Around Bochner-Krall problem

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To Salomon Bochner, one of my mathematical heroes

In the beginning ...

In 1929 S. Bochner published a short paper

Über Sturm-Liouvillesche Polynomsysteme, Math. Z. 29 (1929)

related to orthogonal polynomials and Sturm-Liouville problem.[1] Although after that he left this area for good, the importance of his contribution to the area of orthogonal polynomials is difficult to overestimate and at the moment it has been cited 344 times.

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[1] Salomon Bochner made substantial contributions to harmonic analysis, probability theory, differential geometry as well as history of mathematics. Several notions and results such as the Bochner integral, Bochner theorem on Fourier transforms, Bochner-Riesz means, Bochner-Martinelli formula bear at present his name. For more details on S. Bochner see
www-groups.dcs.stand.ac.uk/ history/Biographies/Bochner.html
He belonged to a sizeable group of European mathematicians of Jewish origin who moved to US before or during the WWII and contributed to an enormous development of mathematics in their new motherland.

Portrait

Frontispiece of Part 1, R. C. Gunning, ed. *Collected Papers of Salomon Bochner*, copyright 1992.
Courtesy of the American Mathematical Society.

Figure : Salomon Bochner in person

Would S. Bochner have a reasonable chance to get a European Research Council grant nowadays?

He published 6 books on various subjects including history of mathematics and about 140 research papers.

Proceedings of National Academy, USA - 32 publications;

Annals of Mathematics - 46 publications;

Acta Mathematica - 3 publications;

Duke Math. Journal - 4 publications.

Unfortunately he left no personal memoirs.

Highly recommended reading of that kind !!!

W. Rudin, The way I remember it, (History of Mathematics, V. 12). AMS, 1996

Main problem

The following classification problem was stated by S. Bochner for order $N = 2$, and H. L. Krall for general order.

Problem

Classify all linear differential operators with real polynomial coefficients of the form:

$$T = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}, \tag{1}$$

such that a) $\deg Q_i(z) \leq i$*; b) there* $\exists$ *a positive integer* $i_0 \leq k$ *with* $\deg Q_{i_0}(z) = i_0$*, satisfying the condition that the set of polynomial solutions* f *of the formal spectral problem*

$$Tf(z) = \lambda f(z), \quad \lambda \in \mathbb{R},$$

form a sequence of polynomials orthogonal with respect to some real bilinear form.

Observations

Additionally, if a real bilinear form comes from a positive measure supported on $\mathbb{R}$, we say that we consider a positive Bochner-Krall problem. Following the terminology used in physics, we call linear differential operators given by (1) *exactly solvable* since all eigenvalues can be found explicitly.

Lemma

Any exactly solvable operator has a unique eigenpolynomial of any sufficiently large degree which makes Problem 1 well-posed.

Let us denote by $\{p_n^T(z)\}$ the sequence of eigenpolynomials of an exactly solvable operator T. (Here $\deg p_n^T(z) = n$ and n runs from some positive integer to $+\infty$.)

An exactly solvable operator which solves Problem 1 will be called a *Bochner-Krall operator*.

Literature

A short and informative summary of work on Problem 1 prior to 1978 can be found in

T. S. Chihara, An Introduction to Orthogonal Polynomials, p.150.

In Problem 1, one may equivalently seek a sequence of moments $\{\mu_j\}_{j=0}^{\infty}$ which permits construction of the orthogonal sequence of polynomials

A. M. Krall, Hilbert Space, Boundary Value Problems, and Orthogonal Polynomials, p. 223.

Let $\langle \cdot, \cdot \rangle$ be a candidate bilinear form, and define a weight on the set of polynomials with respect to the inner product by $\langle w(z), z^n \rangle = \mu_n$ (note such a weight is not necessarily positive or unique). One can show that solutions to Problem 1 exist only if the product wT is equal to its formal adjoint when acting on polynomials, and this immediately implies that the order of T must be even see loc.cit p. 228.

Initial results

Eleven years after Bochner stated and solved Problem 1 for order two differential operators, H. L. Krall settled the order four case in

"On orthogonal polynomials satisfying a certain fourth order differential equation," 1940.

The order two classification has four families, corresponding to the classical Hermite, Laguerre, Jacobi, and Bessel polynomials.

The order four classification has seven families: the four classical families corresponding to iterated order two operators, and three new families (Legendre-type, Laguerre-type and Jacobi-type) which are eigenfunctions of differential operators that do not factor into squares.

W. Hahn showed that the four classical families are the only orthogonal polynomial sequences $\{P_n(z)\}_{n=0}^{\infty}$ for which $\{\frac{d}{dz}P_n(z)\}_{n=0}^{\infty}$ is also an orthogonal polynomial sequence.

An analog of this was proved for the order 4 case by K. H. Kwon, L. L. Littlejohn, J. K. Lee, and B. H. Yoo.

More stuff

An assortment of families corresponding to order six operators have been found in the ensuing decades (see Chapter XVI of the book of A. M. Krall).

All known BKOPS have distributional weights of the form $w = u + v$, where u is a classical orthogonal weight and v consists of some point masses supported on the boundary of the support of u.

It is conjectured that this is true for all BKOPS.

It is also conjectured that the leading coefficient of a BKOPS is a power of either linear or quadratic polynomial.

The most general form of Problem 1 is still widely open for operators of order six or more, but Kwon and Lee have found a satisfactory solution if the polynomials are required to be orthogonal with respect to a compactly supported measure with continuous density.

Theorem (Kwon-Lee)

The only BKOPS with compactly supported positive measure on $\mathbb{R}$ *as the Jacobi-type polynomials, i.e. after an appropriate linear real change of variables they are orthogonal w.r.t*

$$w = (1-x)^{\alpha}(1+x)^{\beta}H(1-x^2) + c\delta(x-1) + d\delta(x+1),$$

where $\alpha, \beta > -1$ *and* $c, d \geq 0$*, and* $H(x)$ *is the Heaviside step function.*

(A weaker result in the same direction was somewhat earlier obtained by H. Rullgård, T. Bergkvist and B.Sh.)

In general the Bochner-Krall problem seems to be quite difficult.

”NO GO” SITUATION!

Where to go from ”no go”?

Observe that orthogonal polynomials with respect to positive measure have real and interlacing roots and special type of assymptotic distributions of its roots.

Some suggestions ...

Illuminating example

Take the operator $T = z(z-1)(z+\frac{1}{2}-I)\frac{d^3}{dz^3}$. What can you say about the roots of its eigenpolynomials?

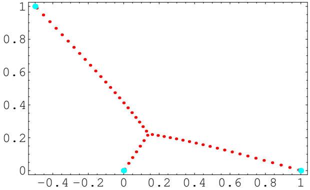


Figure : Root-distribution for $p_{50}(z)$

Lemma

For any operator of the form $T = Q_k(z)\frac{d^k}{dz^k}$ the roots of its eigenpolynomials lie in the convex hull of roots of $Q_k(z)$. In particular, if $Q_k(z)$ has all real roots then every eigenpolynomial will have all real roots.

Problem 1. Given an arbitrary exactly solvable operator, when are all its eigenpolynomials real-rooted.

Apparently this problem is closely related to the notion of hyperbolicity-preserving operators, i.e. $T : \mathbb{R}[x] \to \mathbb{R}[x]$ sending any real-rooted polynomial to a real-rooted polynomial (or 0). These go back to the 1914 paper of Pólya and Schur and were recently completely characterized in

Borcea, J., Brändén, P.: Pólya–Schur master theorems for circular domains and their boundaries. Ann. Math. (2) 170(1), 465–492 (2009)

Theorem (Special case of a result of P. Bränden, 2010)

If T is hyperbolicity-preserving then all its eigenpolynomials are real-rooted.

This result allows to construct an abundance of examples ! I can explain more...

But the converse is not true.

For example,

$$T = z(z+1)\frac{d^2}{dz^2} + 1$$

and

$$T = z(z+1)(1+5z+6z^2)\frac{d^3}{dz^3} + 2z(z+1)(12z+5)\frac{d^2}{dz^2}$$

are not hyperbolicity preserving but have all real-rooted eigenpolynomials. But they are interval hyperbolicity-preserving property, i.e. they send polynomial with all real roots on a certain interval to polynomials with all roots on the same interval.

Suspicion. Any exactly solvable operator with all eigenpolynomials having only real zeros is either hyperbolicity preserving or interval hyperbolicity-preserving.

In any case the problem seems to be tame! Note that the latter class of linear operators does not have a satisfactory description at the moment...

Next step

If we already know that all roots of all eigenpolynomials are real, can we say something about the asymptotic root density (after appropriate scaling)?

I have a rather explicit answer for any exactly solvable operator with all real-rooted eigenpolynomials (to be explained later), but what to compare it with?

Given a sequence $\{p_n(x)\}$ of orthogonal polynomials with respect to a positive measure, shift and scale each $p_n(x)$ so that its rightmost zero is at -1 and the leftmost zero is at 1.

Problem 2. What can one say about the limiting root-counting measure of these scaled polynomials?

(Reminder. The root-counting measure μ_P of a polynomial $P(z) = \prod_{i=1}^{n}(z - z_i)$ is defined as $\mu_P := \frac{1}{n}\sum_{i=1}^{n} \delta(z - z_i)$ where $\delta(z - z_i)$ is the unit mass at z_i.)

Is this problem tame? Maybe it has a reasonable answer for Bochner-Krall polynomials?

This is a challenge for specialists in orthogonal polynomials to whom I do not belong.

Problem 3. Given an arbitrary exactly solvable operator T, find the asymptotic root-counting measure of the sequence (of appropriately scaled) polynomials $\{p_n^T\}$.

Observe that exactly solvable operators split into two large classes, *non-degenerate* and *degenerate*.

Definition. An exactly solvable operator T as above is called *non-degenerate* if $\deg Q_k = k$ and *degenerate*, otherwise.

The main distinction between these two cases is that for any non-degenerate operator T all roots of all its eigenpolynomials $p_n^T(z)$ are contained in some disk, while for any degenerate operator T the union of all roots of its eigenpolynomials is necessarily unbounded, which means that the scaling is need.

Some basic examples

$$T_1 = z(z-1)(z-I)\frac{d^3}{dz^3}$$

$$T_2 = (z-I)(z+I)(z-2+3I)(z-3-2I)\frac{d^4}{dz^4}$$

$$T_3 = (z-I)(z+I)(z-2+3I)(z-3-2I)(z+3)\frac{d^5}{dz^5}$$

$$T_4 = (z^2+1)(z-2+3I)(z-3-2I)(z+3)(z+1+I)\frac{d^6}{dz^6}$$

Roots of eigenpolynomials

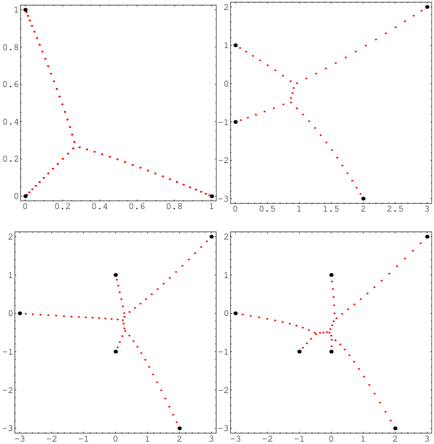


Figure : Roots of $p_{55}(z)$ for the above T's.

Main conjecture on the rate of growth of roots

For a degenerate exactly solvable operator (1), denote by j^T the largest i for which $\deg Q_i = i$. (Since T is a degenerate exactly solvable operator, then $1 \leq j^T < k$.).

Conjecture [T. Bergkvist, 2007]

For any degenerate T, there exists a positive constant $c^T > 0$ such that

$$\lim_{n\to\infty} \frac{r_n^T}{n^{d_T}} = c^T, \tag{2}$$

where

$$d_T := \max_{i\in[j^T+1,k]} \left(\frac{i - j^T}{i - \deg Q_i} \right).$$

Remark. Observe that $0 < d < +\infty$. Some cases clasically known, see Szegö's book. Some new cases settled. There is an explicit conjecture on the value of c^T as well.

(In the non-denegerate case we can set $d_T = 0$.)

The geometric meaning of the power d_T in terms of the Newton polygon of T. The symbol of $T = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$ is a bivariate polynomial $S_T(\mathcal{C}, z) = \sum_{i=1}^{k} Q_i(z)\mathcal{C}^i$.

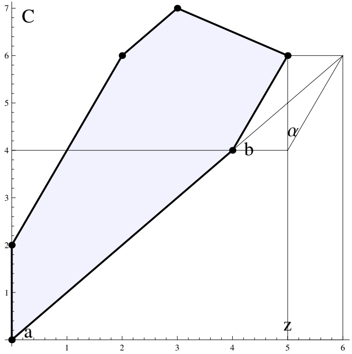


Figure : Newton polygon of a degenerate exactly solvable operator of order 7 and the corresponding d_T.

$$d_T = \cot \alpha.$$

How to get the main conjecture and its consequences

Consider the *scaled eigenpolynomial* $q_n(z) = p_n(n^d z)$, where d is some non-negative number and $p_n(z)$ is the unique and monic nth degree eigenpolynomial of T of suffiiciently large degree n.

The Cauchy tranform $\mathcal{C}_\mu$ of a (complex-valued) measure μ in the complex plane is given by

$$\mathcal{C}_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi}.$$

In particular, the Cauchy transform $\mathcal{C}_P(z)$ of (the root-counting measure of) $P = \prod_{i=1}^{n}(z - z_i)$ with distinct roots equals

$$\mathcal{C}_P(z) = \frac{P'(z)}{nP(z)}.$$

The goal is now to obtain a **well-defined algebraic equation** for the Cauchy transform of the root measure μ_n of the scaled eigenpolynomial q_n when $n \to \infty$, and as we will see that we are forced to choose $d = d_T$ as in Main Conjecture.

Basic assumptions

Such procedure was used earlier by a number of authors.

When performing our calculations we assume that the root-counting measures $\mu_n^{(0)}, \mu_n^{(1)}, \mu_n^{(2)} \dots, \mu_n^{(k-1)}$ of the *scaled eigenpolynomial* $q_n(z)$ and its derivatives up to the kth order exist when $n \to \infty$ and that they are are all weakly convergent to the *same* asymptotic root measure μ.

Thus the corresponding Cauchy transforms are all asymptotically identical, and we define $C(z) := \lim_{n\to\infty} C_{n,j}(z)$ for all $j \in [0, k-1]$, where $C(z)$ is the Cauchy transform of μ and is considered for z's away from the support of μ. Computer experiments strongly indicate that this assumption is true.

Basic derivation

We will use the normalization assumption on (degenerate) exactly solvably T that the polynomial coefficient $Q_{j_T}(z)$ in T is monic. We denote $Q_j(z) = \sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i$. In the above notation consider the eigenvalue equation

$$T p_n(z) = \lambda_n p_n(z),$$

where the eigenvalue λ_n is given by

$$\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_T} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_T} \alpha_{j,j} n(n-1)\cdots(n-j+1).$$

Then our eigenvalue problem is equivalent to

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i \right) p_n^{(j)}(z) = \sum_{j=1}^{j_T} \alpha_{j,j} n(n-1)\cdots(n-j+1) p_n(z).$$

cont...

Substituting $z = n^d z$ in this equation we obtain

$$\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}n^{di}z^i\right)p_n^{(j)}(n^d z) = \sum_{j=1}^{j_T}\alpha_{j,j}n(n-1)\cdots(n-j+1)p_n(n^d z),$$

and with $q_n(z) = p_n(n^d z)$ we get

$$\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\right)q_n^{(j)}(z) = \sum_{j=1}^{j_T}\alpha_{j,j}n(n-1)\cdots(n-j+1)q_n(z).$$

Dividing by $\frac{n!}{(n-j_T)!}q_n(z) = n(n-1)\cdots(n-j_T+1)q_n(z)$

we get

$$\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\right)\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_T+1)q_n(z)} =$$

$$= \sum_{j=1}^{j_T}\alpha_{j,j}\frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_T+1)}. \qquad (3)$$

cont...

Consider the right-hand side of (3). Since $j \leq j_T$, all terms for which $j < j_T$ (if not already zero, which is the case if $\alpha_{j,j} = 0$) tend to zero when $n \to \infty$, and therefore the limit of the right-hand side of (3) equals

$$\lim_{n\to\infty} \sum_{j=1}^{j_T} \alpha_{j,j} \frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_T+1)} = \alpha_{j_T,j_T} = 1,$$

since we assumed that Q_{j_T} is monic. Consider the jth term in the sum on the left-hand side of (3). It equals

$$\begin{aligned}
&\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_T+1)q_n(z)} = \\
= \quad &\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)} \cdot \frac{n\cdots(n-j+1)}{n\cdots(n-j_T+1)} \\
= \quad &\sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}} \cdot \prod_{i=0}^{j-1} C_{n,i}(z) \cdot \frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_T+1)}
\end{aligned}$$

cont...

Taking the limit and using the basic assumption we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\cdot\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_T+1)q_n(z)}\\
=\quad&\lim_{n\to\infty}\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_T-j}}C^j(z)
\end{aligned}$$

for the jth term and thus, taking the limit of the left-hand side of (3) we get

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\right)\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_T+1)q_n(z)}\\
=\quad&\lim_{n\to\infty}\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_T-j}}\right)C^j(z).
\end{aligned}$$

cont...

Adding up, the following equation is satisfied by $C(z)$ for z's away from the support of μ:

$$\lim_{n\to\infty}\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_T-j}}\right)C^j(z)=1. \tag{4}$$

In order to make (4) a well-defined algebraic equation, i.e. to avoid infinities in the denominator when $n\to\infty$, we must impose the condition

$$d(j-i)+j_T-j\geq 0 \quad\Leftrightarrow\quad d\geq\frac{j-j_T}{j-i}$$

on the real number d in the exponent of n, for all $j\in[1,k]$ and all $i\in[0,\deg Q_j]$. Our condition then becomes

$$d=\max_{j\in[1,k]}\Big(\frac{j-j_T}{j-\deg Q_j}\Big).$$

cont...

Finally we observe that since T is degenerate we have $j_T < k$ and thus we need only take this maximum over $j \in [j_T + 1, k]$, since there always exists a positive value on d for any operator of the type we consider. Thus our condition becomes:

$$d = \max_{j \in [j_T+1, k]} \left(\frac{j - j_T}{j - \deg Q_j} \right).$$

In the above notation, we obtain the following well-defined algebraic equation for the conjectural limiting Cauchy transform.

Conjectural corollary. *The Cauchy transform $C(z)$ of the asymptotic root measure μ of the scaled eigenpolynomial $q_n(z) = p_n(n^{d_T} z)$ of an arbitrary exactly-solvable operator T as above satisfies the following algebraic equation for almost all complex z in the usual Lebesgue measure on $\mathbb{C}$:*

$$z^{j_T} C^{j_T}(z) + \sum_{j \in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where A is the set consisting of all j for which the maximum $d_T := \max_{j\in[j_T+1,k]} \left(\frac{j-j_T}{j-\deg Q_j}\right)$ is attained, i.e.

$$A = \{j : (j - j_T)/(j - \deg Q_j) = d_T\}.$$

Which monomials of T are essential for the limiting Cauchy transform?

Only monomials on one side of the Newton polygon are essential for the limiting root-counting measure! See numerical evidence on the next slide.

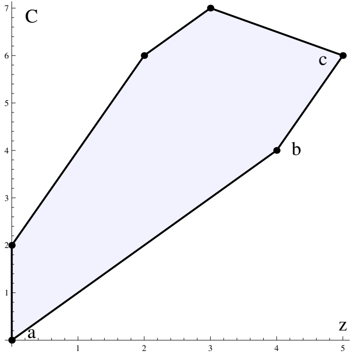
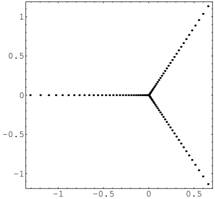
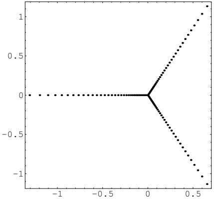


Figure : Newton polygon of a degenerate exactly solvable operator of order 7 and the corresponding d_T.

Numerical evidence clearly illustrates that distinct operators whose scaled eigenpolynomials satisfy the same (conjectural so far) Cauchy transform equation when $n \to \infty$, will yield identical asymptotic zero distributions.



T_4, roots of
$q_{100}(z) = p_{100}(100^{2/3}z)$



$\tilde{T}_4$, roots of
$q_{100}(z) = p_{100}(100^{2/3}z)$

where $T_4 = z^3\frac{d^3}{dz^3} + z^2\frac{d^5}{dz^5}$ and $\widetilde{T}_4 = z^2\frac{d^2}{dz^2} + z^3\frac{d^3}{dz^3} + z\frac{d^4}{dz^4} + z^2\frac{d^5}{dz^5} + \frac{d^6}{dz^6}$. Here $d_{T_4} = d_{\widetilde{T}_4} = 2/3$.

T_4 and $\widetilde{T}_4$ have the same conjectural equation for $\mathcal{C}_\mu$:

$$z^2\mathcal{C}_\mu^5 + z^3\mathcal{C}_\mu^3 = 1.$$

Main results in the non-degenerate case

Assume with loss of generality that $Q_k(z)$ is monic.

Theorem (H. Rullgård and T. Bergkvist, 2004)

For any exactly-solvable non-degenerate operator T, the sequence $\{\mu_n^T\}$ of root-counting measures for $\{p_n^T(z)\}$ converges in the weak sense to a probability measure μ_T whose Cauchy transform $\mathcal{C}_T(z)$ satisfies a.e. in $\mathbb{C}$ the algebraic equation:

$$\mathcal{C}_T^k(z)Q_k(z) = 1 \quad \Leftrightarrow \quad \mathcal{C}_T(z) = \frac{1}{\sqrt[k]{Q_k(z)}}.$$

Theorem (H. Rullgård and T. Bergkvist, 2004)

For any monic polynomial $Q_k(z)$ of degree k, there exists a unique probability measure with Cauchy transform satisfying the latter equation a.e. in $\mathbb{C}$. Additionally its support is a curvilinear tree lying in the convex hull of the roots of $Q_k(z)$ and the endpoints of this support are exactly all roots of $Q_k(z)$.

Illustration

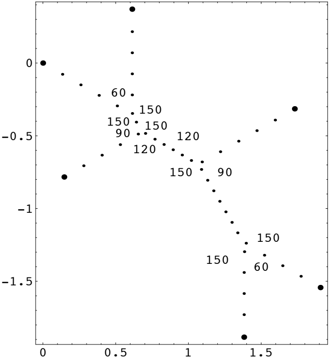


Figure : Small dots show the roots of the eigenpolynomial of degree 50 for the operator $T = Q_6(z)\frac{d^6}{dz^6}$ where $Q_6(z)$ is a sextic polynomial whose roots are the larger dots. (Numbers on the picture are the angles between the respective edges.)

Theorem. [H. Rullgård, 2004] In the above notation

1) supp μ_Q is a curvilinear tree which is straightened out by the analytic mapping

$$\xi(z) = \int_a^z \frac{dz}{\sqrt[k]{Q_k(z)}}.$$

2) supp μ_Q contains all the zeros of $Q_k(z)$ and is contained in the convex hull of those.

3) There is a natural formula for the angles between the branches and the masses of the branches satisfy Kirchhoff law.

Remark for physicists. In WKB-methods in physics one often uses $\int \frac{dz}{\sqrt{Q(z)}}$ as a new variable for Schrödinger-type equations. For equations of order k naturally appears the intergal of the k-th root.

Illustration

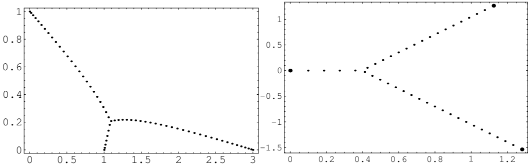


Figure : The measure μ_T before and after the transformation.

Here $T = Q(z)\frac{d^3}{dz^3}$ where $Q(z) = (z-1)(z-3)(z-I)$.

More info

Detailed information about the support of the asymptotic root-counting measure μ_T of a non-degenerate exactly solvable operator T provides necessary conditions for T to be a Bochner-Krall operator.

If we additionally assume that the positive measure belongs to the so-called Nevai class then it is known that the limiting root-counting measure of any sequence of orthogonal polynomials in the Nevai class has the arcsine distribution on some finite interval $[a, b]$.

The latter fact implies that up to a constant the leading coefficient of the corresponding positive BK-operator must be of the form $(x-a)^l(x-b)^l$, where the order k of the operator equals $2l$, see

T. Bergkvist, H. Rullgård and B. Shapiro, On Bochner-Krall orthogonal polynomial systems, Math. Scand. vol 94, issue 1 (2004) 148–154.

Degenerate case

Main Conjecture. Given an arbitrary degenerate operator T as in (1), the Cauchy transform $\mathcal{C}_T(z)$ of the asymptotic root counting measure μ_T for the sequence $\{q_n(z)\}$ of the scaled eigenpolynomials $q_n(z) = p_n(n^{d_T} z)$ satisfies a.e. in $\mathbb{C}$ the algebraic equation

$$z^{j^T}\mathcal{C}_T^{j^T}(z) + \sum_{i\in A_T} \alpha_{i,\deg Q_i} z^{\deg Q_i}\mathcal{C}_T^{i}(z) = 1, \tag{5}$$

where A_T is the set consisting of all i for which the maximum $d := \max_{i\in[j^T+1,k]}\left(\frac{i-j^T}{i-\deg Q_i}\right)$ is attained, i.e. $A_T = \{i : (i-j^T)/(i-\deg Q_i) = d_T\}$. (Observe that $A_T \neq \emptyset$.)

Main challenge in the degenerate case

Problem 4. [Seems doable] Assuming that Bergkvist's conjecture is settled, deduce the information about the support and the density of the measure from the above algebraic equation.

In particular,

Conjecture. The support of the measure satisfying a.e. the algebriac equation of T.Bergkvist is always a curvilinear tree rooted at the origin.

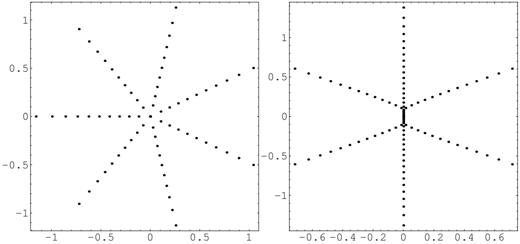


Figure : Degenerate case.

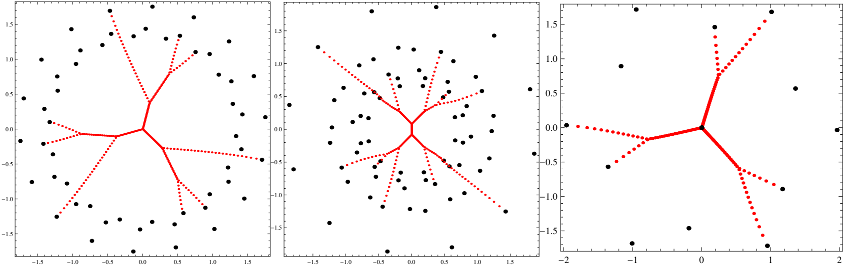


Figure : More examples.

Current progress in the special degenerate case

Joint with Milos Tater (next speaker)

We start our consideration with a simpler case of degenerate exactly solvable operators T for which the set A_T contains a single element $l > j^T$. In this case (5) becomes a simple two-term equation

$$\alpha z^m \mathcal{C}^l + z^{j^T} \mathcal{C}^{j^T} = 1, \tag{6}$$

where $\alpha \in \mathbb{C}^\star$, $0 \leq m < l > j^T \geq 1$. (Observe that, in general, l might be smaller than k which is the order of T. Also by abuse of notation, we surpress the index T everywhere.)

Observe that in this case $d = \frac{l-j}{l-m} > 0$ (where $j = j^T$). We start with the following simple statement.

Lemma

The set of non-vanishing branching points of the projection of the algebraic curve given by (6) *on the z-plane is given by the equation*

$$z^{\frac{j(l-m)}{l-j}} = \frac{l}{l-j}\left(\frac{-j}{\alpha l}\right)^{\frac{l-j}{j}}. \tag{7}$$

Theorem (Not quite settled at the moment)

For any complex number $\alpha \in \mathbb{C}^{\star}$ *and any triple of integers* $0 \leq m < l > j \geq 1$, *there exists a unique probability measure* μ *whose Cauchy transform satisfies equation* (6) *a.e. in* $\mathbb{C}$. *Its support consists of a number of straight segments connecting the origin to some of the branching points of* (6). *Namely, these segments connect the origin with each* $\frac{j}{\mathrm{GCD}[j,l]}$*-th branching point, see Figures on the next slide.*

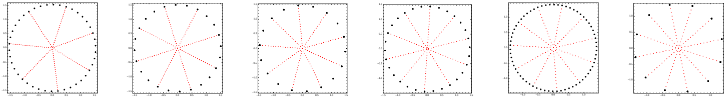


Figure : Measure μ for $j = 6,\ m = 0$ and $l = 7, 8, \ldots, 12$.

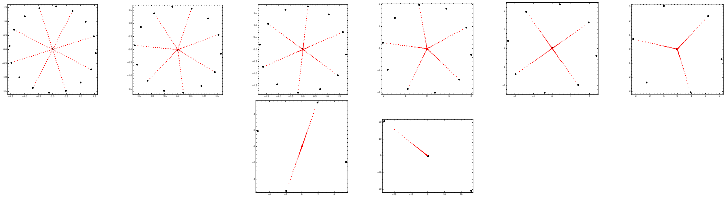


Figure : Measure μ for $j = 6,\ l = 9$ and $m = 1, 2, \ldots, 8$.

Corollary

The total number of legs in the support of the measure satisfying equation (6) *equals $l - m$. In particular, to have a chance to be Bochner-Krall $l - m$ should be* 1 *or* 2 *(Laguerre-like and Hermite-like).*

final question

Problem 5. For which bivariate polynomials $P(\mathcal{C}, z)$, there exists a positive measure μ supported on a finite number of compact curves and points such that its Cauchy transform $\mathcal{C}_\mu$ satisfies:

$$P(\mathcal{C}_\mu, z) = 0,$$

almost everywhere in $\mathbb{C}$?

Partial answer in R. Bøgvad, B.Sh arXiv:1406.1972

Last punch!

Question. If there exists a positive measure solving the later problem is it always unique?

Observe that there are examples of non-uniqueness in analytic setting!

Thank you for your attention and patience