On the Waring problem for polynomial rings

$$f = g_1^k + \cdots + g_r^k$$

How many k-th powers of forms are needed?

Boris Shapiro

joint work with Ralf Fröberg and Giorgio Ottaviani

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Plan of the talk

1 — Classical Waring

$g(k)$, $G(k)$ and rings

2 — Complex forms

powers of forms, generic vs. maximal rank

3 — Main theorem

$r_{\mathrm{gen}} \le k^n$, sharp for large d

4 — The proof

roots of unity & forced factors

5 — Recent progress

2012 – 2026: generic & maximal rank

Waring's problem: from integers to rings

Waring's problem for integers: a short history



Diophantus: sums of squares of rationals

Waring, *Meditationes Algebraicae*: 4 squares, 9 cubes, 19 biquadrates, "and so on"

Lagrange: every n is a sum of four squares

Hilbert: $g(k) < \infty$ for every k

Hardy–Littlewood: circle method, asymptotics, $G(k)$

$g(4) = 19$: Balasubramanian–Deshouillers–Dress

$g(k)$ = least s such that **every** $n \in \mathbb{N}$ is a sum of s k-th powers.

$G(k)$ = least s such that every **sufficiently large** n is a sum of s k-th powers.

The number $g(k)$: small numbers are the hardest

J. A. Euler's lower bound (c. 1772)

Take $q = \lfloor (3/2)^k \rfloor$ and $n = 2^k q - 1 < 3^k$. Only 1^k and 2^k are available, so $n = (q-1)\cdot 2^k + (2^k - 1)\cdot 1^k$ needs $2^k + q - 2$ summands.

Ideal Waring theorem

$$g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$$

Holds whenever $2^k\{(3/2)^k\} + \lfloor (3/2)^k \rfloor \le 2^k$ (Dickson, Pillai, Rubugunday, Niven 1936–44); true for $k \le 4.7 \cdot 10^8$ (Kubina–Wunderlich 1990) and for all large k (Mahler 1957).

k	$g(k)$	proved by
2	4	Lagrange 1770
3	9	Wieferich 1909, Kempner 1912
4	19	Balasubramanian–Deshouillers–Dress 1986
5	37	Chen Jingrun 1964
6	73	Pillai 1940

$23 = 2\cdot 2^3 + 7\cdot 1^3$ needs 9 cubes; $79 = 4\cdot 2^4 + 15\cdot 1^4$ needs 19 fourth powers.

The number $G(k)$: what happens for large n

Exact values are rare

- $k+1 \le G(k) \le g(k)$ (counting / congruences).
- $G(2) = 4$: every $n \equiv 7 \pmod 8$ needs four squares.
- $G(4) = 16$ (Davenport 1939): $31 \cdot 16^m$ needs 16 biquadrates.
- $4 \le G(3) \le 7$ (Linnik 1943); conjecturally $G(3) = 4$.

Upper bounds via the circle method

- Hardy–Littlewood (1920s): $G(k) \le (k-2)2^{k-1} + 5$.
- Vinogradov: $G(k) = O(k \log k)$.
- Wooley (1992): $G(k) \le k \log k + k \log\log k + O(k)$.
- Vaughan–Wooley: $G(5) \le 17$, $G(6) \le 24$, $G(7) \le 33$, $G(8) \le 42$.

The two flavours

$g(k)$ is dictated by a few **exceptional** small numbers; $G(k)$ by the **typical** behaviour. This is the dichotomy $r_{\max}$ vs. r_{gen} that we meet again for polynomials.

The Waring problem for rings

Following Gallardo–Vaserstein (2008)

For a ring A and $k \geq 2$ let $A_k \subset A$ be the set of sums of k-th powers and $w_k(a) = \min\{s : a = a_1^k + \cdots + a_s^k\}$, $a \in A_k$.

Strong Waring problem

Determine $w_k(A) = \sup_{a \in A_k} w_k(a)$ (possibly ∞).

$A = \mathbb{Z}_{\geq 0}$: $g(k)$; forms: the **maximal** rank $r_{\max}$

Weak Waring problem

Determine $\sup w_k(a)$ over *generic* $a \in A_k$.

$A = \mathbb{Z}_{\geq 0}$: $G(k)$; forms: the **generic** rank r_{gen}

★ **Our concrete case:** $A = \mathbb{C}[x_0, \ldots, x_n]$. Here A_k consists of all forms whose degree is divisible by k, and "generic" means "on a Zariski-open set".

Complex-valued forms

A first decomposition

Cancellation is allowed over the complex numbers

$$xy = \left(\frac{x+y}{2}\right)^2 + \left(\frac{i(x-y)}{2}\right)^2$$

💡 Over $\mathbb{C}$ *every* form of even degree is a sum of squares of forms: there is no positivity obstruction.

Question

How many k-th powers are needed for a **general** polynomial? For **every** polynomial?

The summands may be nonlinear

$$S_d = \mathbb{C}[x_0, \ldots, x_n]_d, \qquad f = g_1^k + \cdots + g_r^k, \qquad g_i \in S_d, \quad f \in S_{kd}.$$

One plane sextic, different questions

cubes of quadrics

$f = q_1^3 + \cdots + q_r^3$

$k = 3,\ d = 2$ (this talk)

sixth powers of linear forms

$f = \ell_1^6 + \cdots + \ell_s^6$

$k = 6,\ d = 1$ (classical Waring rank)

The classical case $d = 1$ is the *Waring problem for forms*; letting d vary gives the **k-rank** of f.

Generic rank and maximal rank

Generic

a nonempty Zariski-open set of coefficient vectors

$r_{\mathrm{gen}}(n, k, d)$

Every form

all coefficient vectors, including exceptional ones

$r_{\max}(n, k, d)$

$$r_{\mathrm{gen}} \leq r_{\max}$$

Sylvester: binary forms as sums of powers of linear forms

Odd degree $m = 2s - 1$

A general binary form is

$$f = \ell_1^{2s-1} + \cdots + \ell_s^{2s-1}$$

and this decomposition is **unique**.

Even degree $m = 2s$

A general binary form is

$$f = \ell_1^{2s} + \cdots + \ell_{s+1}^{2s}$$

with a **one-parameter family** of decompositions.

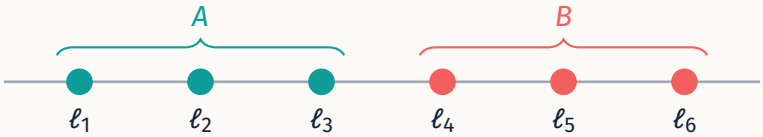
Sylvester's theorem and criterion (1851)

For $\ell_i = \alpha_i x + \beta_i y$ pairwise non-proportional, $f = \sum_{i=1}^{r} \lambda_i \ell_i^m \iff \prod_{i=1}^{r} (\beta_i \partial_x - \alpha_i \partial_y) f = 0$; i.e. a **squarefree** degree-r form lies in the kernel of the $(m - r + 1) \times (r + 1)$ Hankel (catalecticant) matrix of f.

$n = d = 1$: $r_{\text{gen}} = \lceil (m+1)/2 \rceil$, $r_{\max} = m$ (e.g. xy^{m-1}; Comas–Seiguer 2011).

Binary forms: two squares suffice

Factor the binary form and split its roots into two groups



$$f = AB = \left(\frac{A+B}{2}\right)^2 + \left(\frac{i(A-B)}{2}\right)^2$$

Classical (Choi–Lam–Reznick)

Every binary form of degree $2d$ is a sum of **two** squares; a general one in exactly $\binom{2d-1}{d} = \frac{1}{2}\binom{2d}{d}$ ways (up to $O(2,\mathbb{C})$).

What does “three decompositions” count?

A general binary quartic $f = \ell_1\ell_2\ell_3\ell_4$

1

$$A = \ell_1\ell_2$$
$$B = \ell_3\ell_4$$

2

$$A = \ell_1\ell_3$$
$$B = \ell_2\ell_4$$

3

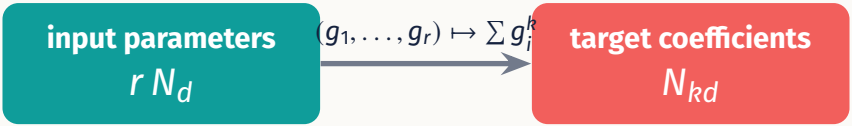
$$A = \ell_1\ell_4$$
$$B = \ell_2\ell_3$$

- Three $O(2,\mathbb{C})$-orbits of decompositions; infinitely many pairs inside each orbit.
- The torus action $(A,B) \mapsto (tA, t^{-1}B)$ does not change the orbit, and

$$\#\{\text{orbits}\} = \tfrac{1}{2}\binom{2d}{d} = \binom{2d-1}{d} \qquad (= 3 \text{ for } d = 2).$$

A necessary parameter count

$$N_e = \dim S_e = \binom{e+n}{n}$$



Lower bound

$$r_{\mathrm{gen}}(n,k,d) \;\geq\; \left\lceil \frac{N_{kd}}{N_d} \right\rceil$$

The classical generic rank: Alexander–Hirschowitz

Theorem (Alexander–Hirschowitz, 1995)

A general form of degree k in $n+1$ variables is a sum of $r_{\mathrm{gen}}(n,k,1) = \left\lceil \frac{1}{n+1}\binom{n+k}{n} \right\rceil$ k-th powers of linear forms — exactly the parameter count — **except** in the cases below.

case	expected	actual
quadrics, $k=2$, any $n \geq 2$	$\lceil \frac{n+2}{2} \rceil$	$n+1$
ternary quartics, $(n,k)=(2,4)$	5	**6**
quaternary quartics, $(3,4)$	9	**10**
quinary quartics, $(4,4)$	14	**15**
quinary cubics, $(4,3)$	7	**8**

Equivalent form (Terracini): h general *double points* in $\mathbb{P}^n$ impose independent conditions on hypersurfaces of degree k, except in the listed cases. E.g. the squared conic through 5 points is a quartic singular at all of them.

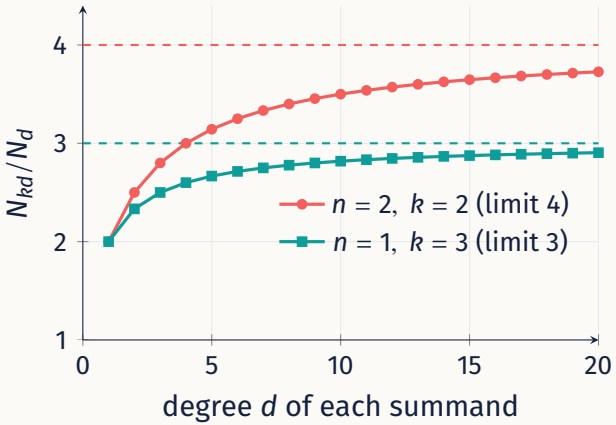
Exceptions known to Clebsch, Richmond, Palatini, Terracini; proof by the *Horace method*; simplified by Brambilla–Ottaviani (2008), Postinghel (2012).

The ratio approaches k^n

$$\frac{N_{kd}}{N_d} = \prod_{j=1}^{n} \frac{kd+j}{d+j} \;<\; k^n,$$

$$\lim_{d\to\infty} \frac{N_{kd}}{N_d} = k^n.$$

So k^n is a natural guess, and the question becomes whether the count is achieved.



The main theorem

A bound independent of the degree

Theorem (Fröberg–Ottaviani–Shapiro, 2012)

Let $n \geq 1$, $k \geq 2$, $d \geq 1$. A **general** form of degree kd in $n+1$ complex variables is a sum of

$$r_{\mathrm{gen}}(n,k,d) \;\leq\; k^n$$

k-th powers of forms of degree d. For fixed n and k, equality holds for all sufficiently large d.

★ New even for $k = 2$ (sums of squares).

∞ The bound does *not* grow with d, unlike the classical Waring rank.

Contrast with powers of linear forms

Remark 3 of the paper

With linear summands ($d = 1$, Alexander–Hirschowitz) a general form of degree kd needs

$$\left\lceil \frac{1}{n+1}\binom{kd+n}{n} \right\rceil \sim \frac{(kd)^n}{(n+1)!}$$

kd-th powers of linear forms — this grows with d.

Degree-d summands

at most k^n of them, for every d

Linear summands

about $(kd)^n/(n+1)!$ of them

Small degrees can behave differently

Squares in three variables: $n = 2$, $k = 2$

Target degree $2d$	**2**	**4**	**6**	**8**	**10**
Naive ceiling $\lceil N_{2d}/N_d \rceil$	2	3	3	3	4
Generic rank	**3**	3	**4**	**4**	4

Lesson

The finite-degree count has to account for the **symmetry** of the problem.

Squares have continuous symmetry

$$g_1^2 + \cdots + g_r^2 = (Qg)_1^2 + \cdots + (Qg)_r^2, \qquad Q^T Q = I, \qquad \dim O(r, \mathbb{C}) = \binom{r}{2}.$$

Corrected count for $k = 2$

effective parameters $= r\,N_d - \binom{r}{2}$

Ternary sextics

$3 \cdot 10 - 3 = 27 < 28 = N_6$

so three squares of cubics **cannot** fill the ternary sextics.

In fact the closure of sums of three squares is a hypersurface of degree 83200 (Blekherman–Hauenstein–Ottem–Ranestad–Sturmfels).

Sharpness can arrive late

Squares in four variables: $n = 3$, $k = 2$ **(Remark 1 of the paper)**

d	input N_d	target N_{2d}	generic rank
≤ 20			**7**
20	1771	12341	**7**
21	2024	14190	**8**

$7 \cdot 2024 = 14168 < 14190$: the bound $2^3 = 8$ is reached only at $d = 21$.

Similarly for $n = 4$, $k = 2$: 15 squares for $d \leq 75$; all $16 = 2^4$ needed from $d = 76$ on.

The proof

The proof in three steps

1

Differentiate

Can small changes of the summands move every coefficient?

2

Use the dual space

Could a nonzero polynomial vanish to very high order?

3

Count forced factors

A roots-of-unity arrangement makes that impossible.

$$\Phi_r(g_1, \ldots, g_r) = \sum_{i=1}^{r} g_i^k$$

$$(g + \varepsilon h)^k = g^k + \varepsilon\, k\, g^{k-1} h + O(\varepsilon^2)$$

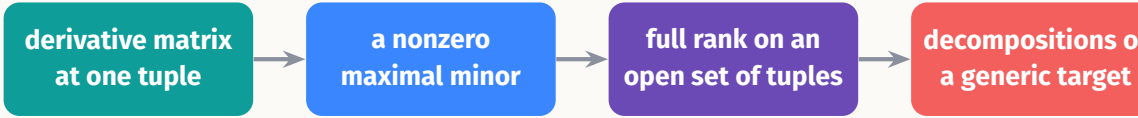
Differential

$$D\Phi_r(h_1, \ldots, h_r) = k \sum_{i=1}^{r} g_i^{k-1} h_i$$

Terracini's lemma, reformulated

A general $f \in S_{kd}$ is a sum of r k-th powers $\iff$ for general $g_i \in S_d$ the ideal $(g_1^{k-1}, \ldots, g_r^{k-1})$ contains all of S_{kd} (it is "kd-regular").

One full-rank derivative is enough



◎ By semicontinuity it suffices to exhibit **one** special tuple $(g_1, \ldots, g_{k^n})$ for which $\sum g_i^{k-1} S_d = S_{kd}$.

A special tuple indexed by roots of unity

ξ is a primitive k-th root of unity,
$a = (a_1, \ldots, a_n) \in \{0, \ldots, k-1\}^n$:

$$\ell_a = x_0 + \xi^{a_1} x_1 + \cdots + \xi^{a_n} x_n, \qquad g_a = \ell_a^d.$$

The spanning statement

$$\sum_a \ell_a^{(k-1)d} S_d = S_{kd}$$

For $k = 2$ the forms $\ell_a = x_0 \pm x_1 \pm \cdots \pm x_n$ have **real** coefficients.

Exactly k^n linear forms $\Rightarrow$ k^n summands.

$$\langle p, F\rangle = p(\partial_y)\, F(y)\big|_{y=0}$$

Apolarity lemma

$F \in S_m^\vee$ is apolar to $\ell^{m-j} S_j \iff$ all derivatives of F of order $\le j$ vanish at the point ℓ.

Reformulation

Spanning fails exactly when there is a nonzero $F \in S_{kd}^\vee$ with

$$\operatorname{mult}_a F \;\ge\; d+1 \qquad \text{for every chosen point } a.$$

Multiplicity means missing Taylor terms

$$F(a+u) = \sum_{\alpha} c_\alpha u^\alpha$$

degree 0	degree 1	...	degree d	degree $\geq d+1$
$= 0$	$= 0$		$= 0$	**possible**

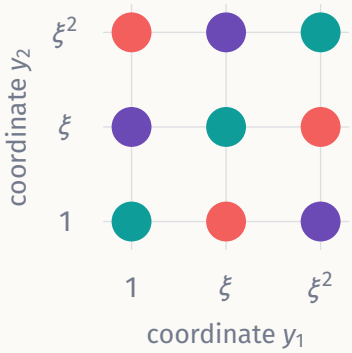
🗜 Restriction to a subspace preserves the vanishing order.

✖ For a product, vanishing orders add.

The roots-of-unity grid

$$Z = \left\{[1 : \xi^{a_1} : \cdots : \xi^{a_n}]\right\}, \qquad |Z| = k^n.$$

Picture: $n = 2$, $k = 3$. Colours mark the hyperplanes $y_1 = \xi^s y_2$; each contains $k^{n-1} = 3$ points.



A stronger vanishing statement

Theorem 5 of the paper

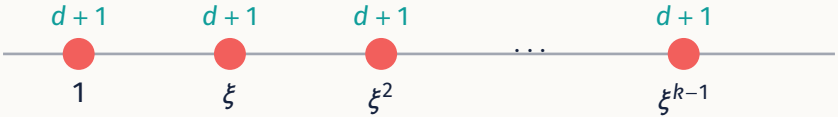
If F is a form in $n+1$ variables with

$$\deg F \le k(d+1)-1 \qquad \text{and} \qquad \operatorname{mult}_a F \ge d+1 \text{ at every point of } Z,$$

then $F \equiv 0$.

The degree kd needed for the main theorem lies inside the range $\le kd+k-1$.

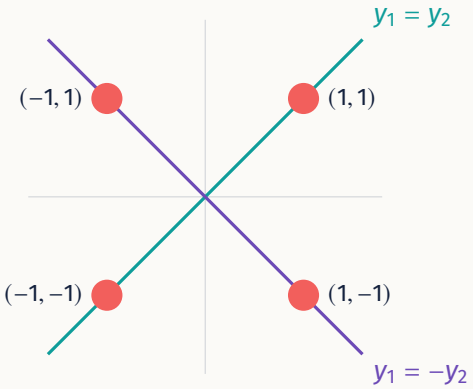
On a line, count roots with multiplicity (case $n = 1$)



$$\prod_{j=0}^{k-1}(t-\xi^j)^{d+1} \,\Big|\, F(t) \qquad \Longrightarrow \qquad F = 0 \;\text{ or }\; \deg F \geq k(d+1).$$

Four points already show the mechanism

$n = 2, k = 2, d = 1$



- Two double zeros on each diagonal, but $\deg(F|_{\text{line}}) \leq 3 < 2 + 2$.
- Hence both diagonals divide F:

$$F = (y_1 - y_2)(y_1 + y_2)\, Q.$$

- Q is linear and vanishes at all four points, so $Q = 0$.

The hyperplanes contain smaller grids

$$H_{ij,s}: \; y_i = \xi^s y_j, \qquad 1 \le i < j \le n, \quad 0 \le s < k.$$

$k\,c$

hyperplanes

k^{n-1}

points on each hyperplane

c

hyperplanes through each point

$c = \binom{n}{2}$. Each $H_{ij,s} \cap Z$ is the **same configuration in one dimension less**.

Restriction forces the whole arrangement

Induction on the dimension

$$F\big|_{H_{ij,s}} = 0 \quad \text{for all } i < j,\ s \qquad \Longrightarrow \qquad \text{every hyperplane equation divides } F.$$

The arrangement polynomial

$$H = \prod_{1 \le i < j \le n} (y_i^k - y_j^k) \quad \text{divides } F, \qquad \deg H = k\,c, \qquad \operatorname{mult}_a H = c.$$

Division gives the same problem at lower degree

$$F = HQ$$

If $d < c$

$\deg F < \deg H$, so divisibility forces $F = 0$.

If $d \geq c$

$\deg Q \leq k(d - c + 1) - 1$, $\operatorname{mult}_a Q \geq (d - c) + 1$.

Repeat with d replaced by $d - c$.

The vanishing lemma completes the theorem



Corollary (of independent interest)

Every form of degree kd is a linear combination of $(x_0+\xi^{a_1}x_1+\cdots+\xi^{a_n}x_n)^{(k-1)d}$ with coefficients of degree d. For $k=2$: of $(x_0 \pm x_1 \pm \cdots \pm x_n)^d$.

Maximal rank, exact generic rank, and recent progress

Every complex form has a uniform bound too

From generic to maximal rank

Let U be the dense open set of forms of rank $\le r_{\mathrm{gen}}$. For any f, two nonempty open sets meet:

$$U \cap (f + U) \neq \varnothing \quad \Longrightarrow \quad f = u - v, \quad u, v \in U.$$

Corollary (cf. Blekherman–Teitler, Math. Ann. 2015)

$$r_{\max}(n, k, d) \;\le\; 2\, r_{\mathrm{gen}}(n, k, d) \;\le\; 2\, k^n$$

Blekherman–Teitler: for any rank w.r.t. a nondegenerate variety, maximal rank $\le 2\times$ generic rank.

Maximal k-rank of binary forms

Conjecture (Shapiro; Lundqvist–Oneto–Reznick–Shapiro 2019)

For every $k \geq 2$, $d \geq 1$, every binary form of degree kd is a sum of at most k k-th powers of forms of degree d:

$$r_{\max}(1, k, d) = k.$$

$d = 1$

Reznick: max Waring rank of binary k-ics is k, attained at $\ell_1 \ell_2^{k-1}$.

$k = 2$

Two squares always suffice ($f = AB$).

$k = 3,\ d = 2$

LORS: every binary sextic is a sum of 3 cubes of quadrics; also $\ell_1 \ell_2^7$ has 4-rank 4.

The conjectural exact generic rank

Conjecture (Ottaviani; stated in Lundqvist–Oneto–Reznick–Shapiro)

Let $N_e = \binom{e+n}{n}$ and $d \geq 2$. Then

$$k \geq 3: \quad r_{\mathrm{gen}}(n,k,d) = \left\lceil \frac{N_{kd}}{N_d} \right\rceil,$$

$$k = 2: \quad r_{\mathrm{gen}}(n,2,d) = \min\left\{ r : rN_d - \binom{r}{2} \geq N_{2d} \right\}.$$

For $d \geq 2$ the varieties of powers are expected to have **no defects** — in sharp contrast with the Alexander–Hirschowitz exceptions for $d = 1$.

Equivalent to **non-defectivity** of all secant varieties of $V^k_{n,d} = \{g^k : g \in S_d\} \subset \mathbb{P}S_{kd}$.

Link with Fröberg's conjecture

Terracini: rank questions ↔ Hilbert functions of power ideals

- $r_{\text{gen}} \le r \iff (g_1^{k-1}, \dots, g_r^{k-1})_{kd} = S_{kd}$ for general $g_i \in S_d$.
- **Fröberg (1985):** predicted Hilbert series of ideals generated by general forms; known in ≤ 3 variables (Fröberg, Anick).
- **Nicklasson (2017):** conjecture that powers g_i^m of general forms behave like general forms of degree md (with explicit exceptions).

Progress

Nenashev (C. R. Math. 2017): Fröberg for many forms of equal degree.

Nicklasson (Comm. Alg. 2017): same, plus new cases.

Boij–Dannetun–Lundqvist (arXiv, May 2026) NEW: Fröberg holds in the second non-trivial degree for forms of degree > 2.

Identifiability: how many decompositions?

Casarotti–Postinghel

For $k \geq 3$: $V^k_{n,d}$ is **not** h**-defective** for

$$h \leq \tfrac{1}{N+1}\binom{N+k-3}{N}, \quad N = \binom{n+d}{d} - 1,$$

via degenerations (Proc. LMS; arXiv 2023); hence general forms of such k-rank have a **unique** decomposition.

Ottaviani–Teixeira Turatti

J. Algebra 2025. Sums of squares $f = \sum f_i^2$: decompositions come in $O(r)$-orbits. Sufficient criteria for a **single orbit**; generic $O(r)$-identifiability of **ternary** forms.

Compare: a general binary form of degree $2d$ has $\binom{2d-1}{d}$ orbits — binary squares are never identifiable for $d \geq 2$.

What is known in October 2026

Case	Result	Reference
Linear summands, $d = 1$	expected rank except quadrics; cubics in 5 vars; quartics in 3,4,5 vars	Alexander–Hirschowitz 1995
Any n, k, d	$r_{\mathrm{gen}} \le k^n$, sharp for $d \gg 0$; $r_{\max} \le 2k^n$	FOS 2012; Blekherman–Teitler
Two variables, every k	$r_{\mathrm{gen}} = \lceil (kd+1)/(d+1) \rceil$ (expected)	LORS 2019
Squares, 3 and 4 variables	exact generic ranks	Oneto 2014
Three variables, $k \ge 3$, $d \ge 2$	complete non-defectivity of $V^k_{2,d}$ **NEW**	Dinh–Hà, arXiv Sept. 2026
$k \ge 3$, small rank	non-defective & identifiable	Casarotti–Postinghel
Binary $r_{\max}$	$= k$ for $d = 1$, $k = 2$, and $(k, d) = (3, 2)$	Reznick; LORS 2019

Still open

The generic-rank conjecture for $n \ge 3$ (and squares for $n \ge 4$); the binary maximal-rank conjecture in general.

Over the reals, the domain matters

Squares in three variables, target degree $2d$

Complex forms

Every form is a sum of squares;

$r_{\mathrm{gen}} \le 4, \qquad r_{\max} \le 8.$

Over $\mathbb{R}$, 2^n is a *typical* rank (FOS, Remark 2).

Real sums of squares

Only forms in the SOS cone; the Pythagoras number satisfies

$$d{+}1 \le \mathrm{py}(3, 2d) \le d{+}2 \quad \text{(Scheiderer)}.$$

Equality $\mathrm{py} = d + 1$ for $2d = 4$ (Hilbert), 6 (Scheiderer), and $8, 10, 12$ (Blekherman–Dunbar–Sinn 2024) NEW.

More degree, the same bound on summands

$$f = g_1^k + \cdots + g_r^k, \qquad r \le k^n$$

generic over $\mathbb{C}$, and sharp once d is large

roots of unity turn the proof into a count of forced factors

Thank you! **Questions?**

Supplementary material

Classical powers of linear forms

supplementary

$d = 1$: $r_{\text{expected}} = \lceil N_k/(n+1) \rceil$ — **the Alexander–Hirschowitz exceptions**

variables	degree	naive ceiling	generic rank
5	3	7	**8**
3	4	5	**6**
4	4	9	**10**
5	4	14	**15**

Quadrics: the generic rank is $n+1$ by matrix rank (not $\lceil (n+2)/2 \rceil$).

Apolarity in one coordinate chart

$$\ell = x_0, \qquad a = [1 : 0 : \cdots : 0], \qquad m = kd.$$

$$F \perp x_0^{m-d} S_d \iff \partial_0^{m-d} F = 0.$$

Allowed monomials

$$y_0^{\alpha_0} y_1^{\alpha_1} \cdots y_n^{\alpha_n} \quad \text{with} \quad \alpha_0 \le m - d - 1, \quad \alpha_1 + \cdots + \alpha_n \ge d + 1$$

i.e. in the affine chart $y_0 = 1$ every Taylor term at a has degree $\ge d + 1$.

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