Around Waring problem for polynomials

Contemporary mathematics Conference, Moscow, December 18-23, 2017

Boris Shapiro, Stockholm University

December 17, 2017

Topics to discuss

1. Waring problem for natural numbers
2. Waring problem for polynomials
3. Proof of the main result
4. Open problems

Main references

(i) R. Fröberg, G. Ottaviani, and B. Shapiro, On the Waring problem for polynomial rings, PNAS, vol 109, issue 15 (2012), 5600–5602.

(ii) S. Lundqvist, A. Oneto, B. Reznick, and B. Shapiro, On generic and maximal k-ranks of binary forms, submitted, arXiv:1711.05014.

(iii) R. Fröberg, S. Lundqvist, A. Oneto, and B. Shapiro, Algebraic stories from one and from the other pockets, in preparation.

In 1770 in his paper Meditationes Algebraicae, the English number theorist E. Waring (1736 – 1798) stated without proof, that every natural number is a sum of at most 9 cubes; every natural number is a sum of at most 19 fourth powers; and so on...
Apparently, he believed that, for every natural number $d \geq 2$, there exists a number N(d) such that every positive integer n can be written as

$$n = a_1^d + ... + a_{N(d)}^d, \quad a_i \in \mathbb{N}.$$

The smallest number with that property is denoted by $g(d)$. It took more than a century to prove this statement.

Theorem (D. Hilbert, 1909)

For any $d \geq 2$, $g(d)$ exists.

In fact, from the famous Lagrange's four-square Theorem (1770), we know that $g(2) = 4$. Later Wieferich and Kempner showed that $g(3) = 9$. In 1986, Balasubramanian, Dress and Deshouillers established that $g(4) = 19$.

Conjecture.

$$g(d) = 2d + \left[(3/2)^d\right] - 2.$$

Mahler proved that there are only finitely many values counterexamples to the equality. Due to massive computer checking, it is believed that it is always true and that the conjectured value of $g(d)$ is correct.

Davenport proved that any sufficiently large integer can be written as the sum of at most 16 fourth powers. For any $d \geq 2$, it is natural to define $G(d)$ as the least integer such that all sufficiently large integers can be expressed as sum of at most $G(d)$ dth powers of integers. Clearly, $G(d) \leq g(d)$.
Gauss observed that every number congruent to 7 mod(8) is a sum of four squares, which proves that $G(2) = g(2) = 4$. But the inequality can be strict. Indeed, it has been shown, for example, that $G(3) \leq 7$ while $g(3) = 9$ and $G(4) = 16$ while $g(4) = 19$.
At present not much is known about numbers G(d) and this problem is currently a very active area of research in number theory.

In case of (homogeneous) polynomials, a natural analog of the Waring problem for natural numbers can be formulated as follows.

Problem. For a given triple (n, k, d) of positive integers, find the minimal number $\sharp(n, k, d)$ such that every (resp. almost every) complex-valued homogeneous polynomial of degree kd in $(n+1)$ variables can be represented as a sum of at most $\sharp(n, k, d)$ many k-th powers of polynomials of degree d.

This vast and currently very active area of mathematical research dealing with additive decompositions of polynomials started with the following classical result on binary forms proven in 1851 by J. J. Sylvester[1].

[1] James Joseph (Sylvester) was born to a Jewish family in London in 1814. His remarkably original and successful mathematical career only partially helped him overcome the pervasive anti-Semitism of his era.

Theorem (Sylvester's Theorem)

(i) A general binary form $p \in \mathbb{C}[x, y]$ of odd degree $k = 2s - 1$ with complex coefficients can be written as

$$p(x, y) = \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^k, \text{ for some } \alpha_j, \beta_j \in \mathbb{C}. \quad (1)$$

(ii) A general binary form $p \in \mathbb{C}[x, y]$ of even degree $k = 2s$ with complex coefficients can be written as

$$p(x, y) = \lambda x^k + \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^k, \text{ for some } \lambda, \alpha_j, \beta_j \in \mathbb{C}. \quad (2)$$

Figure: J. J. Sylvester around 1890.

After Sylvester's work, decompositions of polynomials into sums of powers of linear forms have been widely studied from several perspectives starting with the geometrical point of view by the classic Italian school in algebraic geometry in the beginning of the 20-th century as well as current research by applied mathematicians and engineers in connection with tensor decompositions.

Such presentations are often called *Waring decompositions* and, for a given polynomial f, the smallest length of such a decomposition is called the *Waring rank*, or simply, the *rank* of f. The minimal number of linear forms required to represent a general form of degree k in n variables as a sum of their k-th powers is called the *generic rank* and denoted by $\mathrm{rk}^\circ(n,k)$, while the *maximal rank* $\mathrm{rk}^{\max}(n,k)$ is the minimal number of linear forms required to represent *any* form of degree k in n variables.

Rephrasing Sylvester's Theorem in this terminology, we have that

$$\mathrm{rk}^{\circ}(2,k) = \left\lceil \frac{k+1}{2} \right\rceil .$$

The explicit value of the generic Waring rank for any arbitrary k and n was obtained in the celebrated result of J. Alexander and A. Hirschowitz from 1995. Except for the case of quadrics in all dimensions and four additional exceptions $(n,k) = (3,4),(4,4),(5,3),(5,4)$, the generic rank coincides with its expected value given by

$$\mathrm{rk}^{\circ}(n,k) = \left\lceil \frac{1}{n}\binom{n+k-1}{k} \right\rceil .$$

In the case of quadrics, the generic rank is equal to n, while in all the exceptional cases the generic rank is by 1 bigger than the latter expected value.

Theorem

Additionally, the maximal Waring rank $\mathrm{rk}^{\max}(2, k)$ *of binary forms equals* k. *Also, the maximal value* k *is attained exactly on binary forms of the type* $l_1 l_2^{k-1}$, *where* l_1 *and* l_2 *are linearly independent linear forms.*

This is a classical result known to Sylvester, but it has been recently rigorously reproved by B.Reznick.

Five years ago, jointly with R. Fröberg and G. Ottaviani, we considered, for any triple of positive integers (k, d, n) with $k, n \geq 2$, decompositions of homogeneous polynomials of degree kd in n variables as sums of k-th powers of forms of degree d. Given a form f of degree kd, the smallest length of such a decomposition denoted by $\mathrm{rk}_k(f)$ is called the *k-rank* of f.

Analogously to the classical Waring rank, we define the *generic k-rank* for forms of degree kd in n variables, denoted by $\mathrm{rk}_k^\circ(n, kd)$, and the corresponding *maximal k-rank*, denoted by $\mathrm{rk}_k^{\max}(n, kd)$.

Theorem

For any triple of positive integers (k, d, n),

$$\mathrm{rk}_k^\circ(n, kd) \leq k^n.$$

A remarkable property of this bound is its independence of the parameter d and also its sharpness for any fixed k and n, when $d \gg 0$.

The following general conjecture about $\mathrm{rk}_k^\circ(n, kd)$ was suggested by G. Ottaviani in 2013 (private communication).

Conjecture

For any triple (k, d, n) *of positive integers with* $k, n, d \geq 2$,

$$\mathrm{rk}_k^\circ(n, kd) = \begin{cases} \min\left\{ s \geq 1 \mid s\binom{d+n-1}{n-1} - \binom{s}{2} \geq \binom{2d+n-1}{n-1} \right\}, & \textit{for } k = 2; \\ \min\left\{ s \geq 1 \mid s\binom{d+n-1}{n-1} \geq \binom{kd+n-1}{n-1} \right\}, & \textit{for } k \geq 3. \end{cases} \quad (3)$$

The latter Conjecture is supported by substantial computer experiments performed by G. Ottaviani and by a number of the participants of the problem-solving seminar in Stockholm.

In a recent paper we considered the case of binary forms.

Theorem

For $k, d \geq 2$, the generic k-rank of binary forms of degree kd is

$$\mathrm{rk}_k^\circ(2, kd) = \left\lceil \frac{kd+1}{d+1} \right\rceil .$$

This result extends Sylvester's Theorem to presentations of general binary forms of degree *kd* as sums of *k*-th powers of binary forms of degree *d* and gives a proof of Ottaviani's Conjecture in the case of binary forms.

In Fall 2014, B.Sh. formulated the following conjecture about the maximal rank of binary forms.

Conjecture

For $k \geq 2$, the maximal k-rank of binary forms of degree kd is

$$\mathrm{rk}_k^{\max}(2, kd) = k.$$

The case $d = 1$ is classical and well-known. Moreover, we know that a binary form has maximal rank if and only if it can be decomposed as $\ell_1 \ell_2^{k-1}$. Also, it is easy to prove that any binary polynomial of even degree can be decomposed as a sum two squares.

Theorem

Every binary sextic can be written as a sum of at most three cubes of binary quadratic forms.

One can also suspect that $\mathrm{rk}_k^{\max}(l_1 l_2^{kd-1}) = k$, where l_1 and l_2 are non-proportional linear binary forms, similarly to what happens in the classical case. From a recent results of Carlini-Oneto we know that this is an upper bound, but computing the actual k-rank is a very difficult task.

We recall that for any projective variety X, its p-th secant variety is defined as the Zariski closure of the union of the projective spans $< x_1, \dots, x_p >$, where $x_i \in X$. The following result gives a convenient reformulation of our problem.

Theorem

Given a linear space V, a general polynomial in $S^{kd}V$ is a sum of p k-th powers $g_1^k, \dots g_p^k$, where $g_i \in S^d V$ if and only if for p general forms $g_i \in S^d V$, $i = 1, \dots p$, the ideal generated by $g_1^{k-1}, \dots g_p^{k-1}$ contains $S^{kd}V$. (We shall call such an ideal kd-regular.)

Proof.

The statement is a direct consequence of Terracini's lemma. Consider the subvariety X in the ambient space $\mathcal{P}S^{kd}V$ consisting of the k-th powers of all forms from S^dV. The tangent space to X at $g_i^k \in X$ is of the form $\{g_i^{k-1}f | f \in S^dV\}$. Therefore, the p-secant variety of X coincides with the ambient space $\mathcal{P}S^{kd}V$ if and only if the span of the tangent spaces to X at general g_i^k, (which is equal to $\{\sum_{i=1}^p g_i^{k-1}f_i | f_i \in S^dV\}$), coincides with $\mathcal{P}S^{kd}V$ as well. □

We will show that if V is an $(n+1)$-dimensional linear space, then the ideal generated by k^n general forms of type g_i^{k-1}, where $g_i \in S_n^d V$ is kd-regular, i.e., contains $S^{kd}V$.
In order to do this, it suffices to find k^n specific polynomials $\{g_1, \dots g_{k^n}\}$ of degree d such that the ideal generated by the powers g_i^{k-1} is kd-regular.
Below, we will choose as g_i's powers of certain linear forms. For powers of linear forms one can use a new point of view related to apolarity.

Definition. *A (Macaulay) inverse system is a finite-dimensional space of polynomials which is closed under the differentiation with respect to the variables.*

First, we define a pairing $< \cdot, \cdot >$ between the polynomial rings $\mathbb{C}[V]$ and $\mathbb{C}[V^\vee]$. Let $x_0, ..., x_n$ be a basis of V and $y_0, ..., y_n$ a dual basis of $V^\vee$. Then for each $f(\bar{x}) = f(x_0, ..., x_n) \in \mathbb{C}[V]$, define a differential operator $f(\partial/\partial\bar{y}) := f(\frac{\partial}{\partial y_0}, ..., \frac{\partial}{\partial y_n})$ on $\mathbb{C}[V^\vee]$. Analogously, for each $g(\bar{y}) = g(y_0, ..., y_n) \in \mathbb{C}[V^\vee]$, define a differential operator $g(\partial/\partial\bar{x}) := g(\frac{\partial}{\partial x_0}, ..., \frac{\partial}{\partial x_n})$ on $\mathbb{C}[V]$.

For form f and g of the same degree, we define the pairing

$$< f, g > := f\left(\frac{\partial}{\partial\bar{y}}\right) \cdot g(\bar{y})|_{\bar{y}=0} = g\left(\frac{\partial}{\partial\bar{x}}\right) \cdot f(\bar{x})|_{\bar{x}=0}.$$

Definition. *The inverse system of a homogeneous ideal $I \subset \mathbb{C}[V]$ is its orthogonal complement with respect to the above pairing. In other words, it is given by*

$$I^{\perp} = \left\{ g(\bar{y}) \in \mathbb{C}[V^{\vee}] | f\left(\frac{\partial}{\partial \bar{y}}\right) \cdot g(y) = 0 \quad \text{for any} \quad f(\bar{x}) \in I \right\}.$$

Since $I^{\perp}$ is the space of solutions of a system of homogeneous linear differential equations with constant coefficients, it is a Macaulay inverse system. The space $I^{\perp}$ is graded. The dimension of the quotient algebra $A = \mathbb{C}[V^{\vee}]/I$ equals $\dim I^{\perp}$. Moreover, the dimension of the i-th graded component A_i of A equals the dimension of the i-th graded component of $I^{\perp}$.

Back to our considerations!
The space $T_{g^k}X^{\perp}$ orthogonal to $T_{g^k}X = \{g^{k-1}f | f \in S^dV\}$ is given by $T_{g^k}X^{\perp} = \{h \in S^{kd}V^{\vee} | h \cdot g^{k-1} = 0 \in S^dV^{\vee}\}$, i.e., is the space of polynomials in $V^{\vee}$ apolar to g^{k-1}. Moreover, when $g = \ell^m$, $\ell \in V$, the classical theory of apolarity provides a better result.

Lemma

A form $f \in S^mV^{\vee}$ is apolar to ℓ^{m-k}, i.e., $\ell^{m-k}f = 0$ if and only if all the derivatives of f of order $\leq k$ vanish at $\ell \in V$.

Using the above Lemma one can reduce the Main Theorem to the following statement.

Theorem

For a given integer $k \geq 2$, a form of degree kd in $(n+1)$ variables which has all derivatives of order $\leq d$ vanishing at k^n general points vanishes identically.

Our final effort will be settle the latter Theorem. Denote by $x_0, \ldots, x_n$ a basis of V. Let $\xi_i = e^{2\pi i\sqrt{-1}/k}$ for $i = 0, \ldots, k-1$ be the (set of all) k-th roots of unity. By semicontinuity, it is enough to find k^n special points in $\mathcal{P}V \simeq \mathbf{P}^n$ such that a polynomial of degree kd in $\mathbf{P}^n$ which has all derivatives of order $\leq d$ vanishing at these points must necessarily vanish identically. As such points we choose the points $(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n})$, where $0 \leq i_j \leq k-1$, $1 \leq j \leq n$.

The following result proves even more than was claimed in the above Theorem.

Theorem

For a given integer $k \geq 2$, a form of degree $kd + k - 1$ in $(n+1)$ variables which has all derivatives of order $\leq d$ vanishing at k^n general points vanishes identically.

Proof:
As above we choose as our configuration the k^n points $(1, \xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n})$, where $0 \leq i_j \leq k-1$, $1 \leq j \leq n$. Consider first the case $n = 1$. If a form $f(x_0, x_1)$ of degree $kd + k - 1$ has its derivatives of order $\leq d$ vanishing at all $(1, \xi_i)$, then f should be divisible by $(x_1 - \xi_i x_0)^{d+1}$ for $i = 0, \dots k-1$ and, therefore, if f is not vanishing identically, then its degree should be at least $k(d+1)$, which is a contradiction.

For $n \geq 2$, consider the arrangement of $\binom{n}{2}k$ hyperplanes given by $x_i = \xi_s x_j$, where $1 \leq i < j \leq n$, $0 \leq s \leq k-1$. One can easily check that this arrangement has the property that each hyperplane contains exactly k^{n-1} points. Furthermore, each point is contained in exactly $\binom{n}{2}$ hyperplanes. Indeed, consider, for example, the hyperplane $\mathcal{H}$ given by $x_n = \xi_i x_{n-1}$. The natural parametrization of $\mathcal{H}$ is by $(x_0, \dots, x_{n-1}) \mapsto (x_0, x_1, \dots, x_{n-1}, \xi_i x_{n-1})$ and the k^{n-1} points which lie on $\mathcal{H}$ correspond, according to this parametrization, exactly to $(1, \xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_{n-1}})$ for $0 \leq i_j \leq k-1$, $1 \leq j \leq n-1$. In other words, they correspond exactly to our arrangement of points in the previous dimension n.

Our proof now proceeds by a double induction on the number of variables n and degree d. Assume that the statement holds for all d and up to n variables. (The case $n = 1$ is settled above.) Let us perform a step of induction in d. First we settle the case $d \leq \binom{n}{2} - 1$. Consider a polynomial f of degree $kd + k - 1$ satisfying our assumptions. Restricting f to each of the above $\binom{n}{2}k$ hyperplanes $x_i = \xi_s x_j$, where $1 \leq i < j \leq n$, $0 \leq s \leq k - 1$, we obtain the same situation in dimension n. By the induction hypothesis f vanishes on each such hyperplane and, therefore, must be divisible by H, where H is the product of the linear forms $x_i = \xi_s x_j$ defining all the chosen hyperplanes. (Obviously, $\deg H = \binom{n}{2}k$.) Thus, f vanishes identically since $k\left(\binom{n}{2} - 1\right) + k - 1 < \binom{n}{2}k$.

For higher degrees we argue as follows. Take a form f of degree $kd+k-1$ satisfying our assumptions. Restricting as above f to each of the above $\binom{n}{2}k$ hyperplanes $x_i=\xi_s x_j$, we obtain the same situation in dimension n. Again, by the induction hypothesis f vanishes on each such hyperplane and must be divisible by H. We get

$$f=H\tilde{f},$$

where $\deg\tilde{f}=k\left(d-\binom{n}{2}\right)+k-1$ and $\tilde{f}$ has all derivatives of order $\leq d-\binom{n}{2}$ vanishing at the same k^n points $(1,\xi_{i_1},\xi_{i_2},\dots,\xi_{i_n})$. Indeed, in any affine coordinate system centered at any of these points, f has no terms of degree $\leq d$. Since H has its lowest term in degree $\binom{n}{2}$ it follows that $\tilde{f}$ has no terms in degree $\leq d-\binom{n}{2}$. By the induction hypothesis $\tilde{f}$ is identically zero.

Open problems

Question 1. What about the strong Waring problem for forms of degree kd in $n+1$ variables over $\mathbb{C}$?

By a result of Blekherman –Teitler, the maximal rank over $\mathbb{C}$ is at most twice the generic rank over $\mathbb{C}$, i.e.,

$$rk_d^{max}(n+1,k) \leq 2rk_d(n+1,k)$$

Open problems, cont. 2

Question 2. What about the strong Waring problem over $\mathbb{R}$?

By a recent result of C. Scheiderer, for $n = 2$, $k = 2$ and any d,

$$rk_d^{max,\mathbb{R}}(3,2) = d + 1 \text{ or } d + 2,$$

while $rk_d^{max}(3,2) \leq 8$.

E. Carlini, and A. Oneto. “Monomials as sums of k-th powers of forms." *Communications in Algebra* **43**(2): 650–658 (2015).

R. Fröberg, G. Ottaviani, and B. Shapiro, “On the Waring problem for polynomial rings", *Proceedings of the National Academy of Sciences*, **109**(15): 5600–5602 (2012).

J. J. Sylvester, An Essay on Canonical Forms, Supplement to a Sketch of a Memoir on Elimination, Transformation and Canonical Forms.

J. J. Sylvester, On a remarkable discovery in the theory of canonical forms and of hyperdeterminants, originally in Philosophical Magazine, vol. I, 1851Mathematical Papers, Vol. 1, Chelsea, New York, 1973.