One variation on a theme of Kuijlaars-Van Assche

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Basic Result

In [1] it was proved that given a three-term recurrence relation with varying coefficients of the form:

$$P_{i,n}(z) = (z - a_{i,n})P_{i-1,n}(z) + b_{i,n}P_{i-2,n}(z),$$

with real $a_{i,n}$ and positive $b_{i,n}$, $n = 0, 1, 2, \ldots, 0 \leq i \leq n$, $\deg P_{i,n}(z) = i$, such that $\lim_{i/n \to \tau} a_{i,n} = f(\tau),\ \tau \in [0,1]$ and $\lim_{i/n \to \tau} b_{i,n} = g(\tau),\ \tau \in [0,1]$ for some reasonably behaved functions $f(\tau)$ and $g(\tau)$, one can obtain the density of the asymptotic root distribution for the diagonal polynomial sequence $\{P_{n,n}(z)\}$ by averaging the (standard) densities of the family of the corresponding recurrences with constant coefficients:

$$Q_{i,\tau}(z) = (z - f(\tau))Q_{i-1,\tau}(z) + g(\tau)Q_{i-2,\tau}(z),$$

over $\tau \in [0,1]$.

A. B. J. Kuijlaars, W. Van Assche, *The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory **99** (1999), 167–197.

First generalization

Theorem (A. Kuijlaars - W. Van Assche)

If there exist two continuous functions $f(\tau)$ and $g(\tau)$, $\tau \in [0,1]$, such that

$$\lim_{i/(n+1)\to\tau} a_{i,n} = f(\tau), \qquad \lim_{i/(n+1)\to\tau} b_{i,n} = g(\tau), \qquad \forall\tau \in [0,1],$$

then the asymptotic root-counting measure μ (if it exists) of the polynomial sequence $\{P_{n,n}(\lambda)\}$ and the average M of the arcsine measures given by

$$M = \int_0^1 \omega_{\left[a(\tau)-2\sqrt{b(\tau)},a(\tau)+2\sqrt{b(\tau)}\right]} d\tau,$$

have the same logarithmic potential/Cauchy transform outside the union of their supports. Here ω is the standard arcsine density.

Main problem

Assume that $f(\tau)$ and $g(\tau)$ are complex-valued polynomials with $g(\tau)$ vanishing at 0 and 1. Consider the sequence of polynomials

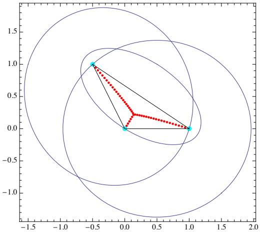
$$P_{i,n}(z) = (z - f\left(\frac{i}{n}\right))P_{i-1,n}(z) + g\left(\frac{i}{n}\right)P_{i-2,n}(z),$$

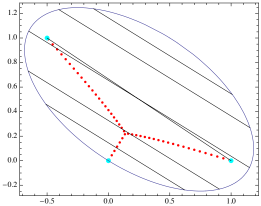
with $P_{-1,n}(z) = 0, \; P_{0,n} = 1.$

describe the limiting root counting measure of the sequence of $\{P_{n,n}(z)\}$.

Example 1

$f(\tau) = -v(1-\tau)^2$; $g(\tau) = -w(1-(1-\tau)^2)(1-\tau^2)$, where v and w are arbitrary complex numbers.





Denote the three roots of $Q(z) = z^3 + vz^2 + wz$ by a_1, a_2, a_3. For $i \in \{1, 2, 3\}$ consider the curve γ_i given as the set of all $b \in \mathbb{C}$ satisfying the relation:

$$\gamma_i : \quad \int_{a_j}^{a_k} \sqrt{\frac{b-t}{(t-a_1)(t-a_2)(t-a_3)}}\,dt \in \mathbb{R}, \qquad (1)$$

here j and k are the remaining two indices in $\{1, 2, 3\}$ in any order and the integration is taken over the straight interval connecting a_j and a_k. One can see that a_i belongs to γ_i and show that these three curves connect all a_i's with the unique common point b_0 lying within Δ_Q. Take the segment of γ_i connecting a_i with the common intersection point b_0 and denote this segment by Γ_i. Finally, denote the union of these three segments by Γ_Q.

Example 2

$f(\tau) = 4b\tau$; $g(\tau) = -16\tau^2(1-\tau)$, where b is arbitrary complex numbers.

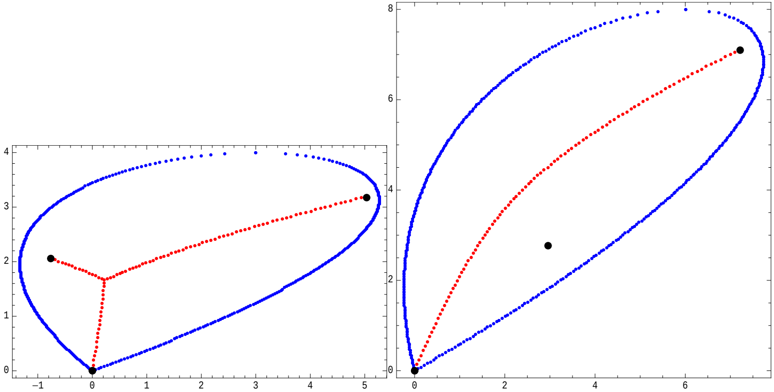


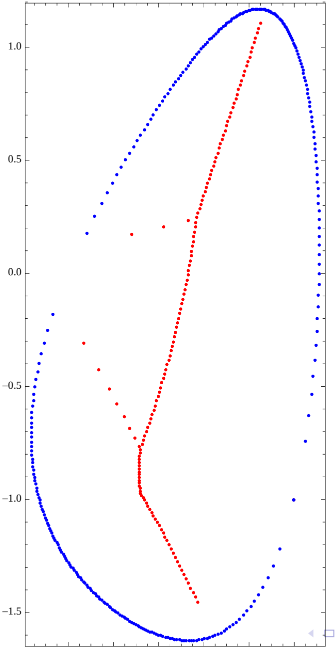
Figure : The spectra for $\widetilde{M}_{100}(b)$ for $b = (3/4 + I)$ (left) and $b = 3/2 + 2I$ (right) with the corresponding ovals and foci.

In our case the corresponding quadratic differential is

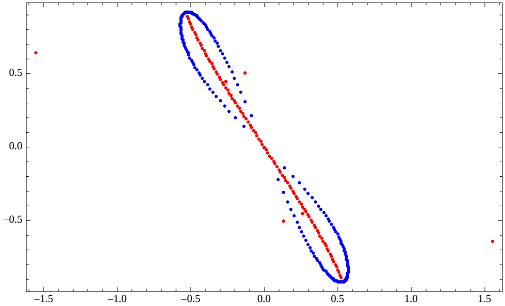
$$\Psi_{b,\Lambda} = -\frac{\Theta(\Theta + b)^2 - 4\Theta - \Lambda}{\Theta} d\Theta^2. \tag{2}$$

Conjecture. The support of the limiting root couting measure consists of all values of Λ for which the above quadratic differential has two short horizontal trajectories connecting two of zeros and the remaining zero with the pole.

More Examples



Even More Examples



THANK YOU FOR YOUR ATTENTION!!!