One variation on a theme of Kuijlaars-Van Assche

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with contributions of A. Kuijlaars, M. Duits, F. Štampach, and M. Tater

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Basic Result

In [1] it was proved that given a three-term recurrence relation with varying coefficients of the form:

$$P_{i,n}(z) = (z - a_{i,n})P_{i-1,n}(z) + b_{i,n}P_{i-2,n}(z),$$

with real $a_{i,n}$ and positive $b_{i,n}$, $n = 0, 1, 2, \ldots,\ 0 \leq i \leq n$, $\deg P_{i,n}(z) = i$, such that $\lim_{i/n \to \tau} a_{i,n} = f(\tau),\ \tau \in [0,1]$ and $\lim_{i/n \to \tau} b_{i,n} = g(\tau),\ \tau \in [0,1]$ for some reasonably behaved functions $f(\tau)$ and $g(\tau)$, one can obtain the density of the asymptotic root distribution for the diagonal polynomial sequence $\{P_{n,n}(z)\}$ by averaging the (standard) densities of the family of the corresponding recurrences with constant coefficients:

$$Q_{i,\tau}(z) = (z - f(\tau))Q_{i-1,\tau}(z) + g(\tau)Q_{i-2,\tau}(z),$$

over $\tau \in [0,1]$.

A. B. J. Kuijlaars, W. Van Assche, *The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory **99** (1999), 167–197.

Complex generalization

Theorem (A. Kuijlaars - W. Van Assche - F. Stampach)

If there exist two continuous functions $f(\tau)$ and $g(\tau)$, $\tau \in [0,1]$, such that

$$\lim_{i/(n+1)\to\tau} a_{i,n} = f(\tau), \qquad \lim_{i/(n+1)\to\tau} b_{i,n} = g(\tau), \qquad \forall\tau \in [0,1],$$

then the asymptotic root-counting measure μ (if it exists) of the polynomial sequence $\{P_{n,n}(\lambda)\}$ and the average M of the arcsine measures given by

$$M = \int_0^1 \omega_{\left[a(\tau)-2\sqrt{b(\tau)},a(\tau)+2\sqrt{b(\tau)}\right]} d\tau,$$

have the same logarithmic potential/Cauchy transform outside the union of their supports. Here ω is the standard arcsine density.

Main problem

Assume that $f(\tau)$ and $g(\tau)$ are analytic functions with no singularities on $[0, 1]$. Consider the sequence of polynomials

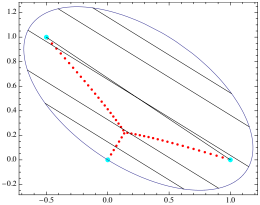
$$P_{i,n}(z) = (z - f\left(\frac{i}{n}\right))P_{i-1,n}(z) + g\left(\frac{i}{n}\right)P_{i-2,n}(z),$$

with $P_{-1,n}(z) = 0, \ P_{0,n} = 1.$

Describe the limiting root counting measure of the sequence of $\{P_{n,n}(z)\}$.

Example 1

$f(\tau) = -v(1-\tau)^2$; $g(\tau) = -w(1-(1-\tau)^2)(1-\tau^2)$, where v and w are arbitrary complex numbers.



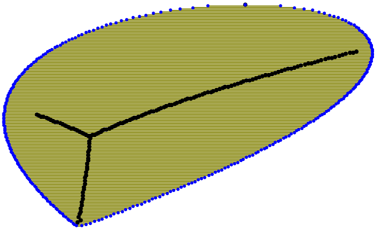
Theorem. Denote the three roots of $Q(z) = z^3 + vz^2 + wz$ by a_1, a_2, a_3. For $i \in \{1,2,3\}$ consider the curve γ_i given as the set of all $b \in \mathbb{C}$ satisfying the relation:

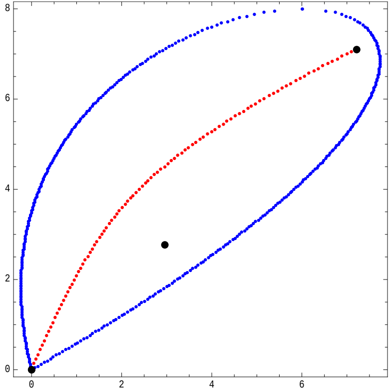
$$\gamma_i : \quad \int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}}\,dt \in \mathbb{R}, \tag{1}$$

here j and k are the remaining two indices in $\{1, 2, 3\}$ in any order and the integration is taken over the straight interval connecting a_j and a_k. One can see that a_i belongs to γ_i and show that these three curves connect all a_i's with the unique common point b_0 lying within Δ_Q. Take the segment of γ_i connecting a_i with the common intersection point b_0 and denote this segment by Γ_i. Finally, denote the union of these three segments by Γ_Q.

Example 2

$f(\tau) = 4b\tau$; $g(\tau) = -16\tau^2(1-\tau)$, where b is arbitrary complex numbers.



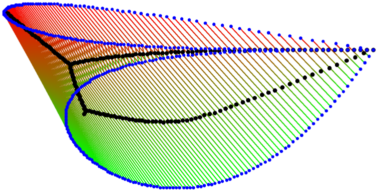


In our case the corresponding quadratic differential is

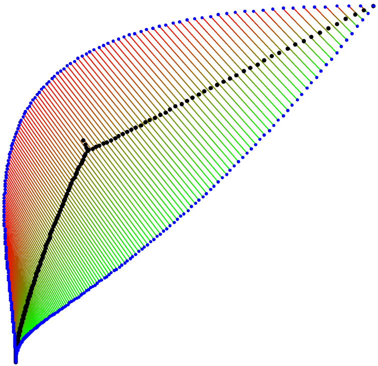
$$\Psi_{b,\Lambda} = -\frac{\Theta(\Theta + b)^2 - 4\Theta - \Lambda}{\Theta} d\Theta^2. \tag{2}$$

Conjecture. The support of the limiting root couting measure consists of all values of Λ for which the above quadratic differential has two short horizontal trajectories connecting two of zeros and the remaining zero with the pole.

More Examples



Even More Examples



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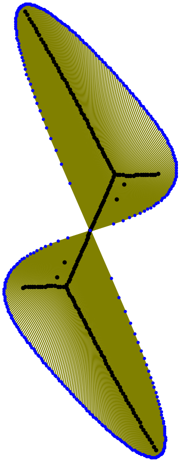

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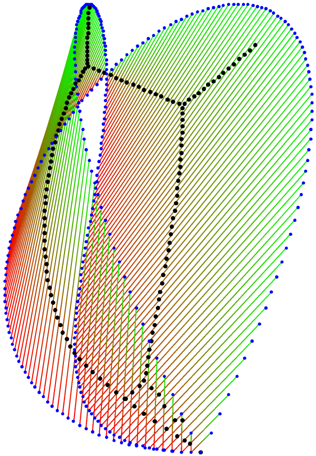

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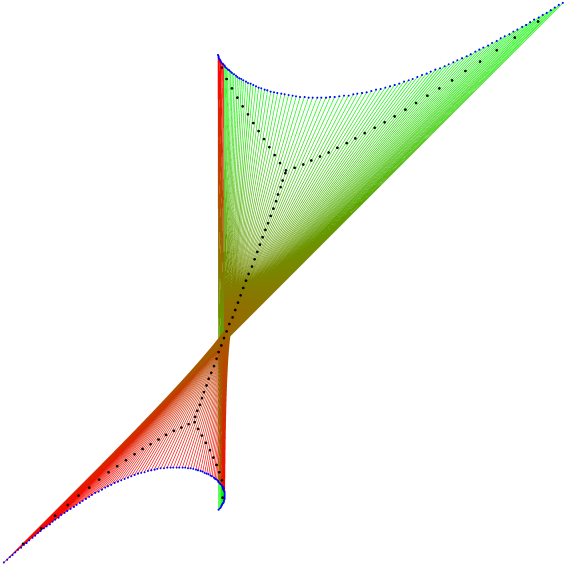

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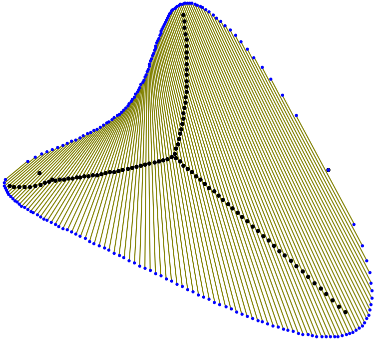

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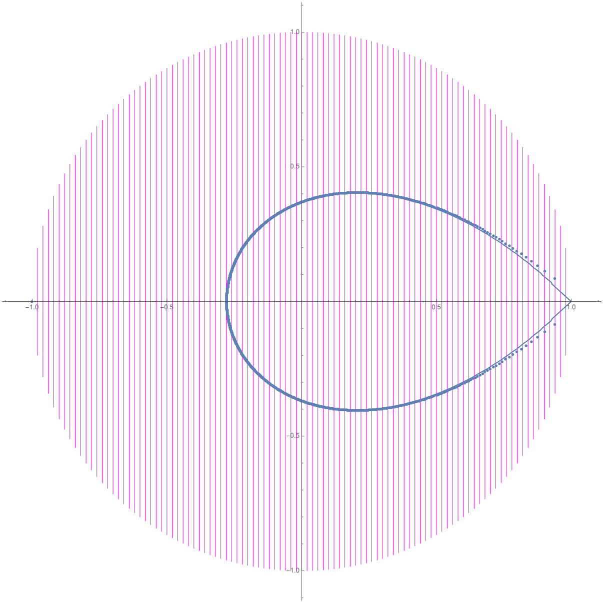

THANK YOU FOR YOUR ATTENTION!!!