Return of the plane evolute

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October 19, 2021

Topics to discuss

Main references

(i) C. Riener, R. Piene, B. Shapiro, Return of the plane evolute, in preparation.

(ii) G. Salmon, A treatise on the higher plane curves: intended as a sequel to “A treatise on conic sections". 3rd ed. Chelsea Publishing Co., New York 1960 xix+395 pp.

(iii) C. Huygens, Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae, (1673).
https://archive.org/details/B-001-004-158/page/n59/mode/2up
(English translation, Richard J Blackwell, Christiaan Huygens' the pendulum clock. Ames : Iowa State University Press, 1986.)

As we normally teach our students in a basic calculus class, the evolute of a curve in the Euclidean plane is the *locus of its centers of curvature*.

Wikipedia, – “Apollonius (c. 200 BC) discussed evolutes in Book V of his treatise Conics.

However, Huygens is sometimes credited with being the first to study them. Huygens formulated his theory of evolutes sometime around 1659 to help solve the problem of finding the tautochrone curve, which in turn helped him construct an isochronous pendulum. This was because the tautochrone curve is a cycloid, and the cycloid has the unique property that its evolute is also a cycloid. The theory of evolutes, in fact, allowed Huygens to achieve many results that would later be found using calculus."

Figure: Christiaan Huygens

CHRISTIANI
HVGENII
ZVLICHEMII, CONST. F.
HOROLOGIVM
OSCILLATORIVM
SIVE
DE MOTV PENDVLORVM
AD HOROLOGIA APTATO
DEMONSTRATIONES
GEOMETRICÆ.

PARISIIS,
Apud F. Muguet, Regis & Illustrissimi Archiepiscopi Typographum,
viâ Citharæ, ad insigne trium Regum.
MDCLXXIII.
CVM PRIVILEGIO REGIS.

Figure: This book freely available on the internet contains a large number of beautiful illustrations including that of evolutes.

Among several dozens of books on (plane) algebraic curves available now only very few by Coolidge, Hilton and Salmon mention evolutes at all, the best of them being Salmon's book first published more than one and half century ago. Some

properties of evolutes have been studied in connection with the so-called 4-vertex theorem of Mukhopadhyaya-Kneser as well as its generalizations. Their definition has been generalized from the case of plane curves to that of plane fronts and also from the case of Euclidean plane to that of Poincaré disk. Singularities of evolutes and involutes have been discussed in details by V. Arnold and his school and more recently by a group of Brazilian mathematicians.

From the computational point of view the most useful presentation of the evolute of a plane curve is as follows. Using a local parametrization of a curve Γ in $\mathbb{R}^2$ one can parameterize its evolute E_Γ as

$$E_\Gamma(t) = \Gamma(t) + \rho(t)\bar{n}(t), \tag{1}$$

where $\rho(t)$ is its curvature radius at the point $\Gamma(t)$ (assumed non-vanishing) and $\bar{n}(t)$ is the unit normal at $\Gamma(t)$ pointing towards the curvature center. In Euclidean coordinates, for $\Gamma(t) = (x(t), y(t))$ and $E_\Gamma(t) = (X(t), Y(t))$, one gets the following explicit expression

$$\begin{cases} X(t) = x(t) - \frac{y'(t)(x'(t)^2+y'(t)^2)}{x'(t)y''(t)-x''(t)y'(t)} \\ Y(t) = y(t) + \frac{x'(t)((x'(t))^2+(y'(t))^2)}{x'(t)y''(t)-x''(t)y'(t)} \end{cases} . \tag{2}$$

If a curve Γ is given by an equation $f(x,y)=0$, then the equation of its evolute can be obtained as follows. Consider the system

$$\begin{cases} f(x,y)=0 \\ X = x + \frac{f'_x((f'_x)^2+(f'_y)^2)}{2f'_xf'_yf''_{xy}.-(f'_y)^2f''_{xx}-(f'_x)^2f''_{yy}} \\ Y = y + \frac{f'_y((f'_x)^2+(f'_y)^2)}{2f'_xf'_yf''_{xy}.-(f'_y)^2f''_{xx}-(f'_x)^2f''_{yy}} \end{cases} . \tag{3}$$

defining the original curve and the family of centers of its curvature circles. Then eliminating the variables (x,y) from (3) one obtains a single equation defining the evolute in variables (X,Y). For concrete bivariate polynomials $f(x,y)$ of lower degrees, such an elimination procedure can be carried out in Macaulay 2.

Example

Two classically known explicit examples of the evolutes are as follows.

For the parabola $\Gamma = (t, t^2)$, its evolute is given by $E_\Gamma = (-4t^3, \frac{1}{2} + 3t^2)$ which is a semicubic parabola satisfying the equation $27X^2 = 16(Y - \frac{1}{2})^3$, see Fig. 4, left.
For the ellipse $\Gamma = (a\cos t, b\sin t)$, the evolute is given by

$$E_\Gamma = (\frac{a^2-b^2}{a}\cos^3 t, \frac{b^2-a^2}{b}\sin^3 t),$$

which is an astroid satisfying the equation $(aX)^{2/3} + (bY)^{2/3} = (a^2 - b^2)^{2/3}$, see Fig. 4, right.

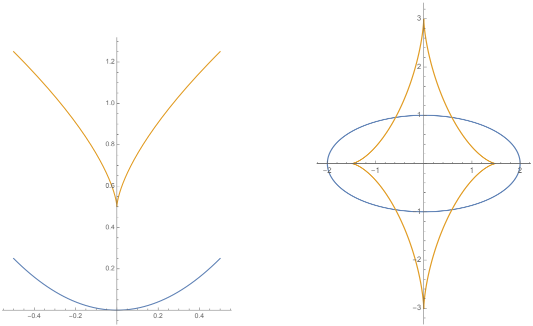

Figure: The evolutes of a parabola and an ellipse

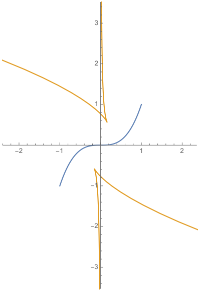

Figure: The evolute near an inflection point

Observe that if Γ is a rational algebraic curve, then the above recipe provides the global parametrization of E_Γ. Given a plane curve Γ, the alternative definition of its evolute E_Γ which will be particularly useful for us is that E_Γ is the *envelope of the family of normals to* Γ, where a normal of Γ is an affine line perpendicular to Γ at some point.

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In other words, each normal to Γ is a tangent line to E_Γ and each tangent to E_Γ is a normal to (the analytic continuation) of Γ. From the definition it follows that the evolute $E_\Gamma \subset \mathbb{R}^2$ is the caustic=critical locus of the projection of the cotangent bundle $T^*\Gamma \subset T^*\mathbb{R}^2$ to the initial curve Γ to the (phase) plane $\mathbb{R}^2$. This circumstance explains, in particular, why the singularities of the evolutes behave differently from those of (generic) plane algebraic curves.

Definition

For a plane algebraic curve $\Gamma \subset \mathbb{R}^2 \subset \mathbb{R}P^2$, define its *curve of normals* $\tilde{N}_\Gamma \subset (\mathbb{R}P^2)^*$ as the curve on the dual projective plane whose points are the normals of Γ. (We start with the quasiprojective curve N_Γ of all normals to Γ and take its projective closure in $(\mathbb{R}P^2)^*$.)

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Similarly to the above, for a (locally) parameterized curve $\Gamma(t) = (x(t), y(t))$ and $N_\Gamma(t) = (U(t), V(t))$, one gets

$$\begin{cases} U(t) = \frac{x'(t)}{y'(t)} \\ V(t) = -\frac{x(t)x'(t)+y(t)y'(t)}{y'(t)} \end{cases} . \qquad (4)$$

(Here we assume that the equation of the normal line to Γ at the point $(x(t), y(x))$ is taken in the form $Y + U(t)X + V(t) = 0$.)

Let us first summarize some complex-algebraic facts about the evolute and the curve of normals mainly borrowed from the classical treatise [Sa].

Proposition (see Art. 111, 112, p. 94–96, in [Sa])

For an affine real-algebraic curve $\Gamma \subset \mathbb{R}^2$ of degree d, in general position with respect to the line at infinity, and having only δ nodes and κ ordinary cusps as singularities, the curves $\tilde{\Gamma}^{\mathbb{C}}$, $\tilde{E}_\Gamma^{\mathbb{C}}$ and $\tilde{N}_\Gamma^{\mathbb{C}}$ are birationally equivalent. The degree of $\tilde{E}_\Gamma$ equals $3d(d-1) - 6\delta - 8\kappa$, while the degree of $\tilde{N}_\Gamma$ equals $d^2 - 2\delta - 3\kappa$.

The genericity assumption for the birationality can be substantially weakened (but not completely removed).

Lemma (see Art. 113, p. 96, in [Sa])

For a generic affine real-algebraic curve $\Gamma \subset \mathbb{R}^2$ of degree d, $\tilde{E}_\Gamma$ has no inflection points.

Proposition (see Art. 113, p. 97, of [Sa])

For an affine real-algebraic curve $\Gamma \subset \mathbb{R}^2$ as in Prop. 1, the only singularities of $\tilde{E}_\Gamma^{\mathbb{C}}$ and $\tilde{N}_\Gamma^{\mathbb{C}}$ are nodes and cusps, except that $\tilde{N}_\Gamma^{\mathbb{C}}$ has an ordinary d-uple point (the line at infinity). The number $\#_c(E)$ of cusps of $\tilde{E}_\Gamma^{\mathbb{C}}$ equals $3(d(2d-3)-4\delta-5\kappa)$. If Γ is nonsingular, the number $\#_n^E$ of nodes of $\tilde{E}_\Gamma^{\mathbb{C}}$ equals $\frac{d}{2}(3d-5)(3d^2-d-6)$, and the number $\#_n^N$ of nodes of $\tilde{N}_\Gamma^{\mathbb{C}}$ equals $\binom{d^2-1}{2}-\binom{d-1}{2}-\binom{d}{2}=\binom{d}{2}(d^2+d-4)$. There are no cusps on $\tilde{N}_\Gamma^{\mathbb{C}}$ (since $\tilde{E}_\Gamma$ has no inflection points).

Remark

Notice that a crunode of N_Γ (i.e., the real node with two real branches) corresponds to the *diameter* of Γ which is a straight segment connecting pairs of points on Γ and which is perdendicular to the tangent lines to Γ at these endpoints. Observe also that a real cusp of E_Γ (resp. an inflection point on N_Γ) corresponds to a *vertex* of Γ which is a critical point of Γ's curvature. (As we mentioned above, vertices of plane appear, for example, in the classical 4-vertex theorem and its numerous generalizations. Beautiful lower bounds on the number of diameters of plane curves, plane wavefronts as well as their higher dimensional generalizations have been obtained in symplectic geometry, see e.g. [Pu1, Pu2].)

To formulate our problems we need to introduce the following notion which deserves to be better known, cf. [LSS].

Definition

Given a real-algebraic hypersurface $H \subset \mathbb{R}^n$, we define its $\mathbb{R}$-*degree* as the supremum of the cardinality of $H \cap L$ taken over all lines $L \subset \mathbb{R}^n$ such that L intersects H transversally. (Observe that we count points in $H \cap L$ without multiplicity.)

In what follows, we denote the $\mathbb{R}$-degree of H by $\mathbb{R}\deg(H)$. For a real-algebraic hypersurface $H \subset \mathbb{R}^n$, one has $\mathbb{R}\deg(H) \leq \deg(H)$ where $\deg(H)$ is the usual degree of H.

Recall that real nodes of real-algebraic curves are classically subdivided into *crunodes* and *acnodes* the former being transversal intersections of two real branches and the latter being transversal intersections of two complex conjugate branches.

In what follows, we discuss four real-algebraic questions related to the evolutes and curves of normals of plane real-algebraic curves.

Problem (1)

For a given positive integer d, what are the maximal possible $\mathbb{R}$-degrees of the evolute E_Γ and of the curve of normals N_Γ where Γ runs over the set of all real-algebraic curves of degree d?

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Problem (2)

For a given positive integer d, what its the maximal possible number of real cusps on E_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of vertices a real-algebraic curve Γ of degree d might have?

Problem (3)

For a given positive integer d, what its the maximal possible number of crunodes on N_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of (real) diameters Γ might have?

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For a given positive integer d, what its the maximal possible number of crunodes on N_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of (real) diameters Γ might have?

Problem (4)

For a given positive integer d, what its the maximal possible number of crunodes on E_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal possible number of points in $\mathbb{R}^2$ which are the centers for at least two distinct (real) curvature circles of Γ?

$\mathbb{R}$-degree of the evolute

Proposition

For any $d \geq 3$, the maximal $\mathbb{R}$-degree among the evolutes of algebraic curves of degree d is not less than $d(d-2)$.

Proof.

Recall that each real inflection point of a real curve corresponds to its evolute going to infinity. Notice that from Klein's theorem follows that a real-algebraic curve of degree d has at most one third of its inflection points real and this bound is achieved. The number of complex inflection points of a generic curve of degree d equals $3d(d-2)$. Thus there exists a smooth real-algebraic curve of degree d with $d(d-2)$ real inflection points. The evolute of such curve hits the line at infinity (transversally) at $d(d-2)$ real points. Thus its $\mathbb{R}$-degree is at least $d(d-2)$. □

Remark

The above lower bound is apparently not sharp. For $d = 2$ the sharp bound is 4. For $d = 3$, taking a small deformation of three lines creating a compact oval one gets an example with $\mathbb{R}$-degree of the evolute greater than or equal to 6 while the number of real inflections is 3. The complex number is $3d(d-1)$ which has leading coefficient 3 while our bound has leading coefficient 1. The correct leading coefficient at d^2 is unknown at the moment.

$\mathbb{R}$-degree of the curve of normals

Our second result solves the second part of Problem (1) about the maximal $\mathbb{R}$-degree of the curve of normals.

Proposition

There exists a real-algebraic curve Γ of degree d and a point $p \in \mathbb{R}^2$ such that all d^2 complex normals to Γ through p are, in fact, real. In other words, the maximal $\mathbb{R}$-degree of N_Γ equals d^2 which is the usual degree of N_Γ.

Proof: A crunode (which is a transversal intersection of two smooth real local branches) admits two types of real smoothing. By theorem of Brusotti, [Br], any (possibly reducible) plane real-algebraic curve with only nodes as singularities admits a small real deformation which realizes independently prescribed smoothing types of all its crunodes.

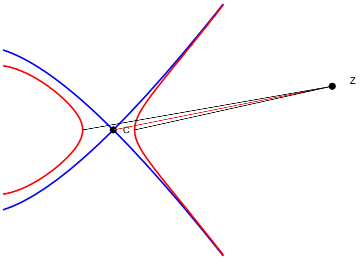

Figure: Resolution of a crunode

Given a crunode and a point z such that the line L through this point and through the crunode is not tangent to the real local branches at the crunode, there exists a smoothing type of the crunode such that, slightly rotating the line L around z, one obtains two real normals to this smoothing. Now take an arrangement $\mathcal{A} \subset \mathbb{R}^2$ of d real lines in general position and a point z outside these lines. By Brusotti, smoothing all $d(d-1)/2$ nodes in an appropriate way we obtain $d(d-1)$ normals close to the lines joining z with the nodes of $\mathcal{A}$. Additional d normals are obtained by small deformations of the altitudes connecting z with each of the d given lines. Thus, there exist d^2 real normals through z implying the $\mathbb{R}$-degree of the curve of normals for the obtained curve is a least d^2. But its usual degree is d^2. The result follows.

Vertices of the initial curve = real cusps of the evolute

Proposition

The number of real cusps for the evolute of an arbitrary small deformation $\mathcal{R}$ of any generic line arrangement $\mathcal{A} \subset \mathbb{R}^2$ consisting of d lines equals $d(d-1)$ plus the number of bounded edges of $\mathcal{A}$ respected by $\mathcal{R}$.

All bounded edges of $\mathcal{A}$ are respected by $\mathcal{R}$ if and only if the small deformation $\mathcal{R}(\mathcal{A})$ is a convex curve; in this case the total number of cusps on its evolute equals $d(2d-3)$ which is the maximal possible number among small deformations of line arrangements and is exactly 1/3 of $\mathbb{C}\,Vert(d) = 3d(2d-3)$.

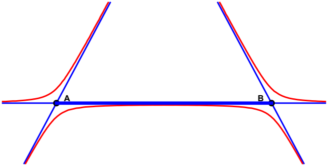

Figure: The resolution respects the the bounded edge

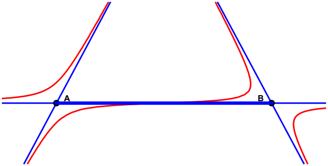

Figure: The resolution twists the the bounded edge

Proof: Consider the complement $\mathbb{R}^2 \setminus \mathcal{A}$. It consists of $2d$ infinite convex polygons and $\binom{d-1}{2}$ bounded convex polygons. Now take any small deformation $\mathcal{R}$ of $\mathcal{A}$. Locally near any vertex v of $\mathcal{A}$ the smooth curve $\mathcal{R}(\mathcal{A})$ will consists of two convex branches for each of which the curvature has a local maximum near v. These local maxima will correspond to two cusps on the evolute of $\mathcal{R}(\mathcal{A})$ which gives totally $2\binom{d}{2} = d(d-1)$ cusps corresponding to local maxima of curvature. Let us now show that every bounded edge of $\mathcal{A}$ respected by $\mathcal{R}$ corresponds to the unique point on $\mathcal{R}(\mathcal{A})$ where the curvature attains its minimum.

Moreover all extremal points of the curvature belong either to the first or to the second types. On the other hand, every twisted edge corresponds to an inflection point on $\mathcal{R}(\mathcal{A})$ which means that the evolute goes to infinity. The total number of bounded edges of any generic arrangement with d lines equals $d(d-2)$. Let us show that one can exist exactly two small deformations for which all bounded edges will be respected and in such case we get $d(d-1)+d(d-2)=d(2d-3)$ extrema of curvature on $\mathcal{R}(\mathcal{A})$. Indeed, observe that the chromatic number of the chart obtained from any generic line arrangement is 2, i.e., we can colored the components of the complement $\mathbb{R}^2 \setminus \mathcal{A}$ into black and white so that every pair of neghbours will get different colors.

Remark

It is not clear that Klein's bound $1/3$ is valid for evolutes which are highly singular curves. We tried to apply Klein's equation to the evolute, but have not got any definite conclusion yet. Thus it is not clear at the moment whether our lower bound is optimal.

Diameters of the original curve = crunodes of the curve of normals

Lemma

Given a strongly generic line arrangement $\mathcal{A}$, the following holds:

(i) Any small resolution of a vertex of $\mathcal{A}$ creates one diameter;

(ii) If an altitude al is admissible w.r.t. a small deformation $\mathcal{R}$ then $\mathcal{R}$ creates two diameters close to al;

(iii) If v_1 and v_2 have each other in sight w.r.t. a small deformation $\mathcal{R}$ then $\mathcal{R}$ creates four diameters close to the segment (v_1, v_2).

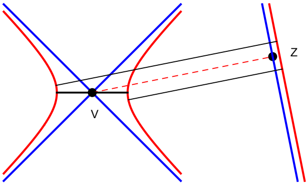

Figure: Resolution of an altitude

Finally, for a given strongly generic $\mathcal{A}$, its two vertices v_1 and v_2 and any resolution $\mathcal{R}$, we say that v_1 and v_2 *have each other in sight w.r.t.* $\mathcal{R}$ if $v_2 \in \mathcal{C}^{\perp}_{v_1}(\mathcal{R})$ and $v_1 \in \mathcal{C}^{\perp}_{v_2}(\mathcal{R})$.

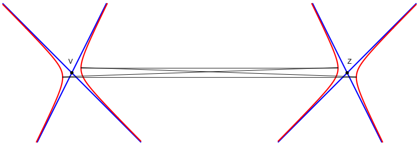

Figure: Two vertices having each other in sight.

Proposition

Given any small resolution $\mathcal{R}$ of a strongly generic arrangement $\mathcal{A}$ consisting of d lines, the number of diameters of the obtained smooth curve $\mathcal{R}(\mathcal{A})$ equals

$$\#_{diam}(\mathcal{R}(\mathcal{A})) = \#_{ver} + 2\#_{adm.alt} + 4\#_{\text{cone pairs in sight}},$$

where $\#_{ver} = \binom{d}{2}$ is the number of vertices of $\mathcal{A}$, $\#_{adm.alt}$ is the number of admissible altitudes w.r.t. $\mathcal{R}$, and $\#_{\text{cone pairs in sight}}$ is the number of pairs having each other in sight.

Proof.

Indeed, firstly, any small resolution of a strongly generic line arrangements creates a short diameter connecting the two sectors of every persistent cone, see Lemma 17(i). Thus there are $\#_{ver} = \binom{d}{2}$ such short diameters. Then each admissible altitude will split into two nearby diameters connecting the two sectors of the persistent cone with a point close to the base point of the altitude, see Lemma 17(ii). Finally, each pairs of dual persistent sectors in proper position centered at the vertices v_1 and v_2 creates 4 diameters close to the straight segment v_1, v_2. Such diameters connect each of the two sectors close to v_1 with each of two sectors close to v_2, see Lemma 17(iii). No other diameters are possible. □

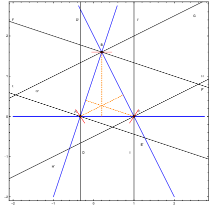

Figure: Three lines (blue), their altitudes (dotted in orange).

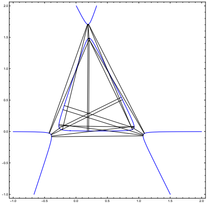

Figure: A small resolution creating 21 diameters (black).

Proposition 18 implies the following lower bound for $\mathbb{R}Diam(d)$.

Proposition

In the above notation,

$$\mathbb{R}Diam(d) \geq \frac{d^4}{2} - d^3 + \frac{d}{2}. \qquad (5)$$

To settle Proposition 20 we need to introduce some class of arrangements. We say that an arrangement $\mathcal{A}$ is *oblate* if the slopes of all lines in $\mathcal{A}$ are close to each other. As a particular example, one can take d lines tangent to the graph of $\arctan x$ for d values of the variable x of the form
$101, 102, 103, \dots, 100 + d$.

Sketch of proof.

For the special small resolution of an oblate arrangement for which is that we will making narrow cones at each vertex as the persistent ones the following diameters will be present. Every pair of persistent cones will be in proper position and contribute 4 diameters and each vertex will contribute 1 diameter. On the other hand, all altitudes will be non-admissible. Thus we get $4\binom{\binom{d}{2}}{2} + \binom{d}{2} = \frac{d^4}{2} - d^3 + \frac{d}{2}$ diameters for this resolution. □

Crunodes of the evolute

Recall that the number of nodes of the evolute of a generic curve of degree d is given by

$$\delta_E(d) = \frac{d}{2}(3d-5)(3d^2-d-6).$$

Denote by $\delta_E^{\mathrm{cru}}(d)$ the maximal number of crunodes for the evolutes of real-algebraic curves of degree d.
At the moment we have only a (not completely proven) lower bound for this number of crunodes.

Proposition

We have $\delta_E^{\mathrm{cru}}(d) \geq (\lfloor \frac{d-2}{2} \rfloor + d - 2)^4 - \frac{1}{2}$.

Final remarks

There is apparently a lot of space for improvement of the suggested bounds (which are very naive) as well as for other real-algebraic problems related to the evolutes, curves of normals and their high-dimensional analogs!

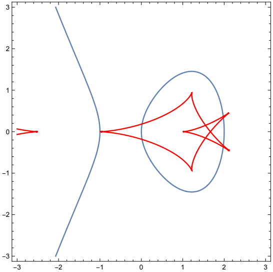

Figure: The Weierstrass cubic in blue and its evolute in red

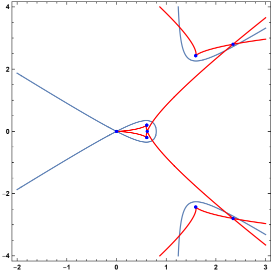

Figure: The nodal cubic $5(x^2 - y^2)(x - 1) + (x^2 + y^2) = 0$ in blue and its evolute in red

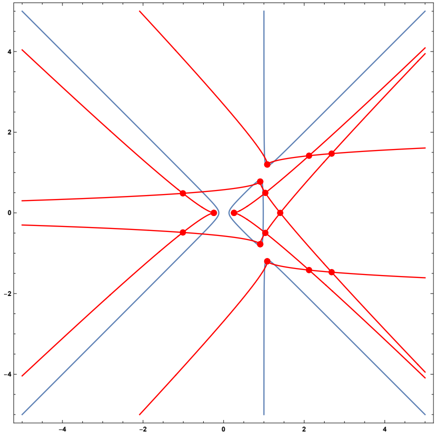

Figure: A non-singular cubic in blue and its evolute in red

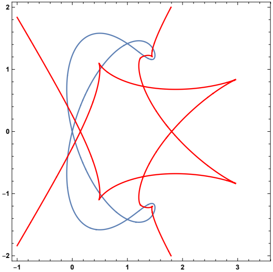

Figure: The ampersand curve in blue and its evolute in red

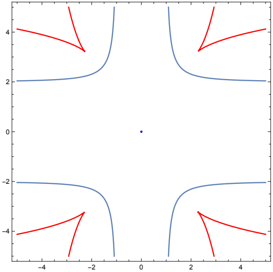

Figure: The cross curve in blue and its evolute in red

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