 2 we see that $r_{\mathbb{R}}(O) \ge d$. Hence $r_{\mathbb{R}}(O) = d$. Set $w := (z+i)/(z-i)$. Since $w^d = -\overline{c}/c$ has $d$ distinct roots, we get that $f_O$ has $d$ distinct roots. It is easy to check that these roots are real, but this is also a consequence of [15], Corollary 1. Since $f_O$ has $d$ distinct real roots, there is a $\epsilon \in \mathbb{R}$, $\epsilon > 0$, such that any $f$ in (5) has $d$ distinct real roots if $c_1 = c$, $|c_j| < \epsilon$ for all $j = 2, \dots, a$ and $|d_h| < \epsilon$ for all $h = 1, \dots, b-2a$. Varying $c$, we get a non-empty open subset $U$ of $M$ formed by real polynomials with $d$ distinct real roots. Hence $f \in U$ has real rank $d$ by [15], Corollary 2.1.

*Remark 4* Take $A$ as in the statement of Theorem 9. We will say that $A$ has type $(b-2a, a)$. We only assumed that $a \ge 1$. With this assumption we get that $d$ is a typical rank for the corresponding real projective space $M$.

*Proof of Theorem 1.* A Zariski open subset of $\sigma_b(X_{1,d}(\mathbb{C}))$ is given by the union of all sets $\langle \nu_d(B)\rangle \setminus (\cup_{B' \subsetneq B} \langle \nu_d(B')\rangle)$ with $B \subset \mathbb{P}^1(\mathbb{C})$ and $\sharp(B) = b$. A Zariski open and non-empty open subset of $\sigma_b(X_{1,d}(\mathbb{C}))(\mathbb{R})$ is obtained taking the union over all $B \subset \mathbb{P}^1(\mathbb{C})$ such that $\sharp(B) = b$ and $\sigma(B) = B$. Only the ones of type $(b, 0)$ are not covered by Theorem 9. Of course, if $B$ as type $(b, 0)$, then $r_{\mathbb{R}}(P) = b$ for all $P \in \langle \nu_d(B)\rangle \setminus (\cup_{B' \subseteq B} \langle \nu_d(B')\rangle)$. □

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