What can be said about representing a polynomial of degree kd as a sum of the k-th powers of polynomials of degree d?

(joint with R.Fröberg, S. Lundqvist, A. Oneto, B. Reznick and all other participants of the problem-solving seminar)

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Topics to discuss

1. Waring problem for natural numbers
2. Waring problem for polynomials

Main references

(i) R. Fröberg, G. Ottaviani, and B. Shapiro, On the Waring problem for polynomial rings, PNAS, vol 109, issue 15 (2012), 5600–5602.

(ii) S. Lundqvist, A. Oneto, B. Reznick, and B. Shapiro, On generic and maximal k-ranks of binary forms, arXiv:1711.05014, submitted.

(iii) R. Fröberg, S. Lundqvist, A. Oneto, and B. Shapiro, Algebraic stories from one and from the other pockets, in preparation.

In 1770 in his paper Meditationes Algebraicae, the English number theorist E. Waring (1736 – 1798) stated without proof, that every natural number is a sum of at most 9 cubes; every natural number is a sum of at most 19 fourth powers; and so on...
Apparently, he believed that, for every natural number $d \geq 2$, there exists a number N(d) such that every positive integer n can be written as

$$n = a_1^d + ... + a_{N(d)}^d, \quad a_i \in \mathbb{N}.$$

The smallest number with that property is denoted by $g(d)$.
It took more than a century to prove this statement.

Theorem (D. Hilbert, 1909)

For any $d \geq 2$, $g(d)$ exists.

In fact, from the famous Lagrange's four-square Theorem (1770), we know that $g(2) = 4$. Later Wieferich and Kempner showed that $g(3) = 9$. In 1986, Balasubramanian, Dress and Deshouillers established that $g(4) = 19$.

Conjecture.

$$g(d) = 2d + \left[(3/2)^d\right] - 2.$$

Mahler proved that there are only finitely many values counterexamples to the equality. Due to massive computer checking, it is believed that it is always true and that the conjectured value of $g(d)$ is correct.

Davenport proved that any sufficiently large integer can be written as the sum of at most 16 fourth powers. For any $d \geq 2$, it is natural to define $G(d)$ as the least integer such that all sufficiently large integers can be expressed as sum of at most $G(d)$ dth powers of integers. Clearly, $G(d) \leq g(d)$.
Gauss observed that every number congruent to 7 mod(8) is a sum of four squares, which proves that $G(2) = g(2) = 4$. But the inequality can be strict. Indeed, it has been shown, for example, that $G(3) \leq 7$ while $g(3) = 9$ and $G(4) = 16$ while $g(4) = 19$.
At present not much is known about numbers G(d) and this problem is currently a very active area of research in number theory.

In case of (homogeneous) polynomials, a natural analog of the Waring problem for natural numbers can be formulated as follows.

Problem. For a given triple (n, k, d) of positive integers, find the minimal number $\sharp(n, k, d)$ such that every (resp. almost every) complex-valued homogeneous polynomial of degree kd in $(n + 1)$ variables can be represented as a sum of at most $\sharp(n, k, d)$ many k-th powers of polynomials of degree d.

This vast and currently very active area of mathematical research dealing with additive decompositions of polynomials started with the following classical result on binary forms proven in 1851 by J. J. Sylvester[1].

[1] James Joseph (Sylvester) was born to a Jewish family in London in 1814. His remarkably original and successful mathematical career only partially helped him overcome the pervasive anti-Semitism of his era.

Theorem (Sylvester's Theorem)

(i) A general binary form $p \in \mathbb{C}[x, y]$ of odd degree $k = 2s - 1$ with complex coefficients can be written as

$$p(x, y) = \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^k, \text{ for some } \alpha_j, \beta_j \in \mathbb{C}. \qquad (1)$$

(ii) A general binary form $p \in \mathbb{C}[x, y]$ of even degree $k = 2s$ with complex coefficients can be written as

$$p(x, y) = \lambda x^k + \sum_{j=1}^{s} (\alpha_j x + \beta_j y)^k, \text{ for some } \lambda, \alpha_j, \beta_j \in \mathbb{C}. \qquad (2)$$

Figure: J. J. Sylvester around 1890.

After Sylvester's work, decompositions of polynomials into sums of powers of linear forms have been widely studied from several perspectives starting with the geometrical point of view by the classic Italian school in algebraic geometry in the beginning of the 20-th century as well as current research by applied mathematicians and engineers in connection with tensor decompositions.

Such presentations are often called *Waring decompositions* and, for a given polynomial f, the smallest length of such a decomposition is called the *Waring rank*, or simply, the *rank* of f. The minimal number of linear forms required to represent a general form of degree k in n variables as a sum of their k-th powers is called the *generic rank* and denoted by $\mathrm{rk}^{\circ}(n, k)$, while the *maximal rank* $\mathrm{rk}^{\max}(n, k)$ is the minimal number of linear forms required to represent *any* form of degree k in n variables.

Rephrasing Theorem 2 in this terminology, we have that

$$\mathrm{rk}^\circ(2,k) = \left\lceil \frac{k+1}{2} \right\rceil.$$

The explicit value of the generic Waring rank for any arbitrary k and n was obtained in the celebrated result of J. Alexander and A. Hirschowitz from 1995. Except for the case of quadrics in all dimensions and four additional exceptions $(n,k) = (3,4),(4,4),(5,3),(5,4)$, the generic rank coincides with its expected value given by

$$\mathrm{rk}^\circ(n,k) = \left\lceil \frac{1}{n}\binom{n+k-1}{k} \right\rceil.$$

In the case of quadrics, the generic rank is equal to n, while in all the exceptional cases the generic rank is by 1 bigger than the latter expected value.

Theorem

Additionally, the maximal Waring rank $\mathrm{rk}^{\max}(2, k)$ *of binary forms equals* k*. Also, the maximal value* k *is attained exactly on binary forms of the type* $l_1 l_2^{k-1}$*, where* l_1 *and* l_2 *are linearly independent linear forms.*

This was probably a classical result known to Sylvester, but it has been recently reproved.

In [FOS12], jointly with R. Fröberg and G. Ottaviani we considered, for any triple of positive integers (k, d, n) with $k, n \geq 2$, decompositions of homogeneous polynomials of degree kd in n variables as sums of k-th powers of forms of degree d. Given a form f of degree kd, the smallest length of such a decomposition denoted by $\mathrm{rk}_k(f)$ is called the *k-rank* of f.

Analogously to the classical Waring rank, we define the *generic k-rank* for forms of degree kd in n variables, denoted by $\mathrm{rk}_k^\circ(n, kd)$, and the corresponding *maximal k-rank*, denoted by $\mathrm{rk}_k^{\max}(n, kd)$.

Theorem

for any triple (k, d, n)*,*

$$\mathrm{rk}^\circ_k(n, kd) \leq k^n.$$

A remarkable property of this bound is its independence of the parameter d and also its sharpness for any fixed k and n, when $d \gg 0$.

The following general conjecture about $\mathrm{rk}_k^\circ(n, kd)$ was suggested by G. Ottaviani in 2013 (private communication).

Conjecture

For any triple (k, d, n) of positive integers with $k, n, d \geq 2$,

$$\mathrm{rk}_k^\circ(n, kd) = \begin{cases} \min\left\{s \geq 1 \mid s\binom{d+n-1}{n-1} - \binom{s}{2} \geq \binom{2d+n-1}{n-1}\right\}, & \text{for } k = 2; \\ \min\left\{s \geq 1 \mid s\binom{d+n-1}{n-1} \geq \binom{kd+n-1}{n-1}\right\}, & \text{for } k \geq 3. \end{cases} \tag{3}$$

Conjecture 5 is supported by substantial computer experiments performed by G. Ottaviani and by a number of the seminar participants.

In a recent paper we considered the case of binary forms.

Theorem

For $k, d \geq 2$, the generic k-rank of binary forms of degree kd is

$$\mathrm{rk}^{\circ}_{k}(2, kd) = \left\lceil \frac{kd+1}{d+1} \right\rceil .$$

This result extends Theorem 2 to presentations of general binary forms of degree kd as sums of k-th powers of binary forms of degree d and gives a proof of Conjecture 5 in the case of binary forms.

In Fall 2014, B.Sh. formulated the following conjecture about the maximal rank of binary forms.

Conjecture

For $k \geq 2$, the maximal k-rank of binary forms of degree kd is

$$\mathrm{rk}_k^{\max}(2, kd) = k.$$

The case $d = 1$ is classical and well-known. Moreover, we know that a binary form has maximal rank if and only if it can be decomposed as $\ell_1 \ell_2^{k-1}$. Also, it is easy to prove that any binary polynomial of even degree can be decomposed as a sum two squares.

Theorem

Every binary sextic can be written as a sum of at most three cubes of binary quadratic forms.

One can also suspect that $\mathrm{rk}_k^{\max}(l_1 l_2^{kd-1}) = k$, where l_1 and l_2 are non-proportional linear binary forms, similarly to what happens in the classical case. From [CO15], we know that this is an upper bound, but computing the actual k-rank is a very difficult task.

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